CAHIERS DE
TOPOLOGIE ET GÉOMÉTRIE DIFFÉRENTIELLE
CATÉGORIQUES

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Cahiers de topologie et géométrie différentielle catégoriques, tome 49, n° 4 (2008), p. 289-306

<http://www.numdam.org/item?id=CTGDC_2008__49_4_289_0>

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Vol. XLIX-4 (2008)

ON SYNCHRONIZED RELATIVELY FULL EMBEDDINGS AND Q-UNIVERSALITY

To Jiří Adámek on his 60th birthday

by V. KOUBEK and J. SICHLER

Abstract

M. E. Adams et W. Dziobiak ont démontré que toute quasi-variété ff-algébrique universelle de systèmes algébriques de signature finie est Q-universelle. Dans cet article on introduit la notion de plongement synchronisé relativement plein qu'on utilise ensuite afin de modifier leur résultat pour les quasi-variétés d'algèbres.

1 Introduction

We aim to show a new connection between two algebraic structures associated with quasivarieties of algebras. All needed definitions are given in the next section.

First, for any quasivariety $\mathbb{Q}$, the homomorphisms between its members form a concrete category. The richness of the categorical structure is reflected in the notion of algebraic universality studied in the monograph [18] by A. Pultr and V. Trnková.

When ordered by inclusion, the subquasivarieties of a given quasivariety $\mathbb{Q}$ form a lattice we denote $\mathrm{QLat}(\mathbb{Q})$. This is the second algebraic structure associated with $\mathbb{Q}$. Questions about the size of $\mathrm{QLat}(\mathbb{Q})$ or lattice identities satisfied in $\mathrm{QLat}(\mathbb{Q})$ motivated M. V. Sapir [19] to define and exhibit Q-universal quasivarieties, and W. Dziobiak [9, 10] to introduce what is now called an A-D family of objects of $\mathbb{Q}$. A survey of these notions and results concerning them is given in [2]. M. E. Adams

1991 Mathematics Subject Classification. 08C15, 18B15.

Key words and phrases. quasivariety, Q-universality, almost full embedding, relatively full embedding, ff-alg-universality.

Both authors gratefully acknowledge the support of the NSERC of Canada and of the project MSM 0021620838 of the Czech Ministry of Education. The first author also acknowledges the support of the grant 201/06/0664 provided by the Grant Agency of Czech Republic.

and W. Dziobiak [3] linked the latter two properties by showing that every quasivariety $\mathbb{Q}$ containing an A-D family is also Q-universal. The converse implication is still an open problem, originally stated by M. E. Adams and W. Dziobiak.

Problem 1.1. Is there a Q-universal quasivariety containing no A-D family?

In [4], M. E. Adams and W. Dziobiak proved the following remarkable and quite surprising result connecting the two algebraic structures associated with a quasivariety of algebraic systems.

Theorem 1.2 [4]. *Any finite-to-finits algebraically universal (ff-alg-universal) quasivariety of algebraic systems of finite similarity type contains an A-D family and hence it is Q-universal.* □

In [16], the present authors extended this result as follows.

Theorem 1.3 [16]. *Any almost ff-alg-universal quasivariety of algebraic systems of finite similarity type contains an A-D family and hence it is Q-universal.* □

Almost universality is a special case of relative universality, see Section 2. Here we aim to modify the latter result for quasivarieties of algebras. We assume that

($*$) $\mathbb{Q}$ is a quasivariety of finitary algebras and $\mathbb{V}$ is a proper subvariety of $\mathbb{Q}$ such that there exists a synchronized $\mathcal{I}(\mathbb{V})$-relatively full embedding F from the category of all undirected graphs into $\mathbb{Q}$ such that Ff is surjective for every graph quotient homomorphism f and $F\mathbf{G}$ is finite for every finite graph $\mathbf{G}$.

Theorem 1.4. *Any quasivariety $\mathbb{Q}$ satisfying ($*$) contains an A-D family and hence it is Q-universal.*

As already noted, all needed notions are reviewed in Section 2 below, and the proof of Theorem 1.4 is given in Section 3. It is based on the fact that any subquasivariety $\mathbb{R}$ of a quasivariety $\mathbb{Q}$ is an epireflective full subcategory of $\mathbb{Q}$. In Section 3, it is also shown how Theorem 1.4 incorporates earlier results of [6, 7, 8].

2 Basic notions and their context

Alg-universality. A category $\mathbb{K}$ is alg-universal if any category of algebras and all homomorphisms between them can be fully embedded into $\mathbb{K}$. This is equivalent to the fact that there exists a full embedding from the category $\mathbb{GRA}$ of all undirected graphs and all graph homomorphisms into $\mathbb{K}$. Moreover, if $\mathbb{K}$ is a concrete category and there exists a full embedding $F : \mathbb{GRA} \to \mathbb{K}$ such that the underlying set of $F\mathbf{G}$ for every finite graph is finite then we say that F preserves finiteness and that $\mathbb{K}$ is ff-alg-universal. If $\mathbb{K}$ is a concrete category then any $\mathbb{K}$-object A with a finite underlying set is called finite. Next we give several well-known properties of alg-universal categories. To do this, we say that a category $\mathbb{K}$ is a monoid universal if for every monoid $\mathbf{M}$ there exists a $\mathbb{K}$-object A such that the endomorphism monoid of A is isomorphic to $\mathbf{M}$.

Theorem 2.1 [18]. *(a) Any concrete alg-universal category $\mathbb{K}$ is monoid universal; and if $\mathbb{K}$ is ff-alg-universal, then for every finite monoid $\mathbf{M}$ there exists a finite $\mathbb{K}$-object A such that the endomorphism monoid of A is isomorphic to $\mathbf{M}$.*

(b) If $\mathbb{K}$ is alg-universal, then for a proper class I there exists a family $\{F_i : \mathbb{K} \to \mathbb{K} \mid i \in I\}$ of full embeddings such that F_iA is not isomorphic to F_jB for any $\mathbb{K}$-objects A and B and for any distinct $i, j \in I$. For any set I there exists a family $\{F_i : \mathbb{K} \to \mathbb{K} \mid i \in I\}$ of full embeddings such that there exists no $\mathbb{K}$-morphism between F_iA and F_jB for any $\mathbb{K}$-objects A and B and for any distinct $i, j \in I$.

(c) If $\mathbb{K}$ is ff-alg-universal and I is a countable set, then there exists a family $\{F_i : \mathbb{K} \to \mathbb{K} \mid i \in I\}$ of full embeddings F_i preserving finiteness such that there exists no $\mathbb{K}$-morphism between F_iA and F_jB for any $\mathbb{K}$-objects A and B and any distinct $i, j \in I$. □

Theorem 2.1 provides a tool for proving that a given category $\mathbb{K}$ is not alg-universal. For example, if $\mathbb{K}$ is a concrete category such that for every set X there exists only a set of non-isomorphic $\mathbb{K}$-objects with a given underlying set X and if there exists a cardinal α such that every $\mathbb{K}$-object whose underlying set has cardinality greater than α has a non-identity endomorphism, then $\mathbb{K}$ is not alg-universal.

Hence for example the variety of lattices or the variety of monoids or the category of topological spaces and continuous mappings are not alg-universal because of the existence of constant morphisms. On the other hand, both the variety of semigroups [13] and the variety of (0,1)-lattices ([11] or [12]) are alg-universal.

Thus we can say that monoids or lattices have sufficiently rich structure to be 'close' to being alg-universal while still permitting constant morphisms, although these categories are not alg-universal in the strict sense. This motivates a notion of almost alg-universality that ignores the constant morphisms. Next we define a more general concept expressing this idea.

