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On synchronized relatively full embeddings and Q-universality


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ON SYNCHRONIZED RELATIVELY FULL EMBEDDINGS AND Q-UNIVERSALITY

To Jiří Adámek on his 60th birthday

by V. KOUBEK and J. SICHLER

Abstract

M. E. Adams et W. Dziobiak ont démontré que toute quasi-variété $ff$-algébrique universelle de systèmes algébriques de signature finie est $Q$-universelle. Dans cet article on introduit la notion de plongement synchronisé relativement plein qu'on utilise ensuite afin de modifier leur résultat pour les quasi-variétés d'algèbres.

1 Introduction

We aim to show a new connection between two algebraic structures associated with quasivarieties of algebras. All needed definitions are given in the next section.

First, for any quasivariety $Q$, the homomorphisms between its members form a concrete category. The richness of the categorical structure is reflected in the notion of algebraic universality studied in the monograph [18] by A. Pultr and V. Trnková.

When ordered by inclusion, the subquasivarieties of a given quasivariety $Q$ form a lattice we denote $QLat(Q)$. This is the second algebraic structure associated with $Q$. Questions about the size of $QLat(Q)$ or lattice identities satisfied in $QLat(Q)$ motivated M. V. Sapir [19] to define and exhibit $Q$-universal quasivarieties, and W. Dziobiak [9, 10] to introduce what is now called an A-D family of objects of $Q$. A survey of these notions and results concerning them is given in [2]. M. E. Adams

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and W. Dziobiak [3] linked the latter two properties by showing that every quasivariety $Q$ containing an A-D family is also $Q$-universal. The converse implication is still an open problem, originally stated by M. E. Adams and W. Dziobiak.

**Problem 1.1.** Is there a $Q$-universal quasivariety containing no A-D family?

In [4], M. E. Adams and W. Dziobiak proved the following remarkable and quite surprising result connecting the two algebraic structures associated with a quasivariety of algebraic systems.

**Theorem 1.2** [4]. Any finite-to-finitis algebraically universal (ff-alg-universal) quasivariety of algebraic systems of finite similarity type contains an A-D family and hence it is $Q$-universal.

In [16], the present authors extended this result as follows.

**Theorem 1.3** [16]. Any almost ff-alg-universal quasivariety of algebraic systems of finite similarity type contains an A-D family and hence it is $Q$-universal.

Almost universality is a special case of relative universality, see Section 2. Here we aim to modify the latter result for quasivarieties of algebras. We assume that

(*) $Q$ is a quasivariety of finitary algebras and $V$ is a proper subvariety of $Q$ such that there exists a synchronized $I(V)$-relatively full embedding $F$ from the category of all undirected graphs into $Q$ such that $Ff$ is surjective for every graph quotient homomorphism $f$ and $FG$ is finite for every finite graph $G$.

**Theorem 1.4.** Any quasivariety $Q$ satisfying (*) contains an A-D family and hence it is $Q$-universal.

As already noted, all needed notions are reviewed in Section 2 below, and the proof of Theorem 1.4 is given in Section 3. It is based on the fact that any subquasivariety $R$ of a quasivariety $Q$ is an epireflective full subcategory of $Q$. In Section 3, it is also shown how Theorem 1.4 incorporates earlier results of [6, 7, 8].

## 2 Basic notions and their context

**Alg-universality.** A category $K$ is alg-universal if any category of algebras and all homomorphisms between them can be fully embedded into $K$. This is equivalent to the fact that there exists a full embedding from the category $\text{GRA}$ of all undirected graphs and all graph homomorphisms into $K$. Moreover, if $K$ is a concrete category
and there exists a full embedding $F : \mathbf{GRA} \rightarrow \mathbf{K}$ such that the underlying set of $FG$ for every finite graph is finite then we say that $F$ preserves finiteness and that $\mathbf{K}$ is $\text{ff-\text{alg-universal}}$. If $\mathbf{K}$ is a concrete category then any $\mathbf{K}$-object $A$ with a finite underlying set is called finite. Next we give several well-known properties of alg-universal categories. To do this, we say that a category $\mathbf{K}$ is a monoid universal if for every monoid $\mathbf{M}$ there exists a $\mathbf{K}$-object $A$ such that the endomorphism monoid of $A$ is isomorphic to $\mathbf{M}$.

**Theorem 2.1** [18]. (a) Any concrete alg-universal category $\mathbf{K}$ is monoid universal; and if $\mathbf{K}$ is $\text{ff-\text{alg-universal}}$, then for every finite monoid $\mathbf{M}$ there exists a finite $\mathbf{K}$-object $A$ such that the endomorphism monoid of $A$ is isomorphic to $\mathbf{M}$.

(b) If $\mathbf{K}$ is alg-universal, then for a proper class $I$ there exists a family $\{F_i : \mathbf{K} \rightarrow \mathbf{K} | i \in I\}$ of full embeddings such that $F_iA$ is not isomorphic to $F_jB$ for any $\mathbf{K}$-objects $A$ and $B$ and for any distinct $i, j \in I$. For any set $I$ there exists a family $\{F_i : \mathbf{K} \rightarrow \mathbf{K} | i \in I\}$ of full embeddings such that there exists no $\mathbf{K}$-morphism between $F_iA$ and $F_jB$ for any $\mathbf{K}$-objects $A$ and $B$ and for any distinct $i, j \in I$.

(c) If $\mathbf{K}$ is $\text{ff-\text{alg-universal}}$ and $I$ is a countable set, then there exists a family $\{F_i : \mathbf{K} \rightarrow \mathbf{K} | i \in I\}$ of full embeddings $F_i$ preserving finiteness such that there exists no $\mathbf{K}$-morphism between $F_iA$ and $F_jB$ for any $\mathbf{K}$-objects $A$ and $B$ and any distinct $i, j \in I$. □

Theorem 2.1 provides a tool for proving that a given category $\mathbf{K}$ is not alg-universal. For example, if $\mathbf{K}$ is a concrete category such that for every set $X$ there exists only a set of non-isomorphic $\mathbf{K}$-objects with a given underlying set $X$ and if there exists a cardinal $\alpha$ such that every $\mathbf{K}$-object whose underlying set has cardinality greater than $\alpha$ has a non-identity endomorphism, then $\mathbf{K}$ is not alg-universal.

Hence for example the variety of lattices or the variety of monoids or the category of topological spaces and continuous mappings are not alg-universal because of the existence of constant morphisms. On the other hand, both the variety of semigroups [13] and the variety of $(0,1)$-lattices ([11] or [12]) are alg-universal.

Thus we can say that monoids or lattices have sufficiently rich structure to be ‘close’ to being alg-universal while still permitting constant morphisms, although these categories are not alg-universal in the strict sense. This motivates a notion of almost alg-universality that ignores the constant morphisms. Next we define a more general concept expressing this idea.

Let $\mathbf{K}$ be a category. A class $\mathcal{C}$ of $\mathbf{K}$-morphisms is an ideal if $f \circ g \in \mathcal{C}$ for $\mathbf{K}$-morphisms $f : a \rightarrow b$, $g : b \rightarrow c$ whenever $f \in \mathcal{C}$ or $g \in \mathcal{C}$. A faithful functor $F : \mathbf{L} \rightarrow \mathbf{K}$ is called $\mathcal{C}$-relatively full embedding if
(•) \( Ff \notin C \) for any \( L \)-morphism \( f \);

(•) if \( f : Fa \to Fb \) is a \( K \)-morphism for \( L \)-objects \( a \) and \( b \) then either \( f \in C \) or \( f = Fg \) for some \( K \)-morphism \( g : a \to b \).

