CAHIERS DE
TOPOLOGIE ET GÉOMÉTRIE DIFFÉRENTIELLE
CATÉGORIQUES

A. H. ROQUE

Protomodular quasivarieties of universal algebras

Cahiers de topologie et géométrie différentielle catégoriques, tome 49, nº 4 (2008), p. 307-314

<http://www.numdam.org/item?id=CTGDC_2008__49_4_307_0>

CAHIERS DE TOPOLOGIE ET
GEOMETRIE DIFFERENTIELLE CATEGORIQUES

Vol. XLIX-4 (2008)

PROTOMODULAR QUASIVARIETIES OF UNIVERSAL ALGEBRAS

Dedicated to Jirí Adámek on the occasion of his sixtieth birthday

by A. H. ROQUE

Abstract

Les variétés protomodulaires d'algèbres universelles ont été caractérisées syntactiquement en [2]. On montre que la même caractérisation s'applique aux quasi-variétés d'algèbres universelles et on présente une condition suffisante pour que les sous-catégories pleines d'une catégorie de structures fermées pour sous-objets et produits soient protomodulaires.

1 Preliminaries

For an object B in a category $\mathbb{C}$, the category $Pt(B)$ of points in $(\mathbb{C} \downarrow B)$ is the category whose objects are triples (A, α, β) with $\alpha : A \to B$, $\beta : B \to A$ morphisms in $\mathbb{C}$, $\alpha \circ \beta = 1_B$, and whose morphisms $f : (A, \alpha, \beta) \to (D, \gamma, \delta)$ are morphisms in $\mathbb{C}$ for which $f \circ \beta = \delta$ and $\gamma \circ f = \alpha$.

When $\mathbb{C}$ has pullbacks, any morphism $p : E \to B$ determines a pullback functor $p^* : (\mathbb{C} \downarrow B) \to (\mathbb{C} \downarrow E)$ which restricts to $p^* : Pt(B) \to Pt(E)$.

Definition 1.1 *A category $\mathbb{C}$ with pullbacks is called protomodular if for each $p : E \to B$, $p^* : Pt(B) \to Pt(E)$ reflects isomorphisms.*

In [2], it is proved that for varieties of universal algebras protomodularity can be reduced to reflection of isomorphisms which are images of monomorphisms in $Pt(B)$ under m^*, with $m : I \to B$ a monomorphism and I an initial object. We begin by showing (in section 2) that this characterization of protomodularity applies to any category with pullbacks, stable $(regular - epi, mono)$ - factorizations and initial objects.

The category $\mathbb{E}$ of structures for a given first order (one sorted) language is regular where regular epimorphisms are the strong surjective homomorphisms and monomorphisms are the injective homomorphisms. A full subcategory $\mathbb{Q}$ of $\mathbb{E}$, closed under subobjects and products is regular (with the same factorizations as in $\mathbb{E}$), so in particular, regular epimorphisms p in $\mathbb{Q}$ are pullback stable and therefore p^* and its restriction to $Pt(B)$, reflect isomorphisms. Moreover, being closed under subobjects and products, such subcategories $\mathbb{Q}$ have free objects over sets. For these reasons, the arguments in [2] for a semantical characterization of protomodularity apply to these subcategories of structures as well, as proved in section 3.

It follows then (see section 4) that the same syntactical characterization of protomodularity as in [2] holds for quasivarieties $\mathbb{Q}$ when the language $\mathcal{L}$ is algebraic, i.e., has no predicate symbols. In fact, in this case, one only needs to prove that protomodularity of $\mathbb{Q}$ implies that
there exist closed terms $k_1, ..., k_n$, terms $t_1, ..., t_n$ with at most two free variables (x, y) and a term t with $n+1$ free variables such that, $\mathbb{Q}$ satisfies the identities

$$t(x, t_1(x, y), ..., t_n(x, y)) \approx y \quad \text{and} \quad t_i(x, x) \approx k_i, \tag{1}$$