Let $\mathbb{K}$ be a category. A class $\mathcal{C}$ of $\mathbb{K}$-morphisms is an ideal if $f \circ g \in \mathcal{C}$ for $\mathbb{K}$-morphisms $f : a \to b$, $g : b \to c$ whenever $f \in \mathcal{C}$ or $g \in \mathcal{C}$. A faithful functor $F : \mathbb{L} \to \mathbb{K}$ is called $\mathcal{C}$-relatively full embedding if

(•) $Ff \notin \mathcal{C}$ for any $\mathbb{L}$-morphism f;

(•) if $f : Fa \to Fb$ is a $\mathbb{K}$-morphism for $\mathbb{L}$-objects a and b then either $f \in \mathcal{C}$ or $f = Fg$ for some $\mathbb{K}$-morphism $g : a \to b$.

Thus F is a full embedding exactly when it is $\mathcal{C}$-relatively full embedding for $\mathcal{C} = \emptyset$. Observe that, if $F : \mathbb{L} \to \mathbb{K}$ is a $\mathcal{C}$-relatively full embedding for some ideal $\mathcal{C}$ then f is an $\mathbb{L}$-isomorphism if and only if Ff is a $\mathbb{K}$-isomorphism. If there exists a $\mathcal{C}$-relatively full embedding $F : \mathbb{GRA} \to \mathbb{K}$ then we say that $\mathbb{K}$ is $\mathcal{C}$-relatively alg-universal. If, moreover, $\mathbb{K}$ is concrete and F preserves finiteness, then $\mathbb{K}$ is called $\mathcal{C}$-relatively ff-alg-universal. Clearly, $\mathbb{K}$ is $\mathcal{C}$-relatively alg-universal (or $\mathcal{C}$-relatively ff-alg-universal) for $\mathcal{C} = \emptyset$ just when $\mathbb{K}$ is alg-universal (or ff-alg-universal, respectively). If $\mathbb{K}$ is conrete category and $\mathcal{C}$ is the ideal consisting of all $\mathbb{K}$-morphisms with constant underlying mapping then we say that $F : \mathbb{L} \to \mathbb{K}$ is almost full embedding instead of $\mathcal{C}$-relatively full embedding and that $\mathbb{K}$ is almost alg-universal or almost ff-alg-universal instead of $\mathcal{C}$-relatively alg-universal or $\mathcal{C}$-relatively ff-alg-universal. The variety of lattices [20] and the variety of monoids [17] or [15] are almost alg-universal but not alg-universal. A second consequence of Theorem 2.1 is that a category $\mathbb{K}$ which is not monoid-universal is not alg-universal. This fact was exploited by M. E. Adams and W. Dziobiak in [5], where they proved that the variety of monadic Boolean algebras is not alg-universal, yet contains a proper class of non-isomorphic algebras whose endomorphism monoids consist of the identity map alone.

Theorem 2.1 naturally leads to the following question.

Problem 2.2. Is there a variety $\mathbb{V}$ of algebras which is monoid universal but not alg-universal?

We shall consider ideals of a special type. Let O be a class of $\mathbb{K}$-objects. Then $\mathcal{I}(O)$ denotes a class of all $\mathbb{K}$-morphisms $f : a \to b$ such that there exist $\mathbb{K}$-morphisms $g : a \to c$ and $h : c \to b$ with $c \in O$ and $f = h \circ g$. Clearly, $\mathcal{I}(O)$ is an ideal of $\mathbb{K}$ called an object ideal of O. In what follows, we shall consider even more specific object ideals.

Q-universality. A class $\mathbb{Q}$ of algebraic systems of a finitary type Δ is a quasivariety if it is closed under all products, all ultraproducts, all subsystems and all isomorphic images. For any class $\mathbb{K}$ of algebraic systems of type Δ, there exists the least quasivariety $\mathbb{Q}$ containing $\mathbb{K}$, which we shall denote $\mathbb{Q} = \mathrm{Qua}(\mathbb{K})$. Quasivarieties will be viewed as categories whose morphisms are all homomorphisms, that is, mappings preserving all operations and relations.

M. V. Sapir [19] defined a quasivariety $\mathbb{Q}$ of finite type Δ as Q-universal if for every quasivariety $\mathbb{R}$ of finite type the lattice $\mathrm{QLat}(\mathbb{R})$ is a homomorphic image of a sublattice of $\mathrm{QLat}(\mathbb{Q})$.

Let $\mathcal{P}(\omega_0)$ be the set of all finite subsets of natural numbers and $\mathcal{P}(\omega) = \mathcal{P}(\omega_0)\setminus\{\emptyset\}$ the set of all finite non-empty subsets of natural numbers. W. Dziobiak [9, 10] studied families $\{\mathbf{S}_A \mid A \in \mathcal{P}(\omega_0)\}$ of finite algebraic systems of a given type Δ we now call Adams-Dziobiak families (or A-D families) defined by these four conditions:

(p1) $\mathbf{S}_\emptyset$ is the terminal algebraic system;

(p2) if $A = B \cup C$ for $A, B, C \in \mathcal{P}(\omega_0)$, then $\mathbf{S}_A \in \mathrm{Qua}(\{\mathbf{S}_B, \mathbf{S}_C\})$;

(p3) if $A \in \mathcal{P}(\omega)$ and $B \in \mathcal{P}(\omega_0)$ with $\mathbf{S}_A \in \mathrm{Qua}(\{\mathbf{S}_B\})$, then $A = B$;

(p4) if $\mathbf{U}, \mathbf{V} \in \mathrm{Qua}(\{\mathbf{S}_A \mid A \in \mathcal{P}\})$ are finite algebraic systems for some finite $\mathcal{P} \subset \mathcal{P}(\omega)$ and if there exists an injective homomorphism $f : \mathbf{S}_A \to \mathbf{U} \times \mathbf{V}$ for some $A \in \mathcal{P}(\omega)$, then there exists an injective homomorphism $g : \mathbf{S}_A \to \mathbf{U}$ or there exists an injective homomorphism $g : \mathbf{S}_A \to \mathbf{V}$ or there exist $B, C \in \mathcal{P}(\omega)$ and injective homomorphisms $g_B : \mathbf{S}_B \to \mathbf{U}$ and $g_C : \mathbf{S}_C \to \mathbf{V}$ with $A = B \cup C$.

We recall some known results.

Theorem 2.3. *(a) If $\mathbb{Q}$ is a Q-universal quasivariety then* $\mathrm{QLat}(\mathbb{Q})$ *has cardinality* $2^{\aleph_0}$ *and the free lattice over a countable set can be embedded into* $\mathrm{QLat}(\mathbb{Q})$*. Thus* $\mathrm{QLat}(\mathbb{Q})$ *satisfies no non-trivial lattice identity* [2].

(b) If a quasivariety $\mathbb{Q}$ contains an A-D family, then the lattice of all ideals of the free lattice over a countable set can be embedded into $\mathrm{QLat}(\mathbb{Q})$ [3]. □

Thus to prove that a quasivariety $\mathbb{Q}$ of finite type is Q-universal, it suffices to prove that $\mathbb{Q}$ has an A-D family. We shall study only quasivarieties $\mathbb{Q}$ of algebras.

In Section 3 we give certain conditions sufficient for the existence of an A-D family in a quasivariety of algebras of finite type. For this we use factorization systems and epireflection.

