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Abstract tangent functors

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Our aim is to axiomatize properties of the tangent functor $T : M \rightarrow M$ on the category of smooth manifolds. The resulting abstract tangent functor $T : C \rightarrow C$ on a category C has the property that there is a well-behaved bracket operation of its sections $A \rightarrow TA$ on any object $A \in C$. Representable abstract tangent functors are closely connected with rings of line type in the sense of Kock - Lawvere.

1. Natural group bundles

Let C be a category, $F, G : C \rightarrow C$ functors and $p : G \rightarrow F$ a natural transformation. Let $\otimes$ denote products in the category of functors over F . It means that

$$\begin{array}{ccc} G \otimes G & \xrightarrow{p_1} & G \\ p_2 \downarrow & & \downarrow p \\ G & \xrightarrow{p} & F \end{array}$$

is a pullback. We say that (G, p) is a natural group bundle over F if it is a group in the category of functors over F . It means that G is equipped with natural transformations

$$+ : G \otimes G \rightarrow G, \quad - : G \rightarrow G, \quad 0 : F \rightarrow G$$

which are over F (i.e. $p \circ + = p \circ p_1$, $p \circ - = p$ and $p \circ 0 = 1$) and satisfy the group axioms. A natural group bundle is a natural group bundle over the identity functor on C .

Let (G, p) be a natural group bundle over F and (H, q) a natural group bundle over E . A homomorphism $(f, g): (G, p) \rightarrow (H, q)$ is a couple of natural transformations $f: G \rightarrow H$ and $g: F \rightarrow E$ such that the following diagrams commute

$$\begin{array}{ccc}
 G & \xrightarrow{f} & H \\
 p \downarrow & & \downarrow q \\
 F & \xrightarrow{g} & E
 \end{array}
 \qquad
 \begin{array}{ccc}
 & f \circ f & \\
 G \circ G & \xrightarrow{\quad} & H \circ H \\
 + \downarrow & & \downarrow + \\
 G & \xrightarrow{f} & H
 \end{array}$$

Let (G, p) be a natural group bundle. If the pullbacks $G \circ G$ and $G \circ G \circ G$ are pointwise then (G^2, pG) is a natural group bundle over G with respect to $+G$, $-G$, OG and $(Gp, p): (G^2, pG) \rightarrow (G, p)$ is a homomorphism. If G preserves $G \circ G$ and $G \circ G \circ G$ then (G^2, Gp) is a natural group bundle over G and $(pG, p): (G^2, Gp) \rightarrow (G, p)$ is a homomorphism.

2. Tangent functors

We say that $T: C \rightarrow C$ is a tangent functor if there are

p, i and m such that

(T1) (T, p) is a natural commutative group bundle such that

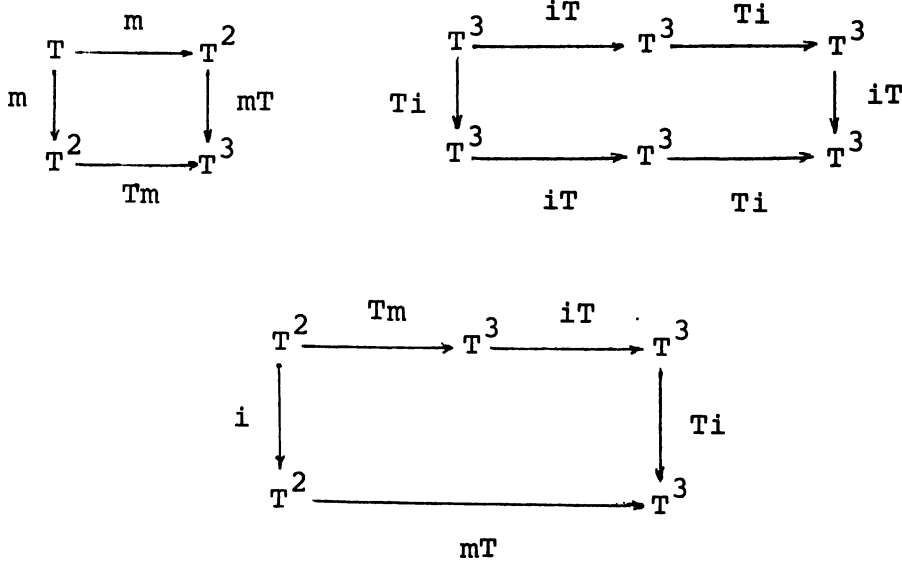
pullbacks $T \circ T$ and $T \circ T \circ T$ are pointwise and preserved by T .

