DIAGRAMMES

I. SAIN

Weak products for universal algebra and model theory

Diagrammes, tome 8 (1982), exp. nº 2, p. S1-S15
<http://www.numdam.org/item?id=DIA_1982__8__A2_0>

Diagrammes, Volume 8, Paris 1982.

WEAK PRODUCTS FOR UNIVERSAL ALGEBRA AND MODEL THEORY

I. Sain

Abstract.

Weak products of arbitrary universal algebras are introduced. The usual notion for groups and rings is a special case. Some universal algebraic properties are proved and applications to cylindric algebras are considered.

Introduction.

The universal algebraic notions of weak products introduced in the literature so far (e. g. <9> et <10>) are not universal algebraic at all. For any similarity type t we denote by M_t the class of all algebras of type t . In any class M_t of similar algebras the weak products as defined e. g. in Grätzer <9> do not exist, except in the trivial case when t consists of a single constant symbol ony (i. e. when $t = \{ \langle c,o \rangle \}$). Further, neither rings with unit element, nor Boolean algebras have weak products in the sense of <9>. At the same time, these weak products play an important role in recent literature see e. g. Monk <19> (while they do not exist in the sense of <9>). Thus we conclude that the universal algebraic

notion of weak products introduced in <9> are highly unsactisfactory (they do not exist in <u>any</u> nontrivial similarity class of algebras !). Here we suggest an improved version which exists in most cases.
Here we would like to mention one misconception of many algebraists. Some of them believe that weak products are used in algebra <u>only</u> to obtain structure theorems. I. e. : they believe that the only purpose of weak products is structure theorems. Perhaps this may hold in group theory, but weak products play an important role in Boolean algebra theory and for Boolean algebras no structure theorem holds with weak products.
Hence the above quoted prejudice of some algebraists is false.

Throughout, t is a similarity type such that in the similarity class M_t of all algebras of type t every algebra has a minimal subalgebra.

<u>Remark</u>:

There are two ways of achieving this: either t contains at least one constant symbol or else the empty algebra is not excluded from M_t . Here we do not care which one is the case.

Let $\Omega_i \in M_t$ for each $i \in I$, for some set I.
$\mathrm{P}_{i \in I}\, \Omega_i$ denotes the product of the algebras in the usual sense, cf. <9> or <10> D.0.3.1.
The following definition generalises R.0.3.60 of <10> p. 104 . It is also a generalisation of <9> p. 139 . Note that the Boolean algebras <u>do not have</u> infinite weak products in the sense of <9> but they do in the sense of Definition 1 below. Weak products of Boolean algebras proved to be rather useful

in e. g. Monk <19> .

DEFINITION 1.

The weak product $\mathop{P^{w}}_{i\in I} \Omega_i$ of the system $< \Omega_i : i \in I >$ of algebras is defined as follows:

- let M denote the universe of the minimal subalgebra of $\mathop{P}_{i\in I} \Omega_i$,
- now $\mathop{P^{w}}_{i\in I} A_i = \{f\in \mathop{P}_{i\in I} A_i : (\exists g \in M)(\{ i\in I : f(i) \neq g(i)\} \text{ finite})\}$
- $\mathop{P^{w}}_{i\in I} \Omega_i$ is defined to be the subalgebra of $\mathop{P}_{i\in I} \Omega_i$ with universe $\mathop{P^{w}}_{i\in I} A_i$.

Clearly, $\mathop{P^{w}}_{i\in I} \Omega_i$ is unique.

PROPOSITION.

(i) Definition 1 is correct in the sense that $\mathop{P^{w}}_{i\in I} A_i$ is a subuniverse of $\mathop{P}_{i\in I} \Omega_i$.

(ii) $\mathop{P^{w}}_{i\in I} \Omega_i$ is a subdirect product, if the minimal subalgebra M of $\mathop{P^{w}}_{i\in I} \Omega_i$ is nonempty, e. g. if t contains a constant symbol.

The proof is left to the reader.

Let $K \subseteq M_t$. I. e. K is a class of algebras of type t.