Factorization systems and epireflections. For a category $\mathbb{K}$, let $\mathcal{E}$ be a class of $\mathbb{K}$-epimorphisms and let $\mathcal{M}$ be a class of $\mathbb{K}$-monomorphisms. We say that $(\mathcal{E}, \mathcal{M})$ is a factorization system of $\mathbb{K}$ if $\mathcal{E}$ and $\mathcal{M}$ are closed under composition, $f \in \mathcal{E} \cap \mathcal{M}$ if and only if f is a $\mathbb{K}$-isomorphism, and for every $\mathbb{K}$-morphism $f : a \to b$ there exist unique, up to a commuting isomorphism, $g : a \to c \in \mathcal{E}$ and $h : c \to b \in \mathcal{M}$ with $f = h \circ g$, see [1]. Any factorization system has the diagonalization property. We formulate it for categories with products. If $\mathbb{K}$ is a category with products and an $(\mathcal{E}, \mathcal{M})$-factorization system, then we write $\{f_i : a \to b_i \mid i \in I\} \in \mathcal{M}$ if the morphism $f : a \to \prod_{i \in I} b_i$ such that $f_i = \pi_i \circ f$ for all $i \in I$ where $\pi_i : \prod_{j \in I} b_j \to b_i$ is the i-th projection belongs to $\mathcal{M}$. Then the diagonalization property says: if $g_i \circ f = k_i \circ h$ for all $i \in I$ where $f : a \to b \in \mathcal{E}$, $\{g_i : b \to c_i \mid i \in I\}$ is a family of $\mathbb{K}$-morphisms, $h : a \to d$ is a $\mathbb{K}$-morphism and $\{k_i : d \to c_i \mid i \in I\} \in \mathcal{M}$ then there exists a $\mathbb{K}$-morphism $l : b \to d$ such that $h = l \circ f$ and $g_i = k_i \circ l$ for all $i \in I$. If $h \in \mathcal{E}$ then $l \in \mathcal{E}$, and if $\{g_i \mid i \in I\} \in \mathcal{M}$ then $l \in \mathcal{M}$.

We say that a family $\{f_i : A \to A_i \mid i \in I\}$ is <u>separating</u> if for distinct $a, b \in A$ there exists $i \in I$ with $f_i(a) \neq f_i(b)$. If $\mathbb{K}$ is a concrete category then a family $\{f_i : a \to b_i \mid i \in I\}$ of $\mathbb{K}$-morphisms is <u>separating</u> if the family of underlying mapping is separating. For concrete categories $\mathbb{K}$ and $\mathbb{L}$ we say that a functor $F : \mathbb{K} \to \mathbb{L}$ <u>preserves separating families</u> if $\{Ff_i : Fa \to Fb_i \mid i \in I\}$ is a separating family in $\mathbb{L}$ whenever $\{f_i : a \to b_i \mid i \in I\}$ is a separating family in $\mathbb{K}$.

For a concrete category $\mathbb{K}$, let $\mathrm{Inj}_{\mathbb{K}}$ consist of all $\mathbb{K}$-homomorphisms such that the underlying mapping is injective and $\mathrm{Surj}_{\mathbb{K}}$ consist of all $\mathbb{K}$-morphisms such that the underlying mapping is surjective. Clearly, every morphism from $\mathrm{Inj}_{\mathbb{K}}$ is a monomorphism of $\mathbb{K}$ and every morphism from $\mathrm{Surj}_{\mathbb{K}}$ is an epimorphism of $\mathbb{K}$. If $(\mathrm{Surj}_{\mathbb{K}}, \mathrm{Inj}_{\mathbb{K}})$ is a factorization system of $\mathbb{K}$ then we say $\mathbb{K}$ has a <u>concrete factorization system</u> and $(\mathrm{Surj}_{\mathbb{K}}, \mathrm{Inj}_{\mathbb{K}})$ is a <u>concrete factorization system</u> of $\mathbb{K}$. Clearly, for every quasivariety $\mathbb{Q}$ of algebras $(\mathrm{Surj}_{\mathbb{Q}}, \mathrm{Inj}_{\mathbb{Q}})$ is a concrete factorization system of $\mathbb{Q}$ (because every bijective homomorphism is an isomorphism). Observe that a family $\{f_i : \mathbf{A} \to \mathbf{B}_i \mid i \in I\}$ of $\mathbb{Q}$-homomorphisms is separating if and only if it belongs to $\mathrm{Inj}_{\mathbb{Q}}$, i.e. if the homomorphism $f : \mathbf{A} \to \prod_{i \in I} \mathbf{B}_i$ with $f_i = f \circ \pi_i$ has an injective underlying mapping where $\pi_i : \prod_{j \in I} \mathbf{B}_j \to \mathbf{B}_i$ is the i-th projection for all $i \in I$. Thus for a concrete category $\mathbb{K}$ we shall say that a family $\{f_i : A \to B_i \mid i \in I\}$ of $\mathbb{K}$-morphisms belong to $\mathrm{Inj}_{\mathbb{K}}$ just when its corresponding family of underlying mappings is separating. A functor $F : \mathbb{Q} \to \mathbb{R}$ between quasivarieties $\mathbb{Q}$ and $\mathbb{R}$ <u>preserves surjectivity</u> if $F(\mathrm{Surj}_{\mathbb{Q}}) \subseteq \mathrm{Surj}_{\mathbb{R}}$.

If $\mathbb{Q}$ is a quasivariety of algebraic systems and $\mathbb{R}$ is a subquasivariety of $\mathbb{Q}$ (of the same type) then, by Theorem 10.1.2 from [14], $\mathbb{R}$ is an epireflective subcategory of $\mathbb{Q}$. This means that for every algebraic system $A \in \mathbb{Q}$ there exists a surjective homomorphism $\rho_A : A \to RA$ where $RA \in \mathbb{R}$ such that for every homomorphism $f : A \to C$ where $C \in \mathbb{R}$ there exists exactly one homomorphism $f^* : RA \to C$ with $f = f^* \circ \rho_A$. Since $\mathbb{R}$ is a full subcategory of $\mathbb{Q}$ then ρ_A is the identity morphism exactly when $A \in \mathbb{R}$. Then $R : \mathbb{Q} \to \mathbb{R}$ such that $Rf = (\rho_B \circ f)^*$ for every homomorphism $f : A \to B$ in $\mathbb{Q}$ is a functor which is a left adjoint to the inclusion functor from $\mathbb{R}$ to $\mathbb{Q}$. We say that R is an epireflection. Observe that $R(\mathrm{Surj}_{\mathbb{Q}}) \subseteq \mathrm{Surj}_{\mathbb{R}}$.

A quasivariety $\mathbb{Q}$ of algebras closed under homomorphic images is a variety. If $\mathbb{Q}$ is a quasivariety of algebras and $\mathbb{V}$ is a subvariety of $\mathbb{Q}$ then a homomorphism $f : \mathbf{A} \to \mathbf{B} \in \mathbb{Q}$ belongs to the ideal $\mathcal{I}(\mathbb{V})$ if and only if $\mathrm{Im}(f) \in \mathbb{V}$.