since by [2], the converse necessarily holds: suppose that $\mathbb{Q}$ is a quasivariety for $\mathcal{L}$ and let $\mathbb{E}$ denote the category of structures for this language. Assume that $\mathbb{Q}$ satisfies identities as in (1). Let $\mathbb{V}$ be the closure of $\mathbb{Q}$ in $\mathbb{E}$ under homomorphic images. Then $\mathbb{V}$ is a variety and, since homomorphisms preserve identities, $\mathbb{V}$ satisfies the same identities. Hence, by [2], $\mathbb{V}$ is protomodular. But the inclusion functor from $\mathbb{Q}$ into $\mathbb{V}$ preserves pullbacks and reflects isomorphisms, so again by [2], $\mathbb{Q}$ must be protomodular.

2 Protomodularity and Factorizations

The following proposition can be found in [1]:

Proposition 2.1 *Let $\mathbb{C}$ be a category with pullbacks. $\mathbb{C}$ is protomodular if and only if for each morphism $p : E \to B$ and each monomorphism f in $Pt(B)$, if $p^*(f)$ is an isomorphism then f is an isomorphism.*

Proof: If $\mathbb{C}$ is protomodular the condition follows. Conversely, assume that for each morphism p and each monomorphism f if $p^*(f)$ is an isomorphism then f is an isomorphism.

Let $p : E \to B$ be a morphism in $\mathbb{C}$ and $f : A \to A'$ be in $Pt(B)$. Let (f_1, f_2) be a kernel pair of f in $\mathbb{C}$ (($\mathbb{C} \downarrow B$) and $Pt(B)$) and consider $\Delta : A' \to A' \times_A A'$. Then, f_1, f_2 and Δ are in $Pt(B)$.

We first show that if $p^*(f)$ is an isomorphism, then $p^*(\Delta)$ is an isomorphism.

Assume that $p^*(f)$ is an isomorphism. Since $p^* : (\mathbb{C} \downarrow B) \to (\mathbb{C} \downarrow E)$ is a right adjoint, it preserves pullback so that (p^*f_1, p^*f_2) is a kernel pair of $p^*(f)$ in $(\mathbb{C} \downarrow E)$ and $Pt(E)$. Since $p^*(f)$ is an isomorphism in $Pt(E)$, $p^*(f_1)$ is an isomorphism. But $p^*(f_1) \circ p^*(\Delta) = p^*(1)$ and so $p^*(\Delta)$ is an isomorphism.

Now assume that f is a monomorphism with $p^*(f)$ an isomorphism. Then $p^*(\Delta)$ is an isomorphism. But Δ is a monomorphism so by hypothesis Δ is an isomorphism. Hence, $f_1 = \Delta^{-1} = f_2$ and since these are a kernel pair of f, f is a monomorphism. Then, again by hypothesis, f is an isomorphism. □

Proposition 2.2 *Let $\mathbb{C}$ be a category with pullbacks and stable $(regular-epi, mono)$ factorizations. $\mathbb{C}$ is protomodular if and only if for each monomorphism $p : E \to B$ and each monomorphism f in $Pt(B)$, if $p^*(f)$ is an isomorphism then f is an isomorphism.*

Proof: If $\mathbb{C}$ is protomodular, the condition is satisfied. Conversely suppose that for each monomorphism m and each monomorphism f if $m^*(f)$ is an isomorphism then f is an isomorphism. For any pullback stable regular epimorphism $p : E \to B$, $p^* : (\mathbb{C} \downarrow B) \to (\mathbb{C} \downarrow E)$ reflects isomorphisms (see [3]), hence so does $p^* : Pt(B) \to Pt(E)$. Therefore, since $p = m \circ e$ with m a monomorphism and e a pullback stable regular epimorphism, p^* reflects isomorphisms if and only if so does m^*. □

An immediate consequence is

Proposition 2.3 *Let $\mathbb{C}$ be a category with pullbacks, stable $(regular - epi, mono)$ factorizations and initial object I. $\mathbb{C}$ is protomodular if and only if for each monomorphism $m : I \to B$ and each monomorphism f in $Pt(B)$, if $m^*(f)$ is an isomorphism then f is an isomorphism.*