Thus \( F \) is a full embedding exactly when it is \( C \)-relatively full embedding for \( C = \emptyset \). Observe that, if \( F : L \to K \) is a \( C \)-relatively full embedding for some ideal \( C \) then \( f \) is an \( L \)-isomorphism if and only if \( Ff \) is a \( K \)-isomorphism. If there exists a \( C \)-relatively full embedding \( F : \text{GRA} \to K \) then we say that \( K \) is \( C \)-relatively alg-universal. If, moreover, \( K \) is concrete and \( F \) preserves finiteness, then \( K \) is called \( C \)-relatively \( \text{ff} \)-alg-universal. Clearly, \( K \) is \( C \)-relatively alg-universal (or \( C \)-relatively \( \text{ff} \)-alg-universal) for \( C = \emptyset \) just when \( K \) is alg-universal (or \( \text{ff} \)-alg-universal, respectively). If \( K \) is concrete category and \( C \) is the ideal consisting of all \( K \)-morphisms with constant underlying mapping then we say that \( F : L \to K \) is almost full embedding instead of \( C \)-relatively full embedding and that \( K \) is almost alg-universal or almost \( \text{ff} \)-alg-universal instead of \( C \)-relatively alg-universal or \( C \)-relatively \( \text{ff} \)-alg-universal. The variety of lattices [20] and the variety of monoids [17] or [15] are almost alg-universal but not alg-universal. A second consequence of Theorem 2.1 is that a category \( K \) which is not monoid-universal is not alg-universal. This fact was exploited by M. E. Adams and W. Dziobiak in [5], where they proved that the variety of monadic Boolean algebras is not alg-universal, yet contains a proper class of non-isomorphic algebras whose endomorphism monoids consist of the identity map alone.

Theorem 2.1 naturally leads to the following question.

**Problem 2.2.** Is there a variety \( \mathcal{V} \) of algebras which is monoid universal but not alg-universal?

We shall consider ideals of a special type. Let \( O \) be a class of \( K \)-objects. Then \( \mathcal{I}(O) \) denotes a class of all \( K \)-morphisms \( f : a \to b \) such that there exist \( K \)-morphisms \( g : a \to c \) and \( h : c \to b \) with \( c \in O \) and \( f = h \circ g \). Clearly, \( \mathcal{I}(O) \) is an ideal of \( K \) called an object ideal of \( O \). In what follows, we shall consider even more specific object ideals.

**\( Q \)-universality.** A class \( \mathcal{Q} \) of algebraic systems of a finitary type \( \Delta \) is a quasivariety if it is closed under all products, all ultraproducts, all subsystems and all isomorphic images. For any class \( K \) of algebraic systems of type \( \Delta \), there exists the least quasivariety \( \mathcal{Q} \) containing \( K \), which we shall denote \( \mathcal{Q} = \text{Qua}(K) \). Quasivarieties will be viewed as categories whose morphisms are all homomorphisms, that is, mappings preserving all operations and relations.

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M. V. Sapir [19] defined a quasivariety $Q$ of finite type $\Delta$ as $Q$-universal if for every quasivariety $R$ of finite type the lattice $QLat(R)$ is a homomorphic image of a sublattice of $QLat(Q)$.

Let $P(\omega_0)$ be the set of all finite subsets of natural numbers and $P(\omega) = P(\omega_0)\setminus \{\emptyset\}$ the set of all finite non-empty subsets of natural numbers. W. Dziobiak [9, 10] studied families $\{S_A \mid A \in \mathcal{P}(\omega_0)\}$ of finite algebraic systems of a given type $\Delta$ we now call Adams-Dziobiak families (or A-D families) defined by these four conditions:

(p1) $S_\emptyset$ is the terminal algebraic system;
(p2) if $A = B \cup C$ for $A, B, C \in P(\omega_0)$, then $S_A \in Qua(\{S_B, S_C\})$;
(p3) if $A \in P(\omega)$ and $B \in P(\omega_0)$ with $S_A \in Qua(\{S_B\})$, then $A = B$;
(p4) if $U, V \in Qua(\{S_A \mid A \in P\})$ are finite algebraic systems for some finite $\mathcal{P} \subset P(\omega)$ and if there exists an injective homomorphism $f : S_A \to U \times V$ for some $A \in P(\omega)$, then there exists an injective homomorphism $g : S_A \to U$ or there exists an injective homomorphism $g : S_A \to V$ or there exist $B, C \in P(\omega)$ and injective homomorphisms $g_B : S_B \to U$ and $g_C : S_C \to V$ with $A = B \cup C$.

We recall some known results.

**Theorem 2.3.** (a) If $Q$ is a $Q$-universal quasivariety then $QLat(Q)$ has cardinality $2^{\aleph_0}$ and the free lattice over a countable set can be embedded into $QLat(Q)$. Thus $QLat(Q)$ satisfies no non-trivial lattice identity [2].

(b) If a quasivariety $Q$ contains an A-D family, then the lattice of all ideals of the free lattice over a countable set can be embedded into $QLat(Q)$ [3].

Thus to prove that a quasivariety $Q$ of finite type is $Q$-universal, it suffices to prove that $Q$ has an A-D family. We shall study only quasivarieties $Q$ of algebras.

In Section 3 we give certain conditions sufficient for the existence of an A-D family in a quasivariety of algebras of finite type. For this we use factorization systems and epireflection.

**Factorization systems and epireflections.** For a category $K$, let $E$ be a class of $K$-epimorphisms and let $M$ be a class of $K$-monomorphisms. We say that $(E, M)$ is a factorization system of $K$ if $E$ and $M$ are closed under composition, $f \in E \cap M$ if and only if $f$ is a $K$-isomorphism, and for every $K$-morphism $f : a \to b$ there
exist unique, up to a commuting isomorphism, \( g : a \to c \in \mathcal{E} \) and \( h : c \to b \in \mathcal{M} \)
with \( f = h \circ g \), see [1]. Any factorization system has the diagonalization property.
We formulate it for categories with products. If \( \mathbb{K} \) is a category with products and
an \((\mathcal{E}, \mathcal{M})\)-factorization system, then we write \( \{ f_i : a \to b_i \mid i \in I \} \in \mathcal{M} \)
if the morphism \( f : a \to \prod_{i \in I} b_i \) such that \( f_i = \pi_i \circ f \) for all \( i \in I \) where \( \pi_i : \prod_{j \in I} b_j \to b_i \)
is the \( i \)-th projection belongs to \( \mathcal{M} \). Then the diagonalization property says: if
\( g_i \circ f = k_i \circ h \) for all \( i \in I \) where \( f : a \to b \in \mathcal{E}, \{ g_i : b \to c_i \mid i \in I \} \) is a family
of \( \mathbb{K} \)-morphisms, \( h : a \to d \) is a \( \mathbb{K} \)-morphism and \( \{ k_i : d \to c_i \mid i \in I \} \in \mathcal{M} \) then
there exists a \( \mathbb{K} \)-morphism \( l : b \to d \) such that \( h = l \circ f \) and \( g_i = k_i \circ l \) for all \( i \in I \).
If \( h \in \mathcal{E} \) then \( l \in \mathcal{E} \), and if \( \{ g_i \mid i \in I \} \in \mathcal{M} \) then \( l \in \mathcal{M} \).

We say that a family \( \{ f_i : A \to A_i \mid i \in I \} \) is separating if for distinct
\( a, b \in A \) there exists \( i \in I \) with \( f_i(a) \neq f_i(b) \). If \( \mathbb{K} \) is a concrete category then
a family \( \{ f_i : a \to b_i \mid i \in I \} \) of \( \mathbb{K} \)-morphisms is separating if the family of
underlying mapping is separating. For concrete categories \( \mathbb{K} \) and \( \mathbb{L} \) we say that
a functor \( F : \mathbb{K} \to \mathbb{L} \) preserves separating families if \( \{ Ff_i : Fa \to Fb_i \mid i \in I \} \) is a
separating family in \( \mathbb{L} \) whenever \( \{ f_i : a \to b_i \mid i \in I \} \) is a separating family in \( \mathbb{K} \).