3 Sufficient conditions for Q-universality

Definition. Let $\mathbb{Q}$ be a quasivariety of finitary algebraic systems, let $\mathbb{V}$ be a proper subvariety of $\mathbb{Q}$ and let $R : \mathbb{Q} \to \mathbb{V}$ be the corresponding epireflection. For any object $\mathbf{A} \in \mathbb{Q}$, let A denote the underlying set of $\mathbf{A}$ and let $\rho_{\mathbf{A}} : \mathbf{A} \to R\mathbf{A}$ denote the surjective $\mathbb{Q}$-morphism from the epitransformation ρ. Let $F : \mathbb{K} \to \mathbb{Q}$ be a $\mathcal{I}(\mathbb{V})$-relatively full embedding. Let $\mathbf{S} \in \mathbb{V}$ be an algebraic system with the underlying set S. We say that F is $\mathbf{S}$-synchronized and call $\mathbf{S}$ its synchronizer if for every $\mathbb{K}$-object k there exists an injective mapping μ_k from S to the underlying set of RFk such that $\mathrm{Im}(\mu_k)$ is an induced subobject of RFk and μ_k is an isomorphism of $\mathbf{S}$ onto the subobject of RFk on the set $\mathrm{Im}(\mu_k)$, and for every $\mathbb{K}$-morphism $f : k_1 \to k_2$ we have

(s1) if Ff is injective on $(\rho_{Fk_1})^{-1}(\mathrm{Im}(\mu_{k_1}))$, then Ff is injective;

(s2) $RFf \circ \mu_{k_1} = \mu_{k_2}$;

(s3) if $Ff \in \mathrm{Surj}_{\mathbb{Q}}$ and A_i is the underlying set of RFk_i for $i = 1, 2$, then every mapping $h : A_2 \to A_1$ such that $RFf \circ h = 1_{A_2}$ is a homomorphism from RFk_2 to RFk_1;

(s4) for every $\mathbb{K}$-object k, if s is an element of the underlying set of RFk such that $s \notin \mathrm{Im}(\mu_k)$ then $\rho_{Fk}^{-1}\{s\}$ is a singleton.

Next we interpret the condition (s3) for algebras.

Proposition 3.1. *Let $\mathbb{Q}$ be a quasivariety of algebras of a finitary similarity type Δ, let $\mathbb{V}$ be a proper subvariety of $\mathbb{Q}$ and let $F : \mathbb{K} \to \mathbb{Q}$ be a functor. Then (s3) holds* *exactly when*

($\bullet$) *if Ff is surjective and for every $s \in A_2$ with $|RFf^{-1}\{s\}| > 1$, if $\sigma_{RFk_2}(a_1, a_2, \dots, a_n) = s$ for an n-ary operation σ and $a_1, a_2, \dots, a_n \in A_2$, then $s = a_{i_0}$ for some $i_0 \in \{1, 2, \dots, n\}$ and $k(s) = \sigma_{RFk_1}(k(a_1), k(a_2), \dots, k(a_n))$ for every mapping $k : \{a_1, a_2, \dots, a_n\} \to A_1$ such that $RFf \circ k(a_i) = a_i$ for all $i \in \{1, 2, \dots, n\}$.*

Proof. Assume (s3). Let $s = \sigma_{RFk_2}(a_1, a_2, \dots, a_n)$ for some $\sigma \in \Delta$, let $a_1, a_2, \dots, a_n, s \in A_2$ and $|RFf^{-1}\{s\}| > 1$. Let $h : A_2 \to A_1$ be a mapping such that $RFf \circ h$ is the identity mapping. Then $h(s) = \sigma_{RFk_1}(h(a_1), h(a_2), \dots, h(a_n))$. If $s \notin \{a_1, a_2, \dots, a_n\}$ then there exists a mapping $h' : A_2 \to A_1$ with $RFf \circ h = RFf \circ h'$, $h(s) \neq h'(s)$ and $h(t) = h'(t)$ for all $t \in A_2 \setminus \{s\}$. Hence $h'(s) \neq \sigma_{RFk_1}(h'(a_1), h'(a_2), \dots, h'(a_n))$ and this contradicts the fact that $h' : RFk_2 \to RFk_1$ is a homomorphism. Thus there exists $i_0 \in \{1, 2, \dots, n\}$ with $a_{i_0} = s$. If $k : \{a_1, a_2, \dots, a_n\} \to A_1$ is a mapping such that $RFf \circ k(a_i) = a_i$ for every $i \in \{1, 2, \dots, n\}$ then there exists a mapping $h : A_2 \to A_1$ such that $RFf \circ h$ is the identity mapping of A_2 and $h(a_i) = k(a_i)$ for all $i = \{1, 2, \dots, n\}$. But $h : RFk_2 \to RFk_1$ is a homomorphism, by (s3), and hence $k(s) = \sigma_{RFk_1}(k(a_1), k(a_2), \dots, k(a_n))$ because $s = a_{i_0}$. Whence the condition ($\bullet$) holds.

For the converse, assume ($\bullet$) and let $h : A_2 \to A_1$ be a mapping such that $RFf \circ h$ is the identity of A_2. Choose an n-ary operation σ of type Δ and $a_1, a_2, \dots, a_n \in A_2$. Write $s = \sigma_{RFk_2}(a_1, a_2, \dots, a_n)$. First we assume that $|RFf^{-1}(s)| > 1$. Then ($\bullet$) gives an $i_0 \in \{1, 2, \dots, n\}$ with $s = a_{i_0}$ and $h(s) = \sigma_{RFk_1}(h(a_1), h(a_2), \dots, h(a_n))$, as required. From $Ff \in \mathrm{Surj}_{\mathbb{Q}}$ we infer that $RFf \in \mathrm{Surj}_{\mathbb{Q}}$, and hence $|RFf^{-1}\{s\}| = 1$ is the only remaining case. If

$$t = \sigma_{RFk_1}(h(a_1), h(a_2), \dots, h(a_n))$$

then

$$\begin{aligned} RFf(t) &= \sigma_{RFk_2}(RFf(h(a_1)), RFf(h(a_2)), \dots, RFf(h(a_n))) \\ &= \sigma_{RFk_2}(a_1, a_2, \dots, a_n) = s \end{aligned}$$

and hence $t = h(s)$. Thus h is a homomorphism, and the proof is complete. $\square$

Remark. Observe that if $F : \mathbb{K} \to \mathbb{Q}$ is an almost full embedding then F is synchronized $\mathcal{I}(\mathbb{T})$-relatively full embedding for the trivial variety $\mathbb{T}$. Indeed, its synchronizer $\mathbf{S}$ is a singleton algebra and μ_k is the identity automorphism of $\mathbf{S}$ for every

$\mathbb{K}$-object k. Clearly, the conditions (s1)-(s4) are satisfied. And $F : \mathbb{K} \to \mathbb{Q}$ is a full embedding exactly when F is an almost full embedding and for every $\mathbb{K}$-object k there exists no $\mathbb{Q}$-morphism from the terminal object of $\mathbb{Q}$ into Fk.

Let $\mathbb{N}_0$ be a poset viewed as a category whose objects are sets from the set $\mathcal{P}(\omega_0)$ of all finite subsets of ω and there exists an $\mathbb{N}_0$-morphism from $A \in \mathcal{P}(\omega_0)$ into $B \in \mathcal{P}(\omega_0)$ if and only if $B \subseteq A$. Let $\mathbb{N}$ be the full subcategory of $\mathbb{N}_0$ whose objects belong to the set $\mathcal{P}(\omega) = \mathcal{P}(\omega_0) \setminus \{\emptyset\}$ of all non-void subsets of ω. For $A, B \in \mathcal{P}(\omega)$ with $B \subseteq A$, let $\eta_{A,B}$ denote the unique $\mathbb{N}$-morphism from A to B.