3 Protomodularity in categories of structures closed under subobjects and products

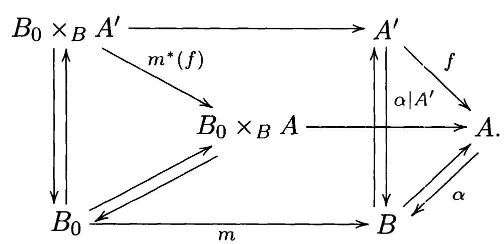
Let $\mathcal{L}$ be any first order (one sorted) language and $\mathbb{E}$ be the category of structures for $\mathcal{L}$. Then $\mathbb{E}$ is a regular category. A full subcategory $\mathbb{Q}$ of $\mathbb{E}$ closed under subobjects and products is regular with the same (regular-epi, mono)-factorizations as in $\mathbb{E}$, i.e., (strong surjective homomorphisms, injective homomorphisms). Closure under subobjects and products guarantees that $\mathbb{Q}$ has free objects over sets. The free object on the empty set $K(\emptyset)$, is an initial object in $\mathbb{Q}$. These subcategories need not be pointed.

Proposition 3.1 *The following are equivalent for a full subcategory $\mathbb{Q}$ of $\mathbb{E}$ closed under subobjects and products:*

1. *$\mathbb{Q}$ is protomodular.*
2. *For A in $\mathbb{Q}$ let $B \subseteq A' \subseteq A$ be a sequence of subobjects of A, $\alpha : A \to B$ a homomorphism such that $\alpha(b) = b$ for $b \in B$, and $\alpha^{-1}(B_0)$ the inverse image under α of the subobject B_0 of B generated by the constants in B. If $\alpha^{-1}(B_0)$ is a subobject of A' then $A' = A$.*
3. *Let A be in $\mathbb{Q}$, B be a subobject of A, $\alpha : A \to B$ a homomorphism such that $\alpha(b) = b$ for $b \in B$, and $\alpha^{-1}(B_0)$ the inverse image under α of the subobject of B generated by the constants in B. Then A is generated by B and $\alpha^{-1}(B_0)$.*

Proof: Factorization in $\mathbb{Q}$ implies that if $m : K(\emptyset) \to B$ is a monomorphism in $\mathbb{Q}$, then $K(\emptyset)$ is isomorphic to B_0. Since monomorphisms may be assumed to be inclusion homomorphisms, by Proposition 2.3, $\mathbb{Q}$ is protomodular if and only if for any A in $\mathbb{Q}$, any sequence $B \subseteq A' \subseteq A$ of subobjects of A (which must be in $\mathbb{Q}$), any $\alpha : A \to B$ such that $\alpha \circ \beta = 1$, with β the inclusion of B in A, and m the inclusion of B_0 in B, if $m^*(f)$ is an isomorphism (with f in $Pt(B)$, the inclusion of A' in A), then f is an isomorphism.
(Note that elements are related in $\alpha^{-1}(B)$ if they are related in A and their images through α are related in B.)

1.⇔ 2. Consider the diagram (in $\mathbb{Q}$) with objects and morphisms as above



The following assertions are equivalent:

(i) $m^*(f)$ is an isomorphism

(ii) $m^*(f)$ is a bijective homomorphism of algebras and for each predicate symbol R, $R^{\alpha^{-1}(B_0)} = R^{(\alpha/A')^{-1}(B_0)}$

(iii) $\alpha^{-1}(B_0) \subseteq A'$ as subalgebra and for each predicate symbol R, $R^{\alpha^{-1}(B_0)} \subseteq R^{A'}$

(iv) $\alpha^{-1}(B_0)$ is a subobject of A'.

Therefore, if $\alpha^{-1}(B_0)$ is a subobject of A', then $m^*(f)$ is an isomorphism and by 1., A is isomorphic to A' i.e., $A = A'$ and 2. is satisfied. If $m^*(f)$ is an isomorphism, then $\alpha^{-1}(B_0)$ is a subobject of A' and by 2., $A = A'$, so 1. is satisfied.