(T2) $(i, 1): (T^2, pT) \rightarrow (T^2, Tp)$ is a homomorphism and $i^2 = 1$.

(T3) $(m, 0): (T, p) \rightarrow (T^2, pT)$ is a homomorphism such that $i.m = m$ and the following diagram is a pointwise equalizer

$$\begin{array}{ccccc}
 T & \xrightarrow{m} & T^2 & \xrightarrow[pT]{pT} & T \\
 & & & \parallel & \\
 & & & 0.p.pT &
 \end{array}$$

(T4) The following diagrams commute



These axioms are satisfied by the tangent functor $T : M \rightarrow M$ on the category of smooth manifolds. Since $T(\mathbb{R}^n) = \mathbb{R}^{2n}$, the following example also indicates the description of i and m in local coordinates.

Example 1 : Let Ab be the category of abelian groups and $T = (-)^2$. Then $p_A(a,b) = a$, $(a,b) +_A (a,c) = (a,b+c)$, $-_A(a,b) = (a,-b)$, $0_A(a) = (a,0)$, $i_A(a,b,c,d) = (a,c,b,d)$ and $m_A(a,b) = (a,0,0,b)$ make from T a tangent functor. We indicate that $T \circ T = (-)^3$ and $(a,b,c,d) +_{TA} (a,b,c',d') = (a,b,c+c',d+d')$ and $(a,b,c,d) \xrightarrow{+_A} (a,b',c,d') = (a,b+b',c,d+d')$. Here $+_{TA}$ and $+_A$ denote the addition in group bundles (T^2_A, p_{TA}) and $(T^2_A, T(p)_A)$ over TA .

Example 2 : Let R be the category of commutative rings. Let $TA = A[x] \setminus x^2$. Then TA consists of polynomials $a + bx$, $(T \circ T)A = A[x,y] \setminus x^2, y^2, xy$ of polynomials $a+bx+cy$ and $T^2A = A[x,y] \setminus x^2, y^2$ of polynomials $a+bx+cy+dxy$. Then T is a tangent functor and its structure is given by the same formulas as in the preceding example.

Example 3 : Let R be a ring of the line type in a cartesian closed category E and $D = \{d \in R \mid d^2 = 0\}$ (see Kock [2]). Let C be a full subcategory of E which consists of all infinitesimally linear objects having the property W . Then $T = (-)^D$ is a tangent functor on C . Here $T \circ T = (-)^{D(2)}$ where $D(2) = \{(d_1, d_2) \in D \times D \mid d_1 \cdot d_2 = 0\}$, $+ : T \circ T \rightarrow T$ is represented by the diagonal $D \rightarrow D(2)$, $T^2 = (-)^{D \times D}$, i is represented by the symmetry $s : D \times D \rightarrow D \times D$ and m by the multiplication $\cdot : D \times D \rightarrow D$.

A concrete example is the category E of functors from the category R_0 of finitely presented commutative rings to the category of sets. R is $R_0(Z, -)$ and D is $R_0(Z[x] \setminus x^2, -)$. We took R_0 to avoid set theoretical difficulties with functor categories. It is evident that Example 2 works for R_0 , too. Hence the tangent functor $(-)^D$ on $C \subset E$ is derived from the tangent functor on R_0 . It is a general phenomenon.