$P^w K$ denotes the class of all weak products of possibly infinite families of algebras in K :

$$P^w K = \{ \underset{i \in I}{P^w} \Omega_i : \{ \Omega_i : i \in I \} \text{ is a subset of } K \} .$$

$Po^w K$ denotes the class of weak powers of elements of K .
We shall use the notations H K , S K , P K as defined in <9> et <10> .
Up K denotes the class of ultraproducts of elements of K see <10> .
We shall consider H , S , P , Up , P^w and Po^w as operators on the class M_t of all universal algebras of some fixed similarity type t . See <17> , <10> p. 89 above T.0.3.17 , <4> , <13> p. 387 , or <9> p. 152 §23 .
Namely, to any class $K \subseteq M_t$ the operator H correlates another class $H K \subseteq M_t$. Juxtaposition of names of operators denotes composition. Namely, HSP is the operator correlating with each $K \subseteq M_t$ the class HSP K , see p. 109 of <10> , or <9> .
The statement " HH = H " means that for every type t and every class $K \subseteq M_t$ we claim HH K = H K .
See T.0.2.23 of <10>. On the other hand, SH $\neq$ HS means that there exist a type t and a class $K \subseteq M_t$ such that SH K $\neq$ HS K , cf. 0.2.19 of <10> .
See also <17> .

Recall from e. g. <13>, <12> or <15> Thm. 3 that HSP and SPUp are the closure operators of generating the smallest variety and generating the smallest quasivariety respectively. I. e. : HSP K and SPUp K are the smallest classes containing K and axiomatisable by equations and equational implications respectively.

PROPOSITION 2.

(o) $P \neq P^W$,

(i) $HSUp\ P^W = HSP = HSU_p\ P$,

$SUp\ P^W = SPUp = SUp\ P$,

(ii) $HSP^W\ Up \neq HSP$,

$SP^W\ Up \neq SPUp$,

(iii) $HSP^W\ K$ is not first order axiomatisable, for some $K \subseteq M_t$,

(iv) HSP^W , SP^W , HP^W , P^W , $HSP^W\ Up$, $SP^W\ Up$ are not closure operators,

though, $HSUp\ P^W$ and $SUp\ P^W$ are closure operators,

(v) $HSPo^W$ preserves the formulas of the following shape:

$$\bigvee_{i < \alpha} \Big(\bigwedge_{i \leq j < \alpha} e_j \Big)$$

where α is an arbitrary ordinal and $\{e_j : j < \alpha\}$ is a set of equations, and $\{e_j : j < \alpha\}$ contains a finite set of variables only (i. e. let β be a formula of the above shape, then $K \models \beta$ implies $HSPo^W\ K \models \beta$),

(vi) SPo^W preserves all the formulas of the shape:

$$\bigwedge_{n \in N} e_n \longrightarrow \bigvee_{i < \alpha} \Big(\bigwedge_{i \leq j < \alpha} e_j \Big)$$

where N is an arbitrary set, α is an ordinal and e_n , e_j are equations (of type t), and $\{e_n , e_j : n \in N , j < \alpha\}$ contains a finite set of variable only.

Proof.

Notation: if Q , Q_1 are operators, then $Q \subseteq Q_1$ means that $Q\,K \subseteq Q_1\,K$ for every K , see <17> .

Proof of (i).

It is known that HSP = HSUp P , see e. g. <10> 0.4.64. To prove SUp P^W = SUp P = SPUp we shall use the following lemma.

Lemma 1.

Let P^f and P^r denote the operators of taking all finite products and all reduced products respectively. Let Q be an operator such that $P^f \subseteq Q \subseteq SP^r$. Then:

$$\text{SUp } Q = \text{SPUp} .$$

Proof of lemma 1.

Some notations:

- let $K \subseteq M_t$, then

$$\text{Univ } K = \{ (\bigwedge_{i \in I} e_i \longrightarrow \bigvee_{j \in J} p_j) : K \models (\bigwedge e_i \longrightarrow \bigvee p_j) \text{ and } \{e_i , p_j : i \in I , j \in J\} \text{ is a finite set of equations of type } t \} ,$$

$$\text{Qeq } K = \{ (\bigwedge_{i\ I} e_i \longrightarrow p) : (\bigwedge e_i \longrightarrow p) \in \text{Univ } K \} ,$$

- if Σ is a set of formulas then Md Σ denotes the class of all models of Σ .