2.$\Leftrightarrow$ 3. Let A' be the subobject of A generated by $\alpha^{-1}(B_0)$ and B. □

4 Syntactical characterization of proto - modularity in quasivarieties of universal algebras

In the following theorem $\mathcal{L}$ is a first order (one sorted) algebraic language and $\mathbb{Q}$ is a quasivariety for this language.

Theorem 4.1 *$\mathbb{Q}$ is protomodular if and only if there exist closed terms $k_1, ..., k_n$, terms $t_1, ..., t_n$ with at most two free variables (x,y) and a term t with $n+1$ free variables such that, $\mathbb{Q}$ satisfies the identities*

$$t(x, t_1(x, y), ..., t_n(x, y)) \approx y \quad \text{and} \quad t_i(x, x) \approx k_i.$$

Proof: Let $K(x, y)$, $K(x)$ and $K(\emptyset)$ be the free algebras in $\mathbb{Q}$ on two and one generators and on the empty set. Let $\alpha : K(x, y) \to K(x)$ and $\beta : K(x) \to K(x, y)$ be the unique homomorphism extensions of $x, y \mapsto x$ and $x \mapsto x$. Then $\alpha \circ \beta = 1$.

By Proposition 3.1, $K(x, y)$ is generated by $K(x) \cup \alpha^{-1}(K(\emptyset))$. Then in $K(x, y)$, $y = t(x, t_1(x.y), ..., t_n(x, y))$ for some n with $\alpha(t_i(x, y))$ in B_0, $i = 1, ..., n$.

Since $t_i(x, x, ..., x) = \alpha(t_i(x, y))$ is in B_0, it must be some closed term k_i, $i = 1, ..., n$.

Hence

$$\mathbb{Q} \models \forall x \forall y \quad y \approx t(x, t_1(x, y), ..., t_n(x, y))$$

$$\mathbb{Q} \models \forall x \bigwedge_{i=1}^{n} t_i(x, x) \approx k_i$$

For the converse let B be a subobject of A and $\alpha : A \to B$ be a homomorphism with $\alpha(b) = b$ for $b \in B$. Let B_0 be the subobject of B generated by the constants in B.

We have that $\mathbb{Q} \models y \approx t(x, t_1(x, y), ..., t_n(x, y))$ and $\mathbb{Q} \models t_i(x, x) \approx k_i$, $i = 1, ..., n$. Then for $a \in A$ with A in $\mathbb{Q}$, $t_i^A(a, a) = k_i^A$. Since α is a homomorphism, $t_i^A(\alpha(a), \alpha(a)) = \alpha(k_i^A) = k_i^B$. But α is the identity in B so $\alpha(\alpha(a)) = \alpha(a)$ and

$$\alpha(t_i^A(\alpha(a), a)) = t_i^B(\alpha(\alpha(a)), \alpha(a)) = t_i^B(\alpha(a), \alpha(a)) = k_i^B.$$

Therefore, $t_i^A(\alpha(a), a) \in \alpha^{-1}(B_0)$. Since in A,

$$a = t^A(\alpha(a), t_1^A(\alpha(a), a), ..., t_n^A(\alpha(a), a)$$

and $\alpha(a)$ is in B, a is in the subobject generated by $B \cup \alpha^{-1}(B_0)$.

Note that when the language has no constants, the identities reduce to $t(x) \approx y$ and the objects in $\mathbb{Q}$ are the empty set and singletons. □

As a consequence, protomodular quasivarieties of algebras are Maltsev, with Maltsev term

$$t(x, t_1(y, z), ..., t_n(y, z)).$$

We now consider again $\mathcal{L}$ any first order (one sorted) language and $\mathbb{E}$ the category of structures for $\mathcal{L}$. Subcategories of $\mathbb{E}$, axiomatizable by formulas of the form $\forall x_1 ... \forall x_n \ (\theta_1 \wedge ... \wedge \theta_k \to u_1 \approx u_2)$ with each θ_i atomic, are closed under subobjects and products. We end with a sufficient condition for such categories to be protomodular.