For a concrete category \( \mathbb{K} \), let \( \text{Inj}_\mathbb{K} \) consist of all \( \mathbb{K} \)-homomorphisms such that
the underlying mapping is injective and \( \text{Surj}_\mathbb{K} \) consist of all \( \mathbb{K} \)-morphisms such
that the underlying mapping is surjective. Clearly, every morphism from \( \text{Inj}_\mathbb{K} \) is
a monomorphism of \( \mathbb{K} \) and every morphism from \( \text{Surj}_\mathbb{K} \) is an epimorphism of
\( \mathbb{K} \). If \( (\text{Surj}_\mathbb{K}, \text{Inj}_\mathbb{K}) \) is a factorization system of \( \mathbb{K} \) then we say \( \mathbb{K} \) has a concrete
factorization system and \( (\text{Surj}_\mathbb{K}, \text{Inj}_\mathbb{K}) \) is a concrete factorization system of \( \mathbb{K} \).
Clearly, for every quasivariety \( \mathcal{Q} \) of algebras \( (\text{Surj}_\mathcal{Q}, \text{Inj}_\mathcal{Q}) \) is a concrete factorization
system of \( \mathcal{Q} \) (because every bijective homomorphism is an isomorphism). Observe that a family \( \{ f_i : A \to B_i \mid i \in I \} \) of \( \mathcal{Q} \)-homomorphisms is separating
if and only if it belongs to \( \text{Inj}_\mathcal{Q} \), i.e. if the homomorphism \( f : A \to \prod_{i \in I} B_i \) with
\( f_i = f \circ \pi_i \) has an injective underlying mapping where \( \pi_i : \prod_{j \in I} B_j \to B_i \)
is the \( i \)-th projection for all \( i \in I \). Thus for a concrete category \( \mathbb{K} \) we shall say that
a family \( \{ f_i : A \to B_i \mid i \in I \} \) of \( \mathbb{K} \)-morphisms belong to \( \text{Inj}_\mathbb{K} \) just when its
corresponding family of underlying mappings is separating. A functor \( F : \mathcal{Q} \to \mathcal{R} \)
between quasivarieties \( \mathcal{Q} \) and \( \mathcal{R} \) preserves surjectivity if \( F(\text{Surj}_\mathcal{Q}) \subseteq \text{Surj}_\mathcal{R} \).

If \( \mathcal{Q} \) is a quasivariety of algebraic systems and \( \mathcal{R} \) is a subquasivariety of \( \mathcal{Q} \) (of
the same type) then, by Theorem 10.1.2 from [14], \( \mathcal{R} \) is an epireflective subcategory
of \( \mathcal{Q} \). This means that for every algebraic system \( A \in \mathcal{Q} \) there exists a surjective
homomorphism \( \rho_A : A \to RA \) where \( RA \in \mathcal{R} \) such that for every homomorphism
\( f : A \to C \) where \( C \in \mathcal{R} \) there exists exactly one homomorphism \( f^* : RA \to C \)
with \( f = f^* \circ \rho_A \). Since \( \mathcal{R} \) is a full subcategory of \( \mathcal{Q} \) then \( \rho_A \) is the identity.
morphism exactly when $A \in R$. Then $R : Q \rightarrow R$ such that $Rf = (\rho_B \circ f)^*$ for every homomorphism $f : A \rightarrow B$ in $Q$ is a functor which is a left adjoint to the inclusion functor from $R$ to $Q$. We say that $R$ is an epireflection. Observe that $R(\text{Surj}_Q) \subseteq \text{Surj}_R$.

A quasivariety $Q$ of algebras closed under homomorphic images is a variety. If $Q$ is a quasivariety of algebras and $V$ is a subvariety of $Q$ then a homomorphism $f : A \rightarrow B \in Q$ belongs to the ideal $\mathcal{I}(V)$ if and only if $\text{Im}(f) \in V$.

### 3 Sufficient conditions for $Q$-universality

**Definition.** Let $Q$ be a quasivariety of finitary algebraic systems, let $V$ be a proper subvariety of $Q$ and let $R : Q \rightarrow V$ be the corresponding epireflection. For any object $A \in Q$, let $A$ denote the underlying set of $A$ and let $\rho_A : A \rightarrow RA$ denote the surjective $Q$-morphism from the epitransformation $\rho$. Let $F : K \rightarrow Q$ be a $\mathcal{I}(V)$-relatively full embedding. Let $S \in V$ be an algebraic system with the underlying set $S$. We say that $F$ is $S$-synchronized and call $S$ its synchronizer if for every $K$-object $k$ there exists an injective mapping $\mu_k$ from $S$ to the underlying set of $RFk$ such that $\text{Im}(\mu_k)$ is an induced subobject of $RFk$ and $\mu_k$ is an isomorphism of $S$ onto the subobject of $RFk$ on the set $\text{Im}(\mu_k)$, and for every $K$-morphism $f : k_1 \rightarrow k_2$ we have

1. if $Ff$ is injective on $(\rho_{FK_1})^{-1}(\text{Im}(\mu_{k_1}))$, then $Ff$ is injective;

2. $RFf \circ \mu_{k_1} = \mu_{k_2}$;

3. if $Ff \in \text{Surj}_Q$ and $A_i$ is the underlying set of $RFk_i$ for $i = 1, 2$, then every mapping $h : A_2 \rightarrow A_1$ such that $RFf \circ h = 1_{A_2}$ is a homomorphism from $RFk_2$ to $RFk_1$;

4. for every $K$-object $k$, if $s$ is an element of the underlying set of $RFk$ such that $s \notin \text{Im}(\mu_k)$ then $\rho_{FK}^{-1}\{s\}$ is a singleton.

Next we interpret the condition (s3) for algebras.

**Proposition 3.1.** Let $Q$ be a quasivariety of algebras of a finitary similarity type $\Delta$, let $V$ be a proper subvariety of $Q$ and let $F : K \rightarrow Q$ be a functor. Then (s3) holds
(●) if $Ff$ is surjective and for every $s \in A_2$ with $|RFf^{-1}\{s\}| > 1$, if $\sigma_{RFf_2}(a_1, a_2, \ldots, a_n) = s$ for an $n$-ary operation $\sigma$ and $a_1, a_2, \ldots, a_n \in A_2$, then $s = a_{i_0}$ for some $i_0 \in \{1, 2, \ldots, n\}$ and $k(s) = \sigma_{RFf_1}(k(a_1), k(a_2), \ldots, k(a_n))$ for every mapping $k : \{a_1, a_2, \ldots, a_n\} \to A_1$ such that $RFf \circ k(a_i) = a_i$ for all $i \in \{1, 2, \ldots, n\}$.

**Proof.** Assume (s3). Let $s = \sigma_{RFf_2}(a_1, a_2, \ldots, a_n)$ for some $\sigma \in \Delta$, let $a_1, a_2, \ldots, a_n, s \in A_2$ and $|RFf^{-1}\{s\}| > 1$. Let $h : A_2 \to A_1$ be a mapping such that $RFf \circ h$ is the identity mapping. Then $h(s) = \sigma_{RFf_1}(h(a_1), h(a_2), \ldots, h(a_n))$. If $s \notin \{a_1, a_2, \ldots, a_n\}$ then there exists a mapping $h' : A_2 \to A_1$ with $RFf \circ h = RFf \circ h'$, $h(s) \neq h'(s)$ and $h(t) = h'(t)$ for all $t \in A_2 \setminus \{s\}$. Hence $h'(s) \neq \sigma_{RFf_1}(h'(a_1), h'(a_2), \ldots, h'(a_n))$ and this contradicts the fact that $h' : RFf_2 \to RFf_1$ is a homomorphism. Thus there exists $i_0 \in \{1, 2, \ldots, n\}$ with $a_{i_0} = s$. If $k : \{a_1, a_2, \ldots, a_n\} \to A_1$ is a mapping such that $RFf \circ k(a_i) = a_i$ for every $i \in \{1, 2, \ldots, n\}$ then there exists a mapping $h : A_2 \to A_1$ such that $RFf \circ h$ is the identity mapping of $A_2$ and $h(a_i) = k(a_i)$ for all $i = \{1, 2, \ldots, n\}$. But $h : RFf_2 \to RFf_1$ is a homomorphism, by (s3), and hence $k(s) = \sigma_{RFf_1}(k(a_1), k(a_2), \ldots, k(a_n))$ because $s = a_{i_0}$. Whence the condition (●) holds.

For the converse, assume (●) and let $h : A_2 \to A_1$ be a mapping such that $RFf \circ h$ is the identity of $A_2$. Choose an $n$-ary operation $\sigma$ of type $\Delta$ and $a_1, a_2, \ldots, a_n \in A_2$. Write $s = \sigma_{RFf_2}(a_1, a_2, \ldots, a_n)$. First we assume that $|RFf^{-1}\{s\}| > 1$. Then (●) gives an $i_0 \in \{1, 2, \ldots, n\}$ with $s = a_{i_0}$ and $h(s) = \sigma_{RFf_1}(h(a_1), h(a_2), \ldots, h(a_n))$, as required. From $Ff \in \text{Sur}_{\mathbb{Q}}$ we infer that $RFf \in \text{Sur}_{\mathbb{Q}}$, and hence $|RFf^{-1}\{s\}| = 1$ is the only remaining case. If

$$t = \sigma_{RFf_1}(h(a_1), h(a_2), \ldots, h(a_n))$$

then

$$RFf(t) = \sigma_{RFf_2}(RFf(h(a_1)), RFf(h(a_2)), \ldots, RFf(h(a_n))) = \sigma_{RFf_2}(a_1, a_2, \ldots, a_n) = s$$

and hence $t = h(s)$. Thus $h$ is a homomorphism, and the proof is complete. 