Theorem 3.2. *Let $\mathbb{Q}$ be a quasivariety of finitary algebras and let $\mathbb{V}$ be a subvariety of $\mathbb{Q}$. If there exists a synchronized $\mathcal{I}(\mathbb{V})$-relatively full embedding $F : \mathbb{N} \to \mathbb{Q}$ such that*

(1) *FA is a finite algebra for every $A \in \mathcal{P}(\omega)$;*

(2) *$F\eta_{A,B} \in \mathrm{Surj}_{\mathbb{Q}}$ for every $A, B \in \mathcal{P}(\omega)$ with $B \subseteq A$ (then $RF\eta_{A,B}$ is a retract);*

(3) *if $A = B \cup C$ for $A, B, C \in \mathcal{P}(\omega)$ then $\{F\eta_{A,B}, F\eta_{A,C}\}$ is a separating family.*

Then $\{\mathbf{S}_A \mid A \in \mathcal{P}(\omega_0)\}$ is an A-D family where $\mathbf{S}_\emptyset$ is a singleton algebra in $\mathbb{Q}$ and $\mathbf{S}_A = FA$ for all $A \in \mathcal{P}(\omega)$.

Proof. We need to prove (p1)–(p4). Clearly, (p1) is satisfied. To prove (p2), consider sets $A, B, C \in \mathcal{P}(\omega)$ with $A = B \cup C$. By (3), $\{F\eta_{A,B}, F\eta_{A,C}\}$ is a separating family and thus FA is a subobject of $FB \times FC$. Hence we obtain $FA \in \mathrm{Qua}\{FB, FC\}$ and the proof of (p2) is complete.

For every $A \in \mathcal{F}(\omega)$, let $\rho_A : FA \to RFA$ denote the epireflection homomorphism of FA into $\mathbb{V}$. Then $\rho_A \in \mathrm{Surj}_{\mathbb{Q}}$.

To prove (p3), let $A, B \in \mathcal{P}(\omega)$ be such that $FA \in \mathrm{Qua}\{FB\}$. By the hypothesis, FB is finite, so that the family of all homomorphisms from FA to FB is separating. Since F is $\mathcal{I}(\mathbb{V})$-relatively full embedding we infer that if $B \not\subseteq A$ then every homomorphism from FA into FB factorizes through ρ_A. Since $FA \notin \mathbb{V}$ and $RFA \in \mathbb{V}$, the mapping ρ_A is not injective and thus $FA \notin \mathrm{Qua}\{FB\}$ – a contradiction. Thus we can assume that $B \subseteq A$. If $f : FA \to FB$ is a homomorphism then either $h = F\eta_{A,B}$ or h factorizes through ρ_A because F is $\mathcal{I}(\mathbb{V})$-relatively full embedding. Since the family of all homomorphisms from FA to FB is separating, the pair $\{F\eta_{A,B}, \rho_A\}$ must be a separating family. We claim that this is impossible when $B \neq A$. Indeed, if $B \neq A$ then $F\eta_{A,B}$ is not injective; this is because from (2) it would follow that $F\eta_{A,B}$ is an isomorphism, contrary to the relative fulness of F. But then $F\eta_{A,B}$ is not injective on $(\rho_A)^{-1}(\mathrm{Im}(\mu_A))$ by (s1) and hence, by (s2), for some $s \in S$ there are distinct $a, b \in \rho_A^{-1}\{s\}$ with $F\eta_{A,B}(a) = F\eta_{A,B}(b)$. Hence $\{F\eta_{A,B}, \rho_A\}$ is not a separating family, a contradiction. Thus $A = B$, and (p3) follows.

To prove (p4), let $\mathcal{F} \subseteq \mathcal{P}(\omega)$ be a finite set and let $\mathbf{B}, \mathbf{C} \in \mathrm{Qua}\{FX \mid X \in \mathcal{F}\}$ be finite algebras such that there exist $A \in \mathcal{P}(\omega)$ and an injective homomorphism $f : FA \to \mathbf{B} \times \mathbf{C}$. Hence there exist finite separating families $\{g_i : \mathbf{B} \to FX_i \mid i \in I\}$ and $\{h_j : \mathbf{C} \to FY_j \mid j \in J\}$ such that $X_i, Y_j \in \mathcal{P}(\omega)$ for all $i \in I$ and $j \in J$. Let $\pi_1 : \mathbf{B} \times \mathbf{C} \to \mathbf{B}$, $\pi_2 : \mathbf{B} \times \mathbf{C} \to \mathbf{C}$ be projections.

First we prove that we can assume that $\pi_1 \circ f, \pi_2 \circ f \in \mathrm{Surj}_{\mathbb{Q}}$. So assume that (p4) is satisfied if $\pi_1 \circ f, \pi_2 \circ f \in \mathrm{Surj}_{\mathbb{Q}}$. By the factorization property, there exist homomorphisms

$$f_1' : FA \to \mathbf{B}' \in \mathrm{Surj}_{\mathbb{Q}}, f_1'' : \mathbf{B}' \to \mathbf{B} \in \mathrm{Inj}_{\mathbb{Q}},$$
$$f_2' : FA \to \mathbf{C}' \in \mathrm{Surj}_{\mathbb{Q}}, f_2'' : \mathbf{C}' \to \mathbf{C} \in \mathrm{Inj}_{\mathbb{Q}}$$

with $\pi_1 \circ f = f_1'' \circ f_1'$ and $\pi_2 \circ f = f_2'' \circ f_2'$. Since f is injective we infer that $\{\pi_1 \circ f, \pi_2 \circ f\}$ is separating and hence $\{f_1', f_2'\}$ is also separating. Thus there exists an injective homomorphism $f' : FA \to \mathbf{B}' \times \mathbf{C}'$ with $\pi_1' \circ f' = f_1'$ and $\pi_2' \circ f' = f_2'$ where $\pi_1' : \mathbf{B}' \times \mathbf{C}' \to \mathbf{B}'$ and $\pi_2' : \mathbf{B}' \times \mathbf{C}' \to \mathbf{C}'$ are projections. Then $\{g_i \circ f_1'' : \mathbf{B}' \to FX_i \mid i \in I\}$ and $\{h_j \circ f_2'' : \mathbf{C}' \to FY_j \mid j \in J\}$ are separating families and, by the assumption, the condition (p4) is satisfied for f', $\mathbf{B}'$ and $\mathbf{C}'$ because $\pi_1' \circ f', \pi_2' \circ f' \in \mathrm{Surj}_{\mathbb{Q}}$. Then (p4) is also satisfied for f, $\mathbf{B}$ and $\mathbf{C}$ because $f_1'' : \mathbf{B}' \to \mathbf{B}, f_2'' : \mathbf{C}' \to \mathbf{C} \in \mathrm{Inj}_{\mathbb{Q}}$. Thus with no loss of generality we can assume that $\pi_1 \circ f, \pi_2 \circ f \in \mathrm{Surj}_{\mathbb{Q}}$.

Let us define $I' = \{i \in I \mid g_i \circ \pi_1 \circ f = F\eta_{A,X_i}\}$ and $J' = \{j \in J \mid g_j \circ \pi_2 \circ f = F\eta_{A,Y_j}\}$. Then $X_i \subseteq A$ and $Y_j \subseteq A$ for all $i \in I$ and $j \in J$. Observe that $g_i \circ \pi_1 \circ f$ and $g_j \circ \pi_2 \circ f$ factorize through ρ_A for all $i \in I \setminus I'$ and $j \in J \setminus J'$ because F is $\mathcal{I}(\mathbb{V})$-relatively full embedding. Hence $I' \neq \emptyset$ or $J' \neq \emptyset$. Set $U = \bigcup_{i \in I'} X_i$ and $V = \bigcup_{j \in J'} Y_j$. Then $U \cup V \subseteq A$, and $g_i \circ \pi_1 \circ f$ factorizes through $F(\eta_{A,U})$ for all $i \in I'$ and $g_j \circ \pi_2 \circ f$ factorizes through $F(\eta_{A,V})$ for all $j \in J'$. Since $\{g_i \circ \pi_1 \circ f \mid i \in I\} \cup \{g_j \circ \pi_2 \circ f \mid j \in J\} \in \mathrm{Inj}_{\mathbb{Q}}$ we infer, by (p3), that if $J' = \emptyset$ then $U = A$, if $I' = \emptyset$ then $V = A$, if $I' \neq \emptyset \neq J'$ then $A = U \cup V$. Assume that $I' \neq \emptyset$. Since $\pi_1 \circ f \in \mathrm{Surj}_{\mathbb{Q}}$, $\{F\eta_{U.X_i} \mid i \in I'\} \in \mathrm{Inj}_{\mathbb{Q}}$ by (3) and $g_i \circ \pi_1 \circ f = F\eta_{U,X_i} \circ F\eta_{A,U}$ for all $i \in I$, by the diagonalization property there exists a homomorphism $\psi : \mathbf{B} \to FU$ with $\psi \circ \pi_1 \circ f = F\eta_{A,U}$ and $F\eta_{U,X_i} \circ \psi = g_i$