Now, SUp Q K = Md Univ Q K , by <10> T.0.3.83 and C.0.3.70 or Thm. 3 (v) of <15> .

It is not hard to prove that:

$$P^f\, K \models (\bigwedge e_i \longrightarrow \bigvee_{j\in J} p_j) \quad \text{and} \quad J \text{ is finite}$$

(*) imply

$$(\exists j\in J)\ \ K \models (\bigwedge e_i \longrightarrow p_j) ,$$

(see e. g. Lemma 5 of <15>).

Now (*) , $P^f \subseteq Q \subseteq SP^r$, and the fact that SP^r preserves quasiequations (i. e. elements of Qeq $\emptyset$) imply that:

$$\text{Md Univ } Q\, K = \text{Md Qeq } K .$$

It is known that Md Qeq K = SPUp K , see e. g. <12> , <15> Thm. 3 (vi).

<u>QED of lemma</u> 1.

Since $P^f \subseteq P \subseteq SP^r$ and $P^f \subseteq P^w \subseteq SP^r$, lemma 1 implies $SUp\ P^w = SUp\ P = SPUp$.

By this (i) is proved.

<u>Proof of (o), (ii) and (iv)</u>.

To prove (o), (ii) and (iv) it is enough to prove:

$$HSP^w\ Up \not\supseteq P \quad \text{and} \quad HSP^w\ Up \not\supseteq P^w\ P^w .$$

We shall fix a class K of algebras for which:

$$HSP^w\ Up\ K \not\supseteq P\ K \quad \text{and} \quad HSP^w\ Up\ K \not\supseteq P^w\ P^w\ K .$$

Let the type t be:

$t = \{ (0,0),(1,0),(f_i,1),(g_i,1) : i \in \omega \}$.

Now:

$K = \{ \Omega \in M_t : A = \{0,1\}$ and for every $i\in \omega$
$\Omega \models f_i\ 0 = 0$ and $\Omega \models g_i\ 1 = 1 \}$.

Lemma 2.

For every element " a " of an arbitrary algebra $\Omega \in HSP^W\, Up\, K$, either $\{f_i a : i \in \omega\}$ is finite or $\{g_i a : i \in \omega\}$ is finite.

Proof of lemma 2.

It is enough to prove lemma 2 for every $\Omega \in P^W K$, since the operator HS "prserves" the above property and $Up\, K = K$.

Let $\Omega = \underset{i \in I}{P^W} \Omega_i$ and $\{\Omega_i : i \in I\} \subseteq K$.

Let $a \in A$, $a = \langle a_i : i \in I \rangle$.

Now, either $\{i \in I : a_i \neq 0\}$ is finite or $\{i \in I : a_i \neq 1\}$ is finite.

Now, $K \models \{f_i 0 = 0\, , g_i 1 = 1 : i \in \omega\}$ completes the proof of the lemma.

QED of lemma 2.

Now we define a system $\langle \Omega_i : i \in \omega + \omega \rangle$ of algebras of K.

Let $i, j \in \omega$.

In the algebra Ω_i we define the operations f_j and g_j as:

$$f_j(1) = \begin{cases} 0 & \text{if } j \leq i \\ 1 & \text{otherwise} \end{cases} \quad , \quad g_j = \text{Identity} \; .$$

In $\Omega_{\omega + i}$ we define f_j and g_j as:

$$g_j(0) = \begin{cases} 1 & \text{if } j \leq i \\ 0 & \text{otherwise} \end{cases} \quad , \quad f_j = \text{Identity} \; .$$

Let $\Omega' = \underset{i \in \omega+\omega}{P} \Omega_i$ and $\Omega'_1 = \underset{i \in \omega}{P^W} \Omega_i \times \underset{i \in \omega}{P^W} \Omega_{\omega + i}$.

Now $\Omega' \in P\,K$, $\Omega'_1 \in P^W P^W K$ and $\Omega'_1 \subseteq \Omega'$.