**Remark.** Observe that if $F : \mathbb{K} \to \mathbb{Q}$ is an almost full embedding then $F$ is synchronized $\mathcal{Z}(\mathbb{T})$-relatively full embedding for the trivial variety $\mathbb{T}$. Indeed, its synchronizer $S$ is a singleton algebra and $\mu_k$ is the identity automorphism of $S$ for every
K-object \( k \). Clearly, the conditions (s1)-(s4) are satisfied. And \( F : K \rightarrow Q \) is a full embedding exactly when \( F \) is an almost full embedding and for every \( K \)-object \( k \) there exists no \( Q \)-morphism from the terminal object of \( Q \) into \( Fk \).

Let \( N_0 \) be a poset viewed as a category whose objects are sets from the set \( \mathcal{P}(\omega_0) \) of all finite subsets of \( \omega \) and there exists an \( N_0 \)-morphism from \( A \in \mathcal{P}(\omega_0) \) into \( B \in \mathcal{P}(\omega_0) \) if and only if \( B \subseteq A \). Let \( N \) be the full subcategory of \( N_0 \) whose objects belong to the set \( \mathcal{P}(\omega) = \mathcal{P}(\omega_0) \setminus \{ \emptyset \} \) of all non-void subsets of \( \omega \). For \( A, B \in \mathcal{P}(\omega) \) with \( B \subseteq A \), let \( \eta_{A,B} \) denote the unique \( N \)-morphism from \( A \) to \( B \).

**Theorem 3.2.** Let \( Q \) be a quasivariety of finitary algebras and let \( \mathcal{V} \) be a subvariety of \( Q \). If there exists a synchronized \( \mathcal{I}(\mathcal{V}) \)-relatively full embedding \( F : N \rightarrow Q \) such that

1. \( FA \) is a finite algebra for every \( A \in \mathcal{P}(\omega) \);
2. \( F\eta_{A,B} \in \text{Surj}_Q \) for every \( A, B \in \mathcal{P}(\omega) \) with \( B \subseteq A \) (then \( RF\eta_{A,B} \) is a retract);
3. if \( A = B \cup C \) for \( A, B, C \in \mathcal{P}(\omega) \) then \( \{ F\eta_{A,B}, F\eta_{A,C} \} \) is a separating family.

Then \( \{ S_A \mid A \in \mathcal{P}(\omega_0) \} \) is an \( \mathcal{A} \)-D family where \( S_0 \) is a singleton algebra in \( Q \) and \( S_A = FA \) for all \( A \in \mathcal{P}(\omega) \).

**Proof.** We need to prove (p1)-(p4). Clearly, (p1) is satisfied. To prove (p2), consider sets \( A, B, C \in \mathcal{P}(\omega) \) with \( A = B \cup C \). By (3), \( \{ F\eta_{A,B}, F\eta_{A,C} \} \) is a separating family and thus \( FA \) is a subobject of \( FB \times FC \). Hence we obtain \( FA \in \text{Qua}\{ FB, FC \} \) and the proof of (p2) is complete.

For every \( A \in \mathcal{F}(\omega) \), let \( \rho_A : FA \rightarrow RFA \) denote the epireflection homomorphism of \( FA \) into \( \mathcal{V} \). Then \( \rho_A \in \text{Surj}_Q \).

To prove (p3), let \( A, B \in \mathcal{P}(\omega) \) be such that \( FA \in \text{Qua}\{ FB \} \). By the hypothesis, \( FB \) is finite, so that the family of all homomorphisms from \( FA \) to \( FB \) is separating. Since \( F \) is \( \mathcal{I}(\mathcal{V}) \)-relatively full embedding we infer that if \( B \not\subseteq A \) then every homomorphism from \( FA \) into \( FB \) factorizes through \( \rho_A \). Since \( FA \notin \mathcal{V} \) and \( RFA \in \mathcal{V} \), the mapping \( \rho_A \) is not injective and thus \( FA \notin \text{Qua}\{ FB \} \) - a contradiction. Thus we can assume that \( B \subseteq A \). If \( f : FA \rightarrow FB \) is a homomorphism then either \( h = F\eta_{A,B} \) or \( h \) factorizes through \( \rho_A \) because \( F \) is \( \mathcal{I}(\mathcal{V}) \)-relatively full embedding. Since the family of all homomorphisms from \( FA \) to \( FB \) is separating, the pair \( \{ F\eta_{A,B}, \rho_A \} \) must be a separating family. We claim that this is impossible.
when $B \neq A$. Indeed, if $B \neq A$ then $F\eta_{A,B}$ is not injective; this is because from (2) it would follow that $F\eta_{A,B}$ is an isomorphism, contrary to the relative fullness of $F$. But then $F\eta_{A,B}$ is not injective on $(\rho_A)^{-1}(\text{Im}(\mu_A))$ by (s1) and hence, by (s2), for some $s \in S$ there are distinct $a, b \in \rho_A^{-1}\{s\}$ with $F\eta_{A,B}(a) = F\eta_{A,B}(b)$. Hence $\{F\eta_{A,B}, \rho_A\}$ is not a separating family, a contradiction. Thus $A = B$, and (p3) follows.

To prove (p4), let $\mathcal{F} \subseteq \mathcal{P}(\omega)$ be a finite set and let $B, C \in \text{Qua}\{FX \mid X \in \mathcal{F}\}$ be finite algebras such that there exist $A \in \mathcal{P}(\omega)$ and an injective homomorphism $f : FA \to B \times C$. Hence there exist finite separating families $\{g_i : B \to FX_i \mid i \in I\}$ and $\{h_j : C \to FY_j \mid j \in J\}$ such that $X_i, Y_j \in \mathcal{P}(\omega)$ for all $i \in I$ and $j \in J$. Let $\pi_1 : B \times C \to B$, $\pi_2 : B \times C \to C$ be projections.

First we prove that we can assume that $\pi_1 \circ f, \pi_2 \circ f \in \text{Surj}_Q$. So assume that (p4) is satisfied if $\pi_1 \circ f, \pi_2 \circ f \in \text{Surj}_Q$. By the factorization property, there exist homomorphisms

$$
f'_1 : FA \to B' \in \text{Surj}_Q, f''_1 : B' \to B \in \text{Inj}_Q,$$

$$f'_2 : FA \to C' \in \text{Surj}_Q, f''_2 : C' \to C \in \text{Inj}_Q$$

with $\pi_1 \circ f = f''_1 \circ f'_1$ and $\pi_2 \circ f = f''_2 \circ f'_2$. Since $f$ is injective we infer that $\{\pi_1 \circ f, \pi_2 \circ f\}$ is separating and hence $\{f'_1, f'_2\}$ is also separating. Thus there exists an injective homomorphism $f' : FA \to B' \times C'$ with $\pi'_1 \circ f' = f'_1$ and $\pi'_2 \circ f' = f'_2$ where $\pi'_1 : B' \times C' \to B'$ and $\pi'_2 : B' \times C' \to C'$ are projections.

Then $\{g_i \circ f'_1' : B' \to FX_i \mid i \in I\}$ and $\{h_j \circ f'_2' : C' \to FY_j \mid j \in J\}$ are separating families and, by the assumption, the condition (p4) is satisfied for $f'$, $B'$ and $C'$ because $\pi'_1 \circ f', \pi'_2 \circ f' \in \text{Surj}_Q$. Then (p4) is also satisfied for $f, B$ and $C$ because $f'_1' : B' \to B$, $f'_2' : C' \to C \in \text{Inj}_Q$. Thus with no loss of generality we can assume that $\pi_1 \circ f, \pi_2 \circ f \in \text{Surj}_Q$.

Let us define $I' = \{i \in I \mid g_i \circ \pi_1 \circ f = F\eta_{A,X_i}\}$ and $J' = \{j \in J \mid g_j \circ \pi_2 \circ f = F\eta_{A,Y_j}\}$. Then $X_i \subseteq A$ and $Y_j \subseteq A$ for all $i \in I$ and $j \in J$. Observe that $g_i \circ \pi_1 \circ f$ and $g_j \circ \pi_2 \circ f$ factorize through $\rho_A$ for all $i \in I \setminus I'$ and $j \in J \setminus J'$ because $F$ is $I(V)$-relatively full embedding. Hence $I' \neq \emptyset$ or $J' \neq \emptyset$.