Proposition 4.2 *Let $\mathbb{Q}$ be full subcategory of $\mathbb{E}$, closed under subobjects and products. $\mathbb{Q}$ is protomodular if there are closed terms $k_1, ..., k_n$, binary terms $t_1, ..., t_n$ and a $n + 1$ - ary term t such that $\mathbb{Q}$ satisfies the identities*

$$t(x, t_1(x, y), ..., t_n(x, y)) \approx y \quad \text{and} \quad t_i(x, x) \approx e_i$$

and for each ($k -$ ary) predicate symbol R, there exist closed terms w_i, $i = 1, ..., k$, for which $\mathbb{Q} \models Ry_1 ... y_n \to \wedge_{i=1}^n y_i \approx w_i$.

Proof: It follows from the fact that interpretations of closed terms in objects and subobjects are the same. □

5 Examples

Quasivarieties of universal algebras are axiomatizable by quasi-identities i.e., sentences of the form

$$\forall x_1 ... \forall x_n \quad (\theta_1 \wedge ... \wedge \theta_k \rightarrow \delta)$$

with all θ_i and δ identities. For these the syntactical characterization holds. It is the case of:

1. Any quasivariety of algebras in a language with a constant c and two binary operations $\circ$ and $*$ satisfying $x \circ (x * y) = y$ and $x * x = c$ is protomodular, with $t_1(x, y) = x * y$ and $t(x, y) = x \circ y$.

2. Any quasivariety of algebras in a language with a constant c and a binary associative operation $\circ$ satisfying $x \circ x = c$ and $c \circ x = x$ is protomodular, where $t_1(x, y) = t(x, y) = x \circ y$.

3. The variety of Heyting semilattices is protomodular (see [4]) with $t_1(x, y) = (y \Rightarrow x)$, $t_2(x, y) = (((y \Rightarrow x) \Rightarrow x) \Rightarrow y)$ and $t(x, y, z) = z \wedge (y \Rightarrow x)$. Thus so are the varieties of Heyting algebras and Boolean algebras and also any quasivariety of Heyting semilattices (Heyting or Boolean algebras).

4. Any quasivariety of algebras in a language with a constant c, one binary associative operation $\circ$ and a unary operation * satisfying $x \circ (x^*) = c = (x^*) \circ x$ and $c \circ x = x$ is protomodular with $t_1(x, y) = x^* \circ y$ and $t(x, y) = (x \circ y))$. In particular any quasivariety of groups. For example the categories of:

   - Torsion free groups
   - Groups with at most one idempotent
   - Rings of characteristic zero
   - Rings with at most one nilpotent element:

   $$(x^2 = 0 \wedge y^2 = 0) \rightarrow x = y.$$

   - Rings where 0 is not a sum of nontrivial squares:

   $$x_1^2 + ... + x_n^2 = 0 \rightarrow (x_1 = 0 \wedge ... \wedge x_n = 0).$$

Any of the above with an extra n-ary relation satisfying $Ry_1...y_n \rightarrow \wedge_{i=1}^{n} y_i \approx c$, c a constant.

Aknowledgements: The author thankfully acknowledges the referee's comments to the first version of this paper.

The author acknowledges partial financial support by UI&D Matemática e Aplicações / FCT.

References

[1] Bourn D., Clementino M. M.; *Categorical and topological aspects of semi-abelian theories*, Haute-Bodeux 2007, http://www.math.york.ca/ tholen/HB07BournClementino.pdf.

[2] Bourn D., Janelidze G.; *Characterization of Protomodular varieties of universal algebras*, Theory and Applications of Categories, 2003.

[3] Janelidze G., Sobral M., Tholen W.; *Beyond Barr Exactness: Effective Descent Morphisms*, Categorical Foundations - Special Topics in Order, Topology, Algebra and Sheaf Theory, Cambridge University Press, 2004.

[4] Johnstone, P.; *A note on the Semiabelian variety of Heyting Semilattices*, Fields Institute Communications, Volume 43, 2004.

Ana Helena Roque
UIMA / Departamento de Matemática
Universidade de Aveiro
Aveiro, Portugal
a.h.roque@ua.pt