for all $i \in I'$. From $F\eta_{A,U} \in \mathrm{Surj}_{\mathbb{Q}}$ it follows that $\psi \in \mathrm{Surj}_{\mathbb{Q}}$.

Since $\{g_i \mid i \in I\}$ is a separating family, for distinct $u, v \in FA$ we have $\pi_1 \circ f(u) \neq \pi_1 \circ f(v)$ if and only if there exists $i \in I$ with $g_i \circ \pi_1 \circ f(u) \neq g_i \circ \pi_1 \circ f(v)$. If $i \in I'$ then $g_i \circ \pi_1 \circ f = F\eta_{A,X_i} = F\eta_{U,X_i} \circ F\eta_{A,U}$. Thus if $F\eta_{A,U}(u) \neq F\eta_{A,U}(v)$ for $u, v \in FA$ then $\pi_1 \circ f(u) \neq \pi_1 \circ f(v)$. If $i \in I \setminus I'$ then $g_i \circ \pi_1 \circ f = h \circ \rho_A$ for some homomorphism h and thus $\pi_1 \circ f(u) \neq \pi_1 \circ f(v)$ implies that $\rho_A(u) \neq \rho_A(v)$ or $F\eta_{A,U}(u) \neq F\eta_{A,U}(v)$ because $\{F\eta_{U,X_i} \mid i \in I'\}$ is a separating family.

Let $\mathbf{S}$ be a synchronizer of F. Consider $t \in \rho_A^{-1}(\mathrm{Im}(\mu_A))$ and $u \in FA \setminus \rho_A^{-1}(\mathrm{Im}(\mu_A))$. Then $\rho_A(t) = \mu_A(s)$ for some $s \in S$. By (s2), $\rho_U \circ \psi \circ \pi_1 \circ f(t) = \rho_U \circ F\eta_{A,U}(t) = \mu_U(s)$ and $\rho_U \circ \psi \circ \pi_1 \circ f(u) = \rho_U \circ F\eta_{A,U}(u) \notin \mathrm{Im}(\mu_U)$. Hence $\psi^{-1}(\rho_U^{-1}(\mathrm{Im}(\mu_U)) = \pi_1 \circ f(\rho_A^{-1}(\mathrm{Im}(\mu_A))$. If we combine this fact with the foregoing argument we conclude that for $u, v \in \rho_A^{-1}(\mathrm{Im}(\mu_A))$ we have $\pi_1 \circ f(u) = \pi_1 \circ f(v)$ if and only if $F\eta_{A,U}(u) = F\eta_{A,U}(v)$. From (s2) it follows that $(RF\eta_{A,U})^{-1}(\mu_U(s)) = \{\mu_A(s)\}$ for all $s \in S$. Thus $(R\psi)^{-1}(\mu_U(s)) = \{R(\pi_1 \circ f(\mu_A(s))\}$ for every $s \in S$ because $\psi \circ \pi_1 \circ f = F\eta_{A,U}$. Since $F\eta_{A,U}$ is surjective, by (s3), every mapping ν' from the underlying set of RFU into the underlying set of RFA such that $RF\eta_{A,U} \circ \nu'$ is the identity mapping is a homomorphism from RFU into RFA. From $\psi \circ \pi_1 \circ f = F\eta_{A,U}$ we conclude $R(\psi \circ \pi_1 \circ f) = RF\eta_{A,U}$. For a homomorphism $\nu' : RFU \to RFA$ such that $RF\eta_{A,U} \circ \nu'$ is the identity automorphism of RFU we set $\nu = R(\pi_1 \circ f) \circ \nu'$ and hence $\nu : RFU \to R\mathbf{B}$ is a homomorphism such that $R\psi \circ \nu$ is the identity homomorphism of RFU. Since ν' exists by (s3), we can assume that we have a homomorphism $\nu : RFU \to R\mathbf{B}$ such that $R\psi \circ \nu$ is the identity homomorphism of RFU.

For every $i \in I \setminus I'$ there exists a homomorphism $\bar{g}_i : RFA \to FX_i$ with $g_i \circ \pi_1 \circ f = \bar{g}_i \circ \rho_A$. By the properties of factorization system, there exist homomorphisms $\sigma : RFA \to \mathbf{D} \in \mathrm{Surj}_{\mathbb{Q}}$ and $\sigma_i : \mathbf{D} \to FX_i$ for $i \in I \setminus I'$ such that $g_i \circ \pi_1 \circ f = \sigma_i \circ \sigma \circ \rho_A$ for all $i \in I \setminus I'$ and $\{\sigma_i \mid i \in I \setminus I'\} \in \mathrm{Inj}_{\mathbb{Q}}$. By the diagonalization property, there exists a homomorphism $\phi' : \mathbf{B} \to \mathbf{D}$ such that $\phi' \circ \pi_1 \circ f = \sigma \circ \rho_A$ and $\sigma_i \circ \phi' = g_i$ for all $i \in I \setminus I'$. From $\rho_A, \sigma \in \mathrm{Surj}_{\mathbb{Q}}$ it follows that $\phi' \in \mathrm{Surj}_{\mathbb{Q}}$. From $RFA \in \mathbb{V}$ and $\sigma : RFA \to \mathbf{D} \in \mathrm{Surj}_{\mathbb{Q}}$ it follows that $\mathbf{D} \in \mathbb{V}$ and if $\rho_{\mathbf{B}} : \mathbf{B} \to R\mathbf{B}$ is the epireflection morphism of $\mathbf{B}$ into $\mathbb{V}$, then there exists a homomorphism $\phi : R\mathbf{B} \to \mathbf{D} \in \mathrm{Surj}_{\mathbb{Q}}$ with $\phi' = \phi \circ \rho_{\mathbf{B}}$. Then

$$\sigma \circ \rho_A = \phi' \circ \pi_1 \circ f = \phi \circ \rho_B \circ \pi_1 \circ f = \phi \circ R(\pi_1 \circ f) \circ \rho_A$$

and $\sigma = \phi \circ R(\pi_1 \circ f)$ follows because $\rho_A \in \mathrm{Surj}_{\mathbb{Q}}$. Since $\{g_i \mid i \in I\} \in \mathrm{Inj}_{\mathbb{Q}}$ we infer that the family $\{\psi, \rho_B\}$ is separating. Hence there exists a homomorphism