Set $U = \bigcup_{i \in I'} X_i$ and $V = \bigcup_{j \in J'} Y_j$. Then $U \cup V \subseteq A$ and $g_i \circ \pi_1 \circ f$ factorizes through $F(\eta_{A,U})$ for all $i \in I'$ and $g_j \circ \pi_2 \circ f$ factorizes through $F(\eta_{A,V})$ for all $j \in J'$. Since $\{g_i \circ \pi_1 \circ f \mid i \in I\} \cup \{g_j \circ \pi_2 \circ f \mid j \in J\} \in \text{Inj}_Q$ we infer, by (p3), that if $J' = \emptyset$ then $U = A$ and $I' = \emptyset$ then $V = A$, if $I' \neq \emptyset \neq J'$ then $A = U \cup V$.

Assume that $I' \neq \emptyset$. Since $\pi_1 \circ f \in \text{Surj}_Q$, $\{F\eta_{U,X_i} \mid i \in I'\} \in \text{Inj}_Q$ by (3) and $g_i \circ \pi_1 \circ f = F\eta_{U,X_i} \circ F\eta_{A,U}$ for all $i \in I$, by the diagonalization property there exists a homomorphism $\psi : B \to FU$ with $\psi \circ \pi_1 \circ f = F\eta_{A,U}$ and $F\eta_{U,X_i} \circ \psi = g_i$.
for all $i \in I'$. From $F \eta_{A,U} \in \text{Surj}_Q$ it follows that $\psi \in \text{Surj}_Q$.

Since $\{g_i \mid i \in I\}$ is a separating family, for distinct $u, v \in FA$ we have

$\pi_1 \circ f(u) \neq \pi_1 \circ f(v)$ if and only if there exists $i \in I$ with $g_i \circ \pi_1 \circ f(u) \neq g_i \circ \pi_1 \circ f(v)$. If $i \in I'$ then $g_i \circ \pi_1 \circ f = F \eta_{A,X_i} = F \eta_{U,X_i} \circ \eta_{A,U}$. Thus if $F \eta_{A,U}(u) \neq F \eta_{A,U}(v)$ for $u, v \in FA$ then $\pi_1 \circ f(u) \neq \pi_1 \circ f(v)$. If $i \in I \setminus I'$ then $g_i \circ \pi_1 \circ f = h \circ \rho_A$ for some homomorphism $h$ and thus $\pi_1 \circ f(u) \neq \pi_1 \circ f(v)$ implies that $\rho_A(u) \neq \rho_A(v)$ or $F \eta_{A,U}(u) \neq F \eta_{A,U}(v)$ because $\{F \eta_{U,X_i} \mid i \in I'\}$ is a separating family.

Let $S$ be a synchronizer of $F$. Consider $t \in \rho_A^{-1}(\text{Im}(\mu_A))$ and $u \in FA \setminus \rho_A^{-1}(\text{Im}(\mu_A))$. Then $\rho_A(t) = \mu_A(s)$ for some $s \in S$. By (s2), $\rho_U \circ \psi \circ \pi_1 \circ f(t) = \rho_U \circ F \eta_{A,U}(t) = \mu_U(s)$ and $\rho_U \circ \psi \circ \pi_1 \circ f(u) = \rho_U \circ F \eta_{A,U}(u) \notin \text{Im}(\mu_U)$. Hence $\psi^{-1}(\rho_U^{-1}(\text{Im}(\mu_U))) = \pi_1 \circ f(\rho_A^{-1}(\text{Im}(\mu_A)))$. If we combine this fact with the foregoing argument we conclude that for $u, v \in \rho_A^{-1}(\text{Im}(\mu_A))$ we have $\pi_1 \circ f(u) = \pi_1 \circ f(v)$ if and only if $F \eta_{A,U}(u) = F \eta_{A,U}(v)$. From (s2) it follows that $(RF \eta_{A,U})^{-1}(\mu_U(s)) = \{\mu_A(s)\}$ for all $s \in S$. Thus $(R\psi)^{-1}(\mu_U(s)) = \{\pi_1 \circ f(\mu_A(s))\}$ for every $s \in S$ because $\psi \circ \pi_1 \circ f = F \eta_{A,U}$. Since $F \eta_{A,U}$ is surjective, by (s3), every mapping $\nu$ from the underlying set of $RFU$ into the underlying set of $RF A$ such that $RF \eta_{A,U} \circ \nu$ is the identity mapping is a homomorphism from $RFU$ into $RF A$. From $\psi \circ \pi_1 \circ f = F \eta_{A,U}$ we conclude $R(\psi \circ \pi_1 \circ f) = RF \eta_{A,U}$. For a homomorphism $\nu' : RFU \to RF A$ such that $RF \eta_{A,U} \circ \nu'$ is the identity automorphism of $RFU$ we set $\nu = R(\pi_1 \circ f) \circ \nu'$ and hence $\nu : RFU \to RB$ is a homomorphism such that $R\psi \circ \nu$ is the identity homomorphism of $RFU$. Since $\nu'$ exists by (s3), we can assume that we have a homomorphism $\nu : RFU \to RB$ such that $R\psi \circ \nu$ is the identity homomorphism of $RFU$.

For every $i \in I \setminus I'$ there exists a homomorphism $\tilde{g}_i : RF A \to FX_i$ with $g_i \circ \pi_1 \circ f = \tilde{g}_i \circ \rho_A$. By the properties of factorization system, there exist homomorphisms $\sigma : RF A \to D \in \text{Surj}_Q$ and $\sigma_i : D \to FX_i$ for $i \in I \setminus I'$ such that $g_i \circ \pi_1 \circ f = \sigma_i \circ \sigma \circ \rho_A$ for all $i \in I \setminus I'$ and $\{\sigma_i \mid i \in I \setminus I'\} \in \text{Inj}_Q$. By the diagonalization property, there exists a homomorphism $\phi' : B \to D$ such that $\phi' \circ \pi_1 \circ f = \sigma \circ \rho_A$ and $\sigma_i \circ \phi' = g_i$ for all $i \in I \setminus I'$. From $\rho_A, \sigma \in \text{Surj}_Q$ it follows that $\phi' \in \text{Surj}_Q$. From $RF A \in V$ and $\sigma : RF A \to D \in \text{Surj}_Q$ it follows that $D \in V$ and if $\rho_B : B \to RB$ is the epireflection morphism of $B$ into $V$, then there exists a homomorphism $\phi : RB \to D \in \text{Surj}_Q$ with $\phi' = \phi \circ \rho_B$. Then

$$\sigma \circ \rho_A = \phi' \circ \pi_1 \circ f = \phi \circ \rho_B \circ \pi_1 \circ f = \phi \circ R(\pi_1 \circ f) \circ \rho_A$$

and $\sigma = \phi \circ R(\pi_1 \circ f)$ follows because $\rho_A \in \text{Surj}_Q$. Since $\{g_i \mid i \in I\} \in \text{Inj}_Q$ we infer that the family $\{\psi, \rho_B\}$ is separating. Hence there exists a homomorphism
\( \omega : B \rightarrow FU \times RB \in \text{Inj}_Q \) such that \( \tau_1 \circ \omega = \psi \) and \( \tau_2 \circ \omega = \rho_B \) where \( \tau_1 : FU \times RB \rightarrow FU \) and \( \tau_2 : FU \times RB \rightarrow RB \) are projections. Then \( \tau_1 \circ \omega \circ \pi_1 \circ f = \psi \circ \pi_1 \circ f = F\eta_{A,U} \) and \( \tau_2 \circ \omega \circ \pi_1 \circ f = \rho_B \circ \pi_1 \circ f = R(\pi_1 \circ f) \circ \rho_A \). Hence for every \( b \in B \) and \( a \in FA \) with \( \pi_1 \circ f(a) = b \) we have \( \omega(b) = (F\eta_{A,U}(a), R(\pi_1 \circ f) \circ \rho_A(a)) \).