For the element $a' = \langle 0 , 0 , \ldots , 1 , 1 , \ldots \rangle = A'_1$ neither $\{ f_i a' : i \in \omega \}$ nor $\{ g_i a' : i \in \omega \}$ is finite (both in Ω'_1 and Ω').
Thus, by lemma 2 , neither Ω' nor Ω'_1 is in $HSP^W \, Up \, K$, proving $HSP^W \, Up \not\supseteq P$ and $HSP^W \, Up \not\supseteq P^W \, P^W$.

By this (o), (ii), and (iv) are proved.

Proof of (iii).

Recall from <10> that Lf_ω denotes the class of all locally finite dimensional cylindric algebras.
$HSP^W \, Lf_\omega = Lf_\omega$ but Lf_ω is not axiomatisable.

Proof of (vi).

Let $\beta = (\bigwedge_{n \in N} e_n \longrightarrow \bigvee_{i < \alpha} (\bigwedge_{i \leq j < \alpha} e_j))$ be a formula of the required shape and let $\{x_1, \ldots ,x_m\}$ be the set of variables occuring in β .
Let $\Omega \models \beta$. We have to prove

$$\mathop{P^W}_{i \in I} \Omega_i = \Omega' \models \beta ,$$

where Ω_i is Ω for every $i \in I$.
Suppose that

$$\mathop{P^W}_{i \in I} \Omega_i \models (\bigwedge_{n \in N} e_n) [a_1, \ldots , a_m] .$$

For every projection function pj_i we denote $pj_i(a_r)$ by $a_r(i)$.
We then have $\Omega \models (\bigwedge_{n \in N} e_n) [a_1(i), \ldots ,a_m(i)]$.
Then, since $\Omega \models \beta$, we have

$$\Omega \models \bigvee_{z \in \alpha} (\bigwedge_{z_i \leq j < \alpha} e_j) [a_1(i), \ldots ,a_m(i)] .$$

Thus for every $i \in I$ there exists $z_i \in \alpha$ such that:

$$\Omega \models \left(\bigwedge_{z_i \leq j < \alpha} e_j \right) [a_1(i), \ldots, a_m(i)] .$$

I. e. such that:

$$\Omega \models \{ e_j : z_i \leq j < \alpha \} [a_1(i), \ldots, a_m(i)] .$$

Since Ω_i is Ω for every $i \in I$ and $a_1, \ldots, a_m \in \underset{i \in I}{P^W} A_i$,

there is a finit $J \subseteq I$ such that:

$$\{ < a_1(i), \ldots, a_m(i) > : i \in I \} \subseteq \{ < a_1(i), \ldots, a_m(i) > : i \in J \}.$$

Let r be the greatest element of $\{ z_i : i \in I \}$ (it exists since J is finite).

Now:

$$\Omega \models \{ e_j : r \leq j < \alpha \} [a_1(i), \ldots, a_m(i)] ,$$

for every $i \in J$, and therefore also for every $i \in I$.

This implies:

$$\underset{i \in I}{P^W} \Omega_i \models \{ e_j : r \leq j < \alpha \} [a_1, \ldots, a_m] ,$$

since subalgebras and direct products preserve equations and $P^W \subseteq SP$.

Therefore:

$$\underset{i \in I}{P^W} \Omega_i \models \left(\bigwedge_{n \in N} e_n \longrightarrow \bigvee_{z < \alpha} \left(\bigwedge_{z \leq j < \alpha} e_j \right) \right) [a_1, \ldots, a_m] .$$

Since $a_1, \ldots, a_m$ was arbitrary, (vi) is proved.

(v) is a consequence of (vi) and the fact that H preserves positive formulas even if they are infinitary.

QED

Remark.

Properties of the operator HSP^f were investigated in <7> and <18> .

Recall that if K contains finite algebras only then $P\,K$ contains <u>no</u> countable algebras.

<u>PROPOSITION</u> 3.

Let t contain a constant symbol.

Let α' be an infinite cardinal such that:

$$(\exists\, \Omega \in K) \quad 1 < |A| \leq \alpha' .$$

Then $P^w K$ contains an algebra of cardinality α'.