$\omega : \mathbf{B} \to FU \times R\mathbf{B} \in \mathrm{Inj}_{\mathbb{Q}}$ such that $\tau_1 \circ \omega = \psi$ and $\tau_2 \circ \omega = \rho_B$ where $\tau_1 : FU \times R\mathbf{B} \to FU$ and $\tau_2 : FU \times R\mathbf{B} \to R\mathbf{B}$ are projections. Then $\tau_1 \circ \omega \circ \pi_1 \circ f = \psi \circ \pi_1 \circ f = F\eta_{A,U}$ and $\tau_2 \circ \omega \circ \pi_1 \circ f = \rho_B \circ \pi_1 \circ f = R(\pi_1 \circ f) \circ \rho_A$. Hence for every $b \in \mathbf{B}$ and $a \in FA$ with $\pi_1 \circ f(a) = b$ we have $\omega(b) = (F\eta_{A,U}(a), R(\pi_1 \circ f) \circ \rho_A(a))$. By the property of products, there exists a homomorphism $\lambda : FU \to FU \times R\mathbf{B}$ such that $\tau_1 \circ \lambda$ is the identity morphism of FU and $\tau_2 \circ \lambda = \nu \circ \rho_U$, hence $\lambda \in \mathrm{Inj}_{\mathbb{Q}}$. Select $u \in FU$. If $\rho_U(u) \in \mathrm{Im}(\mu_U)$ then, by (s2), $\rho_A((F\eta_{A,U})^{-1}(u)) = (RF\eta_{A,U})^{-1}(\rho_U(u))$ is a singleton and hence for every $a \in FA$ with $F\eta_{A,U}(a) = u$ we have $\{R(\pi_1 \circ f) \circ \rho_A(a)\} = (R\psi)^{-1}(\rho_U(u)) = \{\nu(\rho_U(u))\}$. Thus $\lambda(u) = (F\eta_{A,U}(a), R(\pi_1 \circ f) \circ \rho_A(a)) \in \mathrm{Im}(\omega)$. If $\rho_U(u) \notin \mathrm{Im}(\mu_U)$, then there exists $a \in FA$ such that $\rho_{\mathbf{B}} \circ \pi_1 \circ f(a) = \nu(\rho_U(u))$ because $\rho_{\mathbf{B}}, \pi_1 \circ f \in \mathrm{Surj}_{\mathbb{Q}}$. Then

$$R\psi \circ \rho_{\mathbf{B}} \circ \pi_1 \circ f(a) = R\psi \circ \nu(\rho_U(u) = \rho_U(u).$$

Since

$$R\psi \circ \rho_{\mathbf{B}} \circ \pi_1 \circ f = \rho_U \circ \psi \circ \pi_1 \circ f = \rho_U \circ F\eta_{U,A}$$

we conclude that $\rho_U(u) = \rho_U(F\eta_{A,U}(a))$ and, by (s4), $u = F\eta_{A,U}(a)$. Thus $\lambda(u) = (F\eta_{A,U}(a), R(\pi_1 \circ f) \circ \rho_A(a)) \in \mathrm{Im}(\omega)$ because $R(\pi_1 \circ f) \circ \rho_A = \rho_{\mathbf{B}} \circ \pi_1 \circ f$. Thus $\mathrm{Im}(\lambda) \subseteq \mathrm{Im}(\omega)$, so that there exists an injective homomorphism from FU to $\mathbf{B}$.

If $J' \neq \emptyset$ then the same proof gives the existence of an injective $\nu : FV \to \mathbf{C}$, and (p4) follows. □

The technical statement below enables us to prove a generalized version of Theorem 3.2. We say that a surjective homomorphism $f : \mathbf{A} \to \mathbf{B}$ of algebraic systems of similarity type Δ is a quotient if for every relation $r \in \Delta$ we have that $(b_0, b_1, \ldots, b_k) \in r_{\mathbf{B}}$ if and only if there exists $(a_0, a_1, \ldots, a_k) \in r_{\mathbf{A}}$ with $f(a_i) = b_i$ for all $i = 0, 1, \ldots, k$. A quasivariety $\mathbb{Q}$ is closed under quotients if algebraic system $\mathbf{A} \in \mathbb{Q}$ whenever there exist an algebraic system $\mathbf{B} \in \mathbb{Q}$ and a quotient $f : \mathbf{B} \to \mathbf{A}$. Let $\mathrm{Quot}_{\mathbb{Q}}$ denote the class of all quotients of $\mathbb{Q}$. It is well-known [1] that $(\mathrm{Quot}_{\mathbb{Q}}, \mathrm{Inj}_{\mathbb{Q}})$ is a factorization system in $\mathbb{Q}$, and that $\mathrm{Surj}_{\mathbb{Q}} = \mathrm{Quot}_{\mathbb{Q}}$ if $\mathbb{Q}$ is a quasivariety of algebras. If $\mathbb{Q}$ is clear from the context, we write Quot instead of $\mathrm{Quot}_{\mathbb{Q}}$.

Proposition 3.3. *Let $\mathbb{Q}$ be a quasivariety of algebraic systems and let $\mathbb{R}$ be a proper subquasivariety of $\mathbb{Q}$ closed under quotients. If there exists an $\mathcal{I}(\mathbb{R})$-relatively full embedding $F : \mathbb{N} \to \mathbb{Q}$ such that FA is finite for all $A \in \mathcal{P}(\omega)$ and $F\eta_{A,B} \in \mathrm{Quot}_{\mathbb{Q}}$ for all $A, B \in \mathcal{P}(\omega)$ with $B \subseteq A$, then there exists an $\mathcal{I}(\mathbb{R})$-relatively full embedding $G : \mathbb{N} \to \mathbb{Q}$ such that*

(1) *GA is finite for all $A \in \mathcal{P}(\omega)$;*

(2) *if $A, B, C \in \mathcal{P}(\omega)$ satisfy $B \cup C \subseteq A$, then $\{G\eta_{A,B}, G\eta_{A,C}\}$ is a separating family if and only if $A = B \cup C$;*

(3) *$G\eta_{A,B} \in \mathrm{Quot}_{\mathbb{Q}}$ for all $A, B \in \mathcal{P}(\omega)$ with $B \subseteq A$.*

Moreover, if $\mathbb{Q}$ is a quasivariety of algebras and F is synchronized then G is synchronized.

The fairly technical proof of this Proposition can be found in the Appendix.

Proof of Theorem 1.4 completed. Let $\mathbb{GRA}$ denote the (concrete) category of all undirected graphs and compatible mappings. We recall that there exists a full embedding Φ of $\mathbb{N}$ into $\mathbb{GRA}$ such that ΦA is a finite graph of every $A \in \mathbb{N}$ and $\Phi\eta_{A,B} \in \mathrm{Quot}_{\mathbb{GRA}}$ for every $A, B \in \mathcal{P}(\omega)$ with $B \subseteq A$, see [7]. Let $F : \mathbb{GRA} \to \mathbb{Q}$ satisfy the hypothesis of Theorem 1.4. Then the composite $F \circ \Phi : \mathbb{N} \to \mathbb{Q}$ satisfies the hypothesis of Proposition 3.3, and hence $\mathbb{Q}$ contains an A-D family, by Theorem 3.2. This concludes the proof of Theorem 1.4. □

Remark. The embeddings from $\mathbb{GRA}$ into the variety of semigroups generated by $\mathbf{M}_2$ or $\mathbf{M}_3$ or $\mathbf{M}_3^d$ or $\mathbf{M}_4$ or $\mathbf{M}_4^d$ constructed in [6, 7, 8] are synchronized (here for a semigroup $\mathbf{S} = (S, \cdot)$, its dual is defined as $\mathbf{S}^d = (S, \odot)$ with $s \odot t = t \cdot s$ for all $s, t \in S$) and constitute special cases of Theorem 3.2. The semigroups $\mathbf{M}_2$, $\mathbf{M}_3$ and $\mathbf{M}_4$ are defined in Table 1.