Proposition 1 : Let C be a small category and $T : C \rightarrow C$ a tangent functor on C . Let B be the full subcategory of the functor category Set^C consisting of all functors which preserve pullbacks $T \circ T$, $T \circ T \circ T$ and the equalizer from (T3). Then

$$T^*(V) = V \cdot T \quad , \quad T^*(\alpha) = \alpha T$$

yields the tangent functor $T^* : B \rightarrow B$.

Another general construction of new tangent functors is the following one. Let T be a tangent functor on a category C . Consider the comma category $B = C \setminus A$ where $A \in C$. Then

$$\tilde{T}(X, f) = (TX, f \cdot p_X) \quad , \quad \tilde{T}(h) = T(h)$$

yields the tangent functor $\tilde{T} : B \rightarrow B$. Let

$$\begin{array}{ccc} \bar{T}X & \xrightarrow{v_x} & TX \\ & & \xrightarrow[\quad O_A \cdot p_A \cdot T(f) \quad]{\quad T(f) \quad} TA \end{array}$$

be an equalizer. If T preserves this equalizer then

$$\bar{T}(X, f) = (\bar{T}X, f \cdot p_x \cdot v_x)$$

provides the tangent functor $\bar{T} : B \rightarrow B$.

If $T : M \rightarrow M$ is the tangent functor on the category of smooth manifolds then $\bar{T}$ gives the vertical bundle on the category of fibered manifolds.

The following property of tangent functors is very important.

Lemma 1 : Let T be a tangent functor and consider the composition

$$e : T \oplus T \xrightarrow{\langle m \cdot p_2, T(0) \cdot p_1 \rangle} (T \oplus T)T \xrightarrow{+T} T^2$$

Then the diagram

$$\begin{array}{ccc} T \oplus T & \xrightarrow{e} & T^2 \\ & & \xrightarrow[\quad O \cdot p \cdot pT \quad]{\quad pT \quad} T \end{array}$$

is an equalizer.

Since $T(p) \cdot e = p_1$, it says that if $f : S \rightarrow T^2$ is a natural transformation equalizing pT and $O \cdot p \cdot pT$ then there is a unique natural transformation $g : S \rightarrow T$ such that

$$(1) \quad f = T(p \cdot 0) \cdot f +_T m \cdot g.$$

3. Bracket operation

Let T be a tangent functor on a category C and $A \in C$. A morphism $r : A \rightarrow TA$ is called a section of T if $p_A \cdot r = 1$. Hence r is a T -coalgebra in the terminology of Kelly [1].

If $T : M \rightarrow M$ is the usual tangent functor, the sections are vector fields. We want to define the bracket $[r,s]$ of sections $r,s : A \rightarrow TA$ of any tangent functor. For this purpose, we would need the description of the bracket of vector fields in terms of T only and not using functions on manifolds. This description was given by Kolář [3] and we will follow it. Very similar description is stated in White [8].

Let $r,s : A \rightarrow TA$ be sections. Since $T(p)_A \cdot T(s) \cdot r = r = T(p)_A \cdot i_A \cdot T(r) \cdot s$, it is defined the difference

$$v = (T(s) \cdot r)_{T^{-1}A} (i_A \cdot T(i) \cdot s) .$$

It is easy to see that $p_{TA} \cdot v = 0_A$. Following lemma 1, there is a unique morphism $\bar{v} : A \rightarrow (T \oplus T)A$ such that $e_A \cdot \bar{v} = v$. Put

$$[r,s] : A \xrightarrow{\bar{v}} (T \oplus T)A \xrightarrow{(p_2)_A} TA .$$

In example 1, sections $r : A \rightarrow A^2$ correspond to endomorphisms $r : A \rightarrow A$. The bracket is the usual bracket of endomorphisms

$$[r,s] = s \cdot r - r \cdot s .$$

In example 2, sections $r : A \rightarrow A[x] \setminus x^2$ coincide with derivations $r : A \rightarrow A$ and the bracket is the usual bracket of derivations. In example 3, the bracket is the bracket from the synthetic differential geometry (see [2]). It follows from the fact that