<u>Proof</u>.

Let $\Omega \in K$ be such that $1 < |A| \leq \alpha'$.

Let $\Omega' = \underset{i \in \alpha}{P^w} \Omega$.

Now, $|A'| = \alpha'$ because $|A| \geq 2$ and $\alpha' . \alpha' = \alpha'$ (since α' is infinite).

<u>QED</u>.

<u>EXAMPLES</u>.

1. Weak direct products of Boolean algebras have been studied recently by J. D. Monk <19>, <14> , see also p. 20 above question 50 in <6>.

2. In discussions of various special classes of algebras, in particular in the theories of <u>groups</u> and <u>rings</u>, weak products actually play a more important role than ordinary direct products, cf. e. g. <10> p. 105.

3. P^w is specially important for cylindric algebras because $P^w\, Lf_\alpha = Lf_\alpha$ moreover $HSP^w\, Lf_\alpha = Lf_\alpha$ and the class Lf_α is the class of all first order theories when considered as algebras. See Thm. 5.3. of <2> and V.5, VI.5 of <1>.

PROPOSITION 4. $P^W Lf_\alpha = Lf_\alpha$.

Proof.

Let $\Omega_i \in Lf_\alpha$ for every $i \in I$. Let $f \in \underset{i\in I}{P^W} A_i$ be arbitrary. By definition 1 there is a $g \in M \subseteq \underset{i\in I}{P} A_i$, where M is the minimal subalgebra of $\underset{i\in I}{P} \Omega_i$, such that f and g differ only at finitely many places, i. e.:

$$\{ i \in I : f(i) \neq g(i) \} \text{ is finite.}$$

By T.2.4.2. of <10> $\Delta f = \bigcup \{ \Delta f(i) : i \in I \}$.
Also $(\forall i \in I)\ \Delta g(i) \subseteq \Delta g$ and Δg is finite by T.2.1.16 of <10> . Since $(\forall i \in I)\ [\Delta f(i)$ is finite$]$ by $\Omega_i \in Lf_\alpha$, we can conclude that also Δf is finite.

QED.

PROBLEM (cf. <10>).

Denote by $\mathcal{S}$ the class of simple elements of Lf_ω.
$HSP^W \mathcal{S} \subseteq Lf_\omega$, obviously.
Now, is it also true that $HSP^W \mathcal{S} = Lf_\omega$?
The importance and basic properties of the class $\mathcal{S}$ were discussed in <1>, <2>, <3>, <11> and in <16> .

Continuation of examples.

4. Weak direct sum of vector spaces is a special case of weak product P^W as defined here, see <5> p. 42.

5. Direct sums of modules are also a special case of weak products.
   Direct sums of Abelian groups are also a special case, see e. g. <8>.

Recall that for groups, rings, semigroups with zero (annihilator) $P^W P^W = P^W$, see also Prop. 5 (i) below.

PROPOSITION 5.

(i) Let $V = P^W V$ be a class of algebras in which the one-element algebra is initial (i. e. every algebra in V contains a minimal subalgebra and the minimal subalgebra has exactly one element).
Then in V we have $P^W P^W = P^W$, i. e. for every $K \subseteq V$ we have $P^W P^W K = P^W K$.

(ii) For Boolean algebras, P^W , SP^W , SP^W Up , P^W Up are not closure operators.

(iii) For rings $< R ; + , . , 0 , 1 >$ with unit (ii) holds.

Proof.

Proof of (i) is left to the reader.

Proof of (ii).
Let $\underset{\sim}{2} = < 2 ; \cap , \cup , \setminus >$ denote the two-element Boolean algebra, and $K = \{ \underset{\sim}{2} \}$.
Let $\Omega' \in SP^W K$ be arbitrary.
Then $(\forall a' \in A') [\{ x \in A' : x > a' \}$ is finite or $\{ x \in A' : x < a' \}$ is finite$]$.
But this is not true for elements of $P^W P^W K$:
- let $\Omega' = \underset{i \in \omega}{P^W} \underset{\sim}{2}$,
- then $(\exists a' \in A' \times A') [\{ x \in A' \times A' : x > a' \}$ is infinite and $\{ x \in A' \times A' : x < a' \}$ is infinite$]$.