By the property of products, there exists a homomorphism \( \lambda : FU \rightarrow FU \times RB \) such that \( \tau_1 \circ \lambda \) is the identity morphism of \( FU \) and \( \tau_2 \circ \lambda = \nu \circ \rho_U \), hence \( \lambda \in \text{Inj}_Q \). Select \( u \in FU \). If \( \rho_U(u) \in \text{Im} (\mu_U) \) then, by (s2), \( \rho_A((F\eta_{A,U})^{-1}(u)) = (RF\eta_{A,U})^{-1}(\rho_U(u)) \) is a singleton and hence for every \( a \in FA \) with \( F\eta_{A,U}(a) = u \) we have \( \{ R(\pi_1 \circ f) \circ \rho_A(a) \} = (R\psi)^{-1}(\rho_U(u)) = \{ \nu(\rho_U(u)) \} \). Thus \( \lambda(u) = (F\eta_{A,U}(a), R(\pi_1 \circ f) \circ \rho_A(a)) \in \text{Im} (\omega) \). If \( \rho_U(u) \notin \text{Im} (\mu_U) \), then there exists \( a \in FA \) such that \( \rho_B \circ \pi_1 \circ f(a) = \nu(\rho_U(u)) \) because \( \rho_B, \pi_1 \circ f \in \text{Surj}_Q \). Then
\[
R\psi \circ \rho_B \circ \pi_1 \circ f(a) = R\psi \circ \nu(\rho_U(u)) = \rho_U(u).
\]

Since
\[
R\psi \circ \rho_B \circ \pi_1 \circ f = \rho_U \circ \psi \circ \pi_1 \circ f = \rho_U \circ F\eta_{U,A}
\]
we conclude that \( \rho_U(u) = \rho_U(F\eta_{A,U}(a)) \) and, by (s4), \( u = F\eta_{A,U}(a) \). Thus \( \lambda(u) = (F\eta_{A,U}(a), R(\pi_1 \circ f) \circ \rho_A(a)) \in \text{Im} (\omega) \) because \( R(\pi_1 \circ f) \circ \rho_A = \rho_B \circ \pi_1 \circ f \).

Thus \( \text{Im} (\lambda) \subseteq \text{Im} (\omega) \), so that there exists an injective homomorphism from \( FU \) to \( B \).

If \( J' \neq \emptyset \) then the same proof gives the existence of an injective \( \nu : FV \rightarrow C \), and (p4) follows.

The technical statement below enables us to prove a generalized version of Theorem 3.2. We say that a surjective homomorphism \( f : A \rightarrow B \) of algebraic systems of similarity type \( \Delta \) is a quotient if for every relation \( r \in \Delta \) we have that \( (b_0, b_1, \ldots, b_k) \in r_B \) if and only if there exists \( (a_0, a_1, \ldots, a_k) \in r_A \) with \( f(a_i) = b_i \) for all \( i = 0, 1, \ldots, k \). A quasivariety \( Q \) is closed under quotients if algebraic system \( A \in Q \) whenever there exist an algebraic system \( B \in Q \) and a quotient \( f : B \rightarrow A \). Let \( \text{Quot}_Q \) denote the class of all quotients of \( Q \). It is well-known [1] that \( (\text{Quot}_Q, \text{Inj}_Q) \) is a factorization system in \( Q \), and that \( \text{Surj}_Q = \text{Quot}_Q \) if \( Q \) is a quasivariety of algebras. If \( Q \) is clear from the context, we write \( \text{Quot} \) instead of \( \text{Quot}_Q \).

**Proposition 3.3.** Let \( Q \) be a quasivariety of algebraic systems and let \( \mathbb{R} \) be a proper subquasivariety of \( Q \) closed under quotients. If there exists an \( \mathcal{I}(\mathbb{R}) \)-relatively full embedding \( F : \mathbb{N} \rightarrow Q \) such that \( FA \) is finite for all \( A \in \mathcal{P}(\omega) \) and \( F\eta_{A,B} \in \text{Quot}_Q \) for all \( A, B \in \mathcal{P}(\omega) \) with \( B \subseteq A \), then there exists an \( \mathcal{I}(\mathbb{R}) \)-relatively full embedding \( G : \mathbb{N} \rightarrow Q \) such that
(1) \( G A \) is finite for all \( A \in \mathcal{P}(\omega) \);

(2) if \( A, B, C \in \mathcal{P}(\omega) \) satisfy \( B \cup C \subseteq A \), then \( \{G_{\eta_A,B}, G_{\eta_A,C}\} \) is a separating family if and only if \( A = B \cup C \);

(3) \( G_{\eta_A,B} \in \text{Quot}_Q \) for all \( A, B \in \mathcal{P}(\omega) \) with \( B \subseteq A \).

Moreover, if \( Q \) is a quasivariety of algebras and \( F \) is synchronized then \( G \) is synchronized.

The fairly technical proof of this Proposition can be found in the Appendix.

Proof of Theorem 1.4 completed. Let \( \mathcal{G} \mathcal{R}A \) denote the (concrete) category of all undirected graphs and compatible mappings. We recall that there exists a full embedding \( \Phi \) of \( \mathbb{N} \) into \( \mathcal{G} \mathcal{R}A \) such that \( \Phi A \) is a finite graph of every \( A \in \mathbb{N} \) and \( \Phi_{\eta_A,B} \in \text{Quot}_{\mathcal{G} \mathcal{R}A} \) for every \( A, B \in \mathcal{P}(\omega) \) with \( B \subseteq A \), see [7]. Let \( F : \mathcal{G} \mathcal{R}A \rightarrow Q \) satisfy the hypothesis of Theorem 1.4. Then the composite \( F \circ \Phi : \mathbb{N} \rightarrow Q \) satisfies the hypothesis of Proposition 3.3, and hence \( Q \) contains an A-D family, by Theorem 3.2. This concludes the proof of Theorem 1.4. \( \square \)

Remark. The embeddings from \( \mathcal{G} \mathcal{R}A \) into the variety of semigroups generated by \( M_2 \) or \( M_3 \) or \( M_3^d \) or \( M_4 \) or \( M_4^d \) constructed in [6, 7, 8] are synchronized (here for a semigroup \( S = (S, \cdot) \), its dual is defined as \( S^d = (S, \circ) \) with \( s \circ t = t \cdot s \) for all \( s, t \in S \)) and constitute special cases of Theorem 3.2. The semigroups \( M_2, M_3 \) and \( M_4 \) are defined in Table 1.

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<tr>
<th>M_2</th>
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<th>M_3</th>
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<td>0</td>
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Table 1: The semigroups \( M_2, M_3 \) and \( M_4 \)

Finally, we show that for quasivarieties of algebras Theorem 1.4 generalizes Theorem 1.3 of [16]. So let \( Q \) be a quasivariety of algebras and let \( V \) be a proper subvariety of \( Q \). We say that an epireflection \( R : Q \rightarrow V \) is constant on a functor \( F : \mathbb{N} \rightarrow Q \) if the composite \( R \circ F \) is a constant functor. It is then clear that if
the epireflection $R$ is constant on an $\mathcal{I}(\mathcal{V})$-relatively full embedding $F$, then $F$ is synchronized. Thus we immediately obtain

**Corollary 3.4.** Let $\mathbb{Q}$ be a quasivariety of algebras and let $\mathcal{V}$ be a proper subvariety of $\mathbb{Q}$. If $F : \mathbb{N} \rightarrow \mathbb{Q}$ is an $\mathcal{I}(\mathcal{V})$-relatively full embedding such that the epireflection of $\mathbb{Q}$ into $\mathcal{V}$ is constant on $F$, $F\eta_{A,B} \in \text{Sur}_{\mathbb{Q}}$ for all $A, B \in \mathcal{P}(\omega)$ with $B \subseteq A$ and $F\mathcal{A}$ is finite for all $A \in \mathcal{P}(\omega)$ then there exists an $A$-$\mathcal{D}$ family in $\mathbb{Q}$, and thus $\mathbb{Q}$ is $\mathbb{Q}$-universal. □

Thus, in particular, the object ideal $\mathcal{I}(\mathcal{V})$ associated with such an $\mathcal{I}(\mathcal{V})$-relatively full embedding $F$ is *principal* in the sense that it is determined by a single object of $\mathcal{V}$ and includes the case when the synchronizer is a singleton algebra, that is, the case of an almost full embedding.