$$m_A \cdot [r,s] = (+ \cdot k)_{TA} \cdot T^2(+ \cdot k)_A \cdot T(i)_{TA} \cdot -T^3_A \cdot T(-)_{T^2A} \cdot T^3(s) \cdot T^2(r) \cdot T(s) \cdot r$$

and that this formula corresponds to the commutator of infinitesimal transformations. Here $k : T^2 \rightarrow T \oplus T$ is given by

$$p_1 \cdot k = p^T \quad \text{and} \quad p_2 \cdot k = T p .$$

Theorem 1 : Let T be a tangent functor. Then the bracket operation has properties

$$(B1) \quad [r+s, t] = [r, t] + [s, t]$$

$$(B2) \quad [s, r] = -[r, s]$$

$$(B3) \quad [r, [s, t]] + [s, [t, r]] + [t, [r, s]] = 0$$

It is not difficult to calculate (B1) - (B3) for manifolds without using functions (see Vanžurová [7]). In the general case, we are facing the coherence problem for tangent functors. Remark that (B1) and (B2) hold on the basis of axioms (T1) - (T3) only. The proofs from synthetic differential geometry (see Reyes, Wraith [6], Lavendhomme [5] and Kock [2]) do not work in the general case. Our proof is "additive" and consists in calculations with natural group bundles T , T^2 and T^3 over 1 , T and T^2 given by p, pT, Tp, pT^2, pTp and T^2p .

We did not need the R -linear structure on $T : M \rightarrow M$. However, it is present in the general case, too. We say that $h : T \rightarrow T$ is an endomorphism of a tangent functor $T : C \rightarrow C$ if the following diagrams commute

$$\begin{array}{ccc}
 T & \xrightarrow{h} & T \\
 p \searrow & & \swarrow p \\
 & 1 &
 \end{array}
 \qquad
 \begin{array}{ccc}
 T \oplus T & \xrightarrow{h \oplus h} & T \oplus T \\
 + \downarrow & & \downarrow + \\
 T & \xrightarrow{h} & T
 \end{array}$$

$$\begin{array}{ccc}
 T^2 & \xrightarrow{hT} & T^2 \\
 i \downarrow & & \downarrow i \\
 T^2 & \xrightarrow{Th} & T^2
 \end{array}
 \qquad
 \begin{array}{ccc}
 T & \xrightarrow{h} & T \\
 m \downarrow & & \downarrow m \\
 T^2 & \xrightarrow{hT} & T^2
 \end{array}$$

It means that h is an endomorphism of the natural group bundle

T and preserves the tangent structure given by i and m . The set R of all endomorphisms of T is a ring with the composition as the multiplication and the addition

$$g + h : T \xrightarrow{\langle g, h \rangle} T \oplus T \xrightarrow{+} T .$$

The morphisms $h_A : TA \rightarrow TA$ put an R -module structure on TA and T becomes a natural R -module bundle. If $h \in R$ and $r, s : A \rightarrow TA$ are sections of T then it holds

$$[h_A \cdot r, s] = h_A \cdot [r, s] .$$

Kolář [4] proved that if $T : M \rightarrow M$ then natural transformations $h : T \rightarrow T$ with $p \cdot h = p$ are precisely homotheties given by multiplying with $x \in R$. Any homothety is an endomorphism of T and therefore $R = R$ in this case. $R = \mathbb{Z}$ in ex. 1 and 2.

In synthetic differential geometry, any morphism $h : D \rightarrow D$ with $h(0) = 0$ gives an endomorphism $(-)^h : (-)^D \rightarrow (-)^D$. The ring of endomorphism of $(-)^D$ is the ring of 0 preserving morphisms $D \rightarrow D$. However, it is the starting ring R of the line type. To see it one has to realize that the line type property implies that 0 preserving morphism $h : D \rightarrow D$ coincide with morphisms $- \cdot x$, $x \in R$. Hence the line R is determined by its infinitesimal segment D .