Of course, $<$ is understood in $\Omega' \times \Omega'$.
Clearly $K = \text{Up } K$ and thus $SP^W \text{Up } K = SP^W K$.

Proof of (iii) (for rings with unit).

Let $\underset{\sim}{2} = \langle \{0,1\} ; + , \cdot , 0 , 1 \rangle$ be the ring with unit 1 defined by $1 + 1 = 0$ (this is <u>the</u> two-element Boolean ring).

Define $\leq$ by:

$$x \leq y \quad \text{iff} \quad x.y = x .$$

Now the proof given for (ii) works by taking $K = \{ \underset{\sim}{2} \}$.

<u>QED</u>.

<u>PROBLEM</u>.

Find a category-theoretic characterisation of weak products.

<u>REFERENCES</u>.

<1> Andréka H. , Gergely T. , Németi I. : Purely algebraic construction of first order logics. Logic Semester 1973 Warsaw Banach Center. Also: Central Re. Inst. Phis. Hung. Acad. Sci. KFKI-73-71. 1973 Budapest.

<2> Andréka H. , Gergely T. , Németi I. : On Universal Algebraic Construction of Logics. STUDIA LOGICA XXXVI, 1-2, 1977, pp. 9-47.

<3> Andréka H. , Németi I. : A simple, purely algebraic proof of the completeness of some first order logics. ALGEBRA UNIVERSALIS Vol. 5, 1975, pp. 8-15.

<4> Andréka H. , Németi I. : Formulas and ultraproducts in categories. Beiträge zur Algebra u. Geom. 8, 1979, pp. 133-151.

<5> Arbib M. A. , Manes E. G. : Arrows, Structures and Functors: The Categorical Imperative. Ac. Press, 1975.

<6> Van Deuven E. K. , Monk J. D. , Matatyahu R. : Some questions about Boolean algebras. Preprint 1979 Jan. Univ. Colorado Boulder, Colorado 80309 USA.

<7> Eilenberg S. , Schützenberger M. P. : On Pseudovarieties. Advances in Math. Vol. 19, No 3, 1976, pp. 413-418.

<8> Fuchs L. : Infinite Abelian Groups. Ac. Press, 1970.

<9> Grätzer G. : Universal Algebra. Second edition, Springer Verlag, 1979.

<10> Henkin L. , Monk J. D. , Tarski A. : Cylindric Algebras, Part I . North Holland, 1971.

<11> Henkin L. , Monk J. D. , Tarski A. , Andréka H. , Németi I. : Cylindric Set Algebras. Lect. Notes in Math. Vol. 883, Springer Verlag, 1981.

<12> Malcev I. : Algebraic Systems. Akademie Verlag Berlin, 1973.

<13> Monk J. D. : Mathematical Logic. Springer Verlag GTM 37, 1978.

<14> Monk J. D. : On depth of Boolean algebras. Lecture at the Math. Inst. Hung. Acad. Sci., Dec. 1978.

<15> Németi I. , Sain I. : Cone Injectivity and some Birkhoff type Theorems in Categories. Universal Algebra (Proc. Coll. Univ. Alg., Esztergom, 1977). North-Holland, 1981, pp. 535-578.

<16> Andréka H. , Németi I. , Sain I. : Connections between Algebraic Logic and Initial Algebra Semantics of CF Languages. Mathematical Logic in Computer Science (Proc. Coll. Logic in Programming, Salgótarján, 1978). North-Holland, 1981, pp. 25-83 and 561-605.

<17> Pigozzi D. : On some operations on classes of algebras. ALGEBRA UNIVERSALIS Vol. 2, 1972, pp. 346-353.

<18> Rosicky J. : Concerning equational categories. Universal Algebra (Proc. Coll. Univ. Alg., Esztergom 1977). North-Holland, 1981.

<19> Monk J. D. : Cardinal functions on Boolean algebras. Preprint Univ. Colorado Boulder, 1981.

Mathematical Institute of
the Hungarian Academy of Sciences
Budapest, Reáltanoda u. 13-15
H-1053 Hungary.