**Appendix**

**Proof of Proposition 3.3.** Consider a functor $H : \mathbb{N}_0 \rightarrow \mathbb{N}$ defined by $H\emptyset = \{0\}$ and $HA = \{0\} \cup \{n + 1 \mid n \in A\}$ for all $A \in \mathcal{P}(\omega)$ and $H\eta_{A,B} = \eta_{HA,HB}$ for $A, B \in \mathcal{P}(\omega_0)$ with $B \subseteq A$. Then $H$ is a full embedding (since $A \subseteq B$ if and only if $HA \subseteq HB$ for $A, B \in \mathcal{P}(\omega_0)$, it is correctly defined). Thus the composite $F' = F \circ H : \mathbb{N}_0 \rightarrow \mathbb{Q}$ is an $\mathcal{I}(\mathbb{R})$-relatively full embedding such that $F'A$ is finite for all $A \in \mathcal{P}(\omega)$ and $F'\eta_{A,B} = F\eta_{HA,HB} \in \text{Quot}_{\mathbb{Q}}$ for all $A, B \in \mathcal{P}(\omega_0)$ with $B \subseteq A$.

Since $F'$ is an $\mathcal{I}(\mathbb{R})$-relatively full embedding, $F'A \notin \mathbb{R}$ for all $A \in \mathcal{P}(\omega_0)$. For $n \in \omega$, set $G\{n\} = F'\{n\}$. For $A \in \mathcal{P}(\omega)$, define $\Pi(A) = \prod_{a \in A} F'\{a\}$ and let $\pi_a : \Pi(A) \rightarrow F'\{a\}$ be the $a$-th projection for each $a \in A$. By the universal property of products, there exists a unique homomorphism $\tau_A : F'A \rightarrow \Pi(A)$ such that $F'\eta_{A,\{a\}} = \pi_a \circ \tau_A$ for every $a \in A$. Factorizing $\tau_A$ in $\mathbb{Q}$ in the factorization system $(\text{Quot}_{\mathbb{Q}}, \text{Inj}_{\mathbb{Q}})$, we obtain homomorphisms (unique up to an isomorphism) $\chi_A : F'A \rightarrow GA \in \text{Quot}_{\mathbb{Q}}$ and $\mu_A : GA \rightarrow \Pi(A) \in \text{Inj}_{\mathbb{Q}}$ such that $\tau_A = \mu_A \circ \chi_A$. Since the underlying set of $F'A$ is finite and since $\chi_A$ is a quotient, the underlying set of $GA$ is finite for all $A \in \mathcal{P}(\omega)$. This proves (1).

Consider $A, B \in \mathcal{P}(\omega)$ with $B \subseteq A$. By the universal property of products, there exists a unique homomorphism $\Pi(\eta_{A,B}) : \Pi(A) \rightarrow \Pi(B)$ such that $\pi_b = \kappa_b \circ \Pi(\eta_{A,B})$ for all $b \in B \subseteq A$, where $\kappa_b : \Pi(B) \rightarrow F'\{b\}$ is the $b$-th projection for $b \in B$. Then for every $b \in B$ we have

$$
\kappa_b \circ \Pi(\eta_{A,B}) \circ \tau_A = \pi_b \circ \tau_A = F'\eta_{A,\{b\}} = F'\eta_{B,\{b\}} \circ F'\eta_{A,B}
$$

$$
= \kappa_b \circ \tau_B \circ F'\eta_{A,B}
$$
because \( \kappa_b \circ \tau_B = F'\eta_B,\{b\} \), and hence

\[
\Pi(\eta_{A,B}) \circ \mu_A \circ \chi_A = \Pi(\eta_{A,B}) \circ \tau_A = \tau_B \circ F'\eta_{A,B} = \mu_B \circ \chi_B \circ F'\eta_{A,B}
\]

because the family \( \{\kappa_b \mid b \in B\} \) of projections is separating.

By the diagonalization property, there exists a homomorphism \( G\eta_{A,B} : GA \to GB \) with \( G\eta_{A,B} \circ \chi_A = \chi_B \circ F'\eta_{A,B} \) and \( \Pi(\eta_{A,B}) \circ \mu_A = \mu_B \circ G\eta_{A,B} \) because \( \mu_B \in \text{Inj} \) and \( \chi_A \in \text{Quot} \). From \( \chi_B \circ F'\eta_{A,B} \in \text{Quot} \) it follows that \( \chi_B \circ F'\eta_{A,B} \in \text{Quot} \) and \( G\eta_{A,B} \in \text{Quot} \), and (3) is proved. Note the diagram below, commuting for every \( b \in B \subseteq A \).

\[
\begin{array}{ccc}
F' A & \xrightarrow{\chi_A} & GA & \xrightarrow{\mu_A} & \Pi(A) & \xrightarrow{\pi_b} & G\{b\} = F'\{b\} \\
F'\eta_{A,B} \downarrow & & \downarrow G\eta_{A,B} & & \downarrow \Pi(\eta_{A,B}) & & \\
F' B & \xrightarrow{\chi_B} & GB & \xrightarrow{\mu_B} & \Pi(B) & \xrightarrow{\kappa_b} & G\{b\} = F'\{b\}
\end{array}
\]

To prove that \( G \) is a functor, let \( A, B, C \in P(\omega) \) satisfy \( C \subseteq B \subseteq A \). Then

\[
G\eta_{B,C} \circ G\eta_{A,B} \circ \chi_A = G\eta_{B,C} \circ \chi_B \circ F'\eta_{A,B} = \chi_C \circ F'\eta_{B,C} \circ F'\eta_{A,B} = \chi_C \circ F'\eta_{A,C} \circ \chi_A
\]

and because \( \chi_A \in \text{Quot} \) we conclude that \( G\eta_{B,C} \circ G\eta_{A,B} = G\eta_{A,C} \). Since \( F'\eta_{A,A} \) is the identity homomorphism, from \( G\eta_{A,A} \circ \chi_A = \chi_A \circ F'\eta_{A,A} = \chi_A \in \text{Quot} \) it follows that \( G\eta_{A,A} \) is also the identity homomorphism. Altogether, \( G \) is a functor.

We turn to (2). Note that \( F'\eta_{A,\{a\}} = \pi_A \circ \tau_A = \pi_A \circ \mu_A \circ \chi_A \) and \( G\eta_{A,\{a\}} \circ \chi_A = F'\eta_{A,\{a\}} \) for every \( a \in A \) because \( \chi_{\{a\}} \) is the identity morphism of \( F'\{a\} = G\{a\} \). From \( \chi_A \in \text{Quot} \) we then obtain \( G\eta_{A,\{a\}} = \pi_A \circ \mu_A \) for every \( a \in A \). But then \( \{G\eta_{A,\{a\}} \mid a \in A\} \) is a separating family because \( \mu_B \in \text{Inj} \) and the family \( \{\pi_a \mid a \in A\} \) of projections is separating. Hence \( \{G\eta_{A,B}, G\eta_{A,C}\} \) is a separating family for any \( A, B, C \in P(\omega) \) with \( A = B \cup C \). Conversely, assume that \( B \cup C \subseteq A \) and \( \{G\eta_{A,B}, G\eta_{A,C}\} \) is a separating family. Then \( \{G\eta_{A,\{a\}} \mid a \in B \cup C\} \) is clearly a separating family. Set \( A' = B \cup C \). Then \( G\eta_{A,A'} \in \text{Inj} \) and thus from the already proved (3) it follows that \( G\eta_{A,A'} \) is an isomorphism. Choose \( a \in A \setminus A' \). Since \( G\eta_{A,\{a\}} \circ \chi_A = F'\eta_{A,\{a\}} \in \text{Quot} \), we have \( G\eta_{A,\{a\}} = \pi_A \circ \mu_A \in \text{Quot} \).

But then \( \pi_A \circ \mu_A \circ (G\eta_{A,A'})^{-1} \circ \chi_{A'} : F'\eta_{A'} \to F'\{a\} \) is a quotient because \( (G\eta_{A,A'})^{-1} \circ \chi_{A'} \in \text{Quot} \). This is a contradiction because \( F' \) is an \( I(R) \)-relatively full embedding, \( F'\{a\} \) does not belong to \( \mathbb{R} \) and \( \{a\} \not\subseteq A' \). Hence (2) follows.