4. Representable tangent functors

Let $\mathcal{C}$ be a cartesian closed category and $D \in \mathcal{C}$ such that $T = (-)^D : \mathcal{C} \rightarrow \mathcal{C}$ is a tangent functor. Let $p, +, -, 0, i$ and m be represented by $0 : 1 \rightarrow D$, $\delta : D \rightarrow D * D$, $- : D \rightarrow D$, $D \rightarrow 1$, $\cup : D^2 \rightarrow D^2$ and $\cdot : D^2 \rightarrow D$.

Here,

$$\begin{array}{ccc}
 1 & \xrightarrow{0} & D \\
 0 \downarrow & & \downarrow \pi_1 \\
 D & \xrightarrow{\pi_2} & D * D
 \end{array}$$

is a pushout. There is a unique morphism $+$: $D * D \rightarrow D$ such that $+$. $\pi_1 = +$. $\pi_2 = 1$.

Proposition 2 : $(D, .)$ is a semigroup with the zero 0 such that $d^2 = 0$ for any $d \in D$.

Categorical logic justifies the set theoretical terminology. The last assertion means that

$$\begin{array}{ccc}
 D & \xrightarrow{\Delta} & D^2 \\
 \downarrow & & \downarrow . \\
 1 & \xrightarrow{0} & D
 \end{array}$$

commutes where Δ is the diagonal. It follows from the fact that Δ is the composition

$$D \xrightarrow{\delta} D * D \xrightarrow{\kappa} D^2$$

and that

$$\begin{array}{ccc}
 D * D & \xrightarrow{\kappa} & D^2 \\
 \downarrow & & \downarrow . \\
 1 & \xrightarrow{0} & D
 \end{array}$$

commutes. The morphism κ represents $k : T^2 \rightarrow T \oplus T$, i.e.

$\kappa . \pi_1(d) = (0, d)$ and $\kappa . \pi_2(d) = (d, 0)$ for any $d \in D$. Hence $\kappa(D * D) \subseteq D(2) = \{(d_1, d_2) \in D^2 \mid d_1 . d_2 = 0\}$.

From now on, assume that T is represented in such a way that ι is the symmetry s . Then $(D, \cdot)$ is commutative because $i.m = m$.

Let an endomorphism of T be represented by a morphism $h : D \rightarrow D$. Then $h(0) = 0$, $\delta.h = (h^*h).\delta$ and

$$h(d_1.d_2) = h(d_1).d_2$$

for any $d_1, d_2 \in D$. Let $\bar{R}$ be the ring of these endomorphisms $h : D \rightarrow D$. The exponential transpose $\wedge : D \rightarrow D^D$ of $\cdot : D^2 \rightarrow D$ yields a morphism $j : D \rightarrow \bar{R}$.

Proposition 3 : For any morphism $f : D \rightarrow \bar{R}$ there is a unique element $h \in \bar{R}$ such that

$$f(d) = f(0) + h.j(d)$$

holds for any $d \in D$.

It is the translation of formula (1). Hence $\bar{R}^D \cong \bar{R}^2$ and $\bar{R}$ thus has some properties of a ring of a line type. We are missing the commutativity of $\bar{R}$ and the fact that

$$D = \{h \in \bar{R} \mid h^2 = 0\}.$$

Since $0 \times D \hat{=} 0$ where 0 is an initial object, the tangent functors from examples 1 and 2 are not representable. The tangent functor from example 1 is not a restriction of a representable tangent functor on a subcategory closed with respect to a terminal object. It follows from the fact that 0 is not a unique natural transformation $1 \rightarrow T$. On the other hand, there are very convenient extensions of $T : M \rightarrow M$ to a representable tangent functor (see Kock [2]).

This paper had a rather slow development due to difficulties with the proof of theorem 1. The other material was completed

in 1982. Proofs will appear elsewhere. I profited from discussions with G. Wraith and A. Kock. However, I am especially indebted to I. Kolář who patiently introduced me into basic differential geometry.

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