To prove that \( G \) is an \( I(R) \)-relatively full embedding consider \( A, B \in P(\omega) \) with \( B \subseteq A \). Then \( \eta_{A,B} \) is a morphism of \( \mathbb{N} \) and we must prove that \( \text{Im}(G\eta_{A,B}) \notin \).
For every $b \in B$, $G\eta_{B,(b)} \in \text{Quot}$ and $G\{b\} \notin \mathbb{R}$. Since $\mathbb{R}$ is closed under Quot we infer that $GB \notin \mathbb{R}$ and because $G\eta_{A,B} \in \text{Quot}$ we conclude that $\text{Im}(G\eta_{A,B}) \notin \mathbb{R}$. Conversely, let $f : GA \to GB$ for $A, B \in \mathcal{P}(\omega)$ be a homomorphism such that $\text{Im}(f) \notin \mathbb{R}$. To complete the proof it suffices to prove that $B \subseteq A$ and $f = G\eta_{A,B}$. Let $f' : A \to C \in \text{Quot}$ and $f'' : C \to B \in \text{Inj}$ be homomorphisms with $f = f'' \circ f'$ then $C$ is isomorphic to $\text{Im}(f)$. Since $\{G\eta_{B,(b)} \mid b \in B\}$ is a separating family, we infer that $\{G\eta_{B,(b)} \circ f'' \mid b \in B\}$ is a separating family and, by the universal property of products, the morphism $h : C \to \prod_{b \in B} \text{Im}(G\eta_{B,(b)} \circ f'') \in \text{Inj}$. Since $\text{Im}(f) \notin \mathbb{R}$ we conclude that $\prod_{b \in B} \text{Im}(G\eta_{B,(b)} \circ f'') \notin \mathbb{R}$, and thus there exists $b \in B$ such that $\text{Im}(G\eta_{B,(b)} \circ f'') = \text{Im}(G\eta_{B,(b)} \circ f) \notin \mathbb{R}$. Thus $G\eta_{B,(b)} \circ f \notin \mathcal{I}(\mathbb{R})$. Since $\chi_B \in \text{Quot}$ we conclude that $G\eta_{B,(b)} \circ f \circ \chi_A : F''A \to F''\{b\} \notin \mathcal{I}(\mathbb{R})$ and thus $b \in A$ and $G\eta_{B,(b)} \circ f \circ \chi_A = G\eta_{A,(b)}$ because $F'$ is an $\mathcal{I}(\mathbb{R})$-relatively full embedding. Then

$$F'\eta_{(b),0} \circ G\eta_{B,(b)} \circ f \circ \chi_A = F'\eta_{(b),0} \circ F'\eta_{A,(b)} = F'\eta_{A,0}.$$ 

Since for every $b' \in B$ we have

$$F'\eta_{(b'),0} \circ G\eta_{B,(b')} \circ \chi_B = F'\eta_{(b'),0} \circ F'\eta_{B,(b')} = F'\eta_{B,0} \circ F'\eta_{B,(b')} \circ \chi_B = F'\eta_{(b),0} \circ G\eta_{B,(b)} \circ \chi_B$$

we infer that $F'\eta_{(b'),0} \circ G\eta_{B,(b')} = F'\eta_{(b),0} \circ G\eta_{B,(b)}$ for all $b' \in B$ because $\chi_B \in \text{Quot}$. From this it follows that

$$F'\eta_{A,0} = F'\eta_{(b),0} \circ G\eta_{B,(b)} \circ f \circ \chi_A = F'\eta_{(b),0} \circ G\eta_{B,(b')} \circ f \circ \chi_A$$

for all $b' \in B$. Since $F'\eta_{A,0} \notin \mathcal{I}(\mathbb{R})$ we conclude that $G\eta_{B,(b')} \circ f \circ \chi_A \notin \mathcal{I}(\mathbb{R})$ for all $b' \in B$ because $F'\eta_{(b'),0} \circ G\eta_{B,(b')} \circ f \circ \chi_A$ and $\mathbb{R}$ is closed under Quot. Hence $b' \in A$ and $G\eta_{B,(b')} \circ f \circ \chi_A = F'\eta_{A,(b')} = G\eta_{A,(b')} \circ f \circ \chi_A$ for all $b' \in B$ because $F'$ is an $\mathcal{I}(\mathbb{R})$-relatively full embedding. Thus $B \subseteq A$ and

$$G\eta_{B,(b')} \circ G\eta_{A,B} \circ \chi_A = G\eta_{B,(b')} \circ \chi_B \circ F'\eta_{A,B} = F'\eta_{B,(b')} \circ F'\eta_{A,B} = F'\eta_{A,(b')} = G\eta_{B,(b')} \circ f \circ \chi_A$$

for all $b' \in B$. By (2), $\{G\eta_{B,(b')} \mid b' \in B\} \in \text{Inj}$ and thus $G\eta_{A,B} \circ \chi_A = f \circ \chi_A$. But $\chi_A \in \text{Quot}$, and this completes the proof that $f = G\eta_{A,B}$.

It remains to prove that if $\mathcal{Q}$ is a quasivariety of algebras and $F$ is synchronized then also $G$ is synchronized. First observe that $F'$ is also synchronized. For $A \in$
Let \( S \) be an algebra and for \( A \in \mathcal{P}(\omega) \) let \( \nu_A : S \to RF'A \) witness the fact that \( F' \) is synchronized. Since for every \( a \in A \) we have \( F'\eta_{A,\{a\}} = G\eta_{A,\{a\}} \circ \chi_A \) we conclude that \( RF'\eta_{A,\{a\}} = R(G\eta_{A,\{a\}} \circ \chi_A) \). Set \( \zeta_A = R\chi_A \circ \nu_A : S \to RGA \), then the property that for every \( s \in S \) and \( A, B \in \mathcal{P}(\omega) \) with \( B \subseteq A \) we have \( RF'\eta_{A,B}(\nu_A(s)) = \nu_B(s) \) implies \( RG\eta_{A,B}(\zeta_A(s)) = \zeta_B(s) \) and the fact that \( \nu_A \) is injective for every \( A \in \mathcal{P}(\omega) \) and \( \chi_{\{a\}} \) is the identity mapping for every \( a \in \omega \) imply that \( \zeta_A \) is injective for all \( A \in \mathcal{P}(\omega) \). The validity of (s1) and (s2) for \( F' \) implies that \( G \) also satisfies (s1) and (s2). From the facts that \( F' \) satisfies (s4) and \( \zeta_{\{a\}} = \nu_{\{a\}} \) for all \( a \in \omega \) and \( \{G\eta_{A,\{a\}} \mid a \in A\} \) is a separating family for all \( A \in \mathcal{P}(\omega) \) it follows that (s4) holds for \( G \). Indeed, if \( u \) and \( v \) are distinct elements of \( RGA \) with \( \rho_{G,A}(u), \rho_{G,A}(v) \notin \text{Im}(\zeta_A) \) then there exists \( a \in A \) with \( F'\eta_{A,\{a\}}(u) \neq F'\eta_{A,\{a\}}(v) \) and hence \( \rho_{G,A}(u) \circ F'\eta_{A,\{a\}}(u) \neq \rho_{G,A}(v) \circ F'\eta_{A,\{a\}}(v) \). Then \( \rho_{G,A}(u) \circ F'\eta_{A,\{a\}} = RF'\eta_{A,\{a\}} \circ \rho_{G,A} \) implies that \( \rho_{G,A}(u) \neq \rho_{G,A}(v) \). If \( u \) and \( v \) are elements of \( RGA \) with \( \rho_{G,A}(u) \notin \text{Im}(\zeta_A) \) and \( v \in \text{Im}(\zeta_A) \) then, by the same argument, we obtain that \( \rho_{G,A}(u) \neq \rho_{G,A}(v) \) and hence \( GA \) satisfies (s4). To prove (s3) consider \( A, B \in \mathcal{P}(\omega) \) with \( B \subseteq A \). Choose \( b \in B \). Since \( F' \) satisfies (s3), the condition (●) from Proposition 3.1 is satisfied for \( F'\eta_{A,B} \) and \( F'\eta_{B\{b\}} \). Since \( \chi \) is a surjective natural transformation from \( F' \) onto \( G \) and since \( F'\{b\} = G\{b\} \) we conclude, by Proposition 3.1, that every mapping \( h \) from the underlying set of \( RGB \) into \( RGA \) such that \( RG\eta_{A,B} \circ h \) is the identity mapping is a homomorphism from \( RGA \) into \( RGB \). Thus \( G \) satisfies (s3) and whence \( G \) is synchronized.

References


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