$\mathbf{M}_2$	a	b	c	0
a	0	c	0	0
b	c	0	0	0
c	0	0	0	0
0	0	0	0	0

$\mathbf{M}_3$	d	a	b	c
d	a	a	a	b
a	a	a	a	a
b	b	b	b	b
c	c	c	c	c

$\mathbf{M}_4$	t	u	v	s	0
t	t	u	s	s	0
u	t	u	0	s	0

Table 1: The semigroups $\mathbf{M}_2$, $\mathbf{M}_3$ and $\mathbf{M}_4$

Finally, we show that for quasivarieties of algebras Theorem 1.4 generalizes Theorem 1.3 of [16]. So let $\mathbb{Q}$ be a quasivariety of algebras and let $\mathbb{V}$ be a proper subvariety of $\mathbb{Q}$. We say that an epireflection $R : \mathbb{Q} \to \mathbb{V}$ is constant on a functor $F : \mathbb{N} \to \mathbb{Q}$ if the composite $R \circ F$ is a constant functor. It is then clear that if the epireflection R is constant on an $\mathcal{I}(\mathbb{V})$-relatively full embedding F, then F is synchronized. Thus we immediately obtain

Corollary 3.4. *Let $\mathbb{Q}$ be a quasivariety of algebras and let $\mathbb{V}$ be a proper subvariety of $\mathbb{Q}$. If $F : \mathbb{N} \to \mathbb{Q}$ is an $\mathcal{I}(\mathbb{V})$-relatively full embedding such that the epireflection of $\mathbb{Q}$ into $\mathbb{V}$ is constant on F, $F\eta_{A,B} \in \mathrm{Surj}_{\mathbb{Q}}$ for all $A, B \in \mathcal{P}(\omega)$ with $B \subseteq A$ and FA is finite for all $A \in \mathcal{P}(\omega)$ then there exists an A-D family in $\mathbb{Q}$, and thus $\mathbb{Q}$ is Q-universal.* $\square$

Thus, in particular, the object ideal $\mathcal{I}(\mathbb{V})$ associated with such an $\mathcal{I}(\mathbb{V})$-relatively full embedding F is *principal* in the sense that it is determined by a single object of $\mathbb{V}$ and includes the case when the synchronizer is a singleton algebra, that is, the case of an almost full embedding.

Appendix

Proof of Proposition 3.3. Consider a functor $H : \mathbb{N}_0 \to \mathbb{N}$ defined by $H\emptyset = \{0\}$ and $HA = \{0\} \cup \{n+1 \mid n \in A\}$ for all $A \in \mathcal{P}(\omega)$ and $H\eta_{A,B} = \eta_{HA,HB}$ for $A, B \in \mathcal{P}(\omega_0)$ with $B \subseteq A$. Then H is a full embedding (since $A \subseteq B$ if and only if $HA \subseteq HB$ for $A, B \in \mathcal{P}(\omega_0)$, it is correctly defined). Thus the composite $F' = F \circ H : \mathbb{N}_0 \to \mathbb{Q}$ is an $\mathcal{I}(\mathbb{R})$-relatively full embedding such that $F'A$ is finite for all $A \in \mathcal{P}(\omega)$ and $F'\eta_{A,B} = F\eta_{HA,HB} \in \mathrm{Quot}_{\mathbb{Q}}$ for all $A, B \in \mathcal{P}(\omega_0)$ with $B \subseteq A$.

Since F' is an $\mathcal{I}(\mathbb{R})$-relatively full embedding, $F'A \notin \mathbb{R}$ for all $A \in \mathcal{P}(\omega_0)$. For $n \in \omega$, set $G\{n\} = F'\{n\}$. For $A \in \mathcal{P}(\omega)$, define $\Pi(A) = \prod_{a \in A} F'\{a\}$ and let $\pi_a : \Pi(A) \to F'\{a\}$ be the a-th projection for each $a \in A$. By the universal property of products, there exists a unique homomorphism $\tau_A : F'A \to \Pi(A)$ such that $F'\eta_{A,\{a\}} = \pi_a \circ \tau_A$ for every $a \in A$. Factorizing τ_A in $\mathbb{Q}$ in the factorization system $(\mathrm{Quot}_{\mathbb{Q}}, \mathrm{Inj}_{\mathbb{Q}})$, we obtain homomorphisms (unique up to an isomorphism) $\chi_A : F'A \to GA \in \mathrm{Quot}_{\mathbb{Q}}$ and $\mu_A : GA \to \Pi(A) \in \mathrm{Inj}_{\mathbb{Q}}$ such that $\tau_A = \mu_A \circ \chi_A$. Since the underlying set of $F'A$ is finite and since χ_A is a quotient, the underlying set of GA is finite for all $A \in \mathcal{P}(\omega)$. This proves (1).

Consider $A, B \in \mathcal{P}(\omega)$ with $B \subseteq A$. By the universal property of products, there exists a unique homomorphism $\Pi(\eta_{A,B}) : \Pi(A) \to \Pi(B)$ such that $\pi_b = \kappa_b \circ \Pi(\eta_{A,B})$ for all $b \in B \subseteq A$, where $\kappa_b : \Pi(B) \to F'\{b\}$ is the b-th projection for $b \in B$. Then for every $b \in B$ we have

$$\begin{aligned}\kappa_b \circ \Pi(\eta_{A,B}) \circ \tau_A = \pi_b \circ \tau_A = F'\eta_{A,\{b\}} &= F'\eta_{B,\{b\}} \circ F'\eta_{A,B} \\ &= \kappa_b \circ \tau_B \circ F'\eta_{A,B}\end{aligned}$$

because $\kappa_b \circ \tau_B = F'\eta_{B,\{b\}}$, and hence

$$\Pi(\eta_{A,B}) \circ \mu_A \circ \chi_A = \Pi(\eta_{A,B}) \circ \tau_A = \tau_B \circ F'\eta_{A,B} = \mu_B \circ \chi_B \circ F'\eta_{A,B}$$

because the family $\{\kappa_b \mid b \in B\}$ of projections is separating.

By the diagonalization property, there exists a homomorphism $G\eta_{A,B} : GA \to GB$ with $G\eta_{A,B} \circ \chi_A = \chi_B \circ F'\eta_{A,B}$ and $\Pi(\eta_{A,B}) \circ \mu_A = \mu_B \circ G\eta_{A,B}$ because $\mu_B \in \mathrm{Inj}$ and $\chi_A \in \mathrm{Quot}$. From $\chi_B \circ F'\eta_{A,B} \in \mathrm{Quot}$ it follows that $\chi_B \circ F'\eta_{A,B} \in \mathrm{Quot}$ and $G\eta_{A,B} \in \mathrm{Quot}$, and (3) is proved. Note the diagram below, commuting for every $b \in B \subseteq A$.

$$\begin{array}{ccccccc}
F'A & \xrightarrow{\chi_A} & GA & \xrightarrow{\mu_A} & \prod(A) & \xrightarrow{\pi_b} & G\{b\} = F'\{b\} \\
{\scriptstyle F'\eta_{A,B}}\downarrow & & {\scriptstyle G\eta_{A,B}}\downarrow & & {\scriptstyle \prod(\eta_{A,B})}\downarrow & & \| \\
F'B & \xrightarrow{\chi_B} & GB & \xrightarrow{\mu_B} & \prod(B) & \xrightarrow{\kappa_b} & G\{b\} = F'\{b\}
\end{array}$$

To prove that G is a functor, let $A, B, C \in \mathcal{P}(\omega)$ satisfy $C \subseteq B \subseteq A$. Then

$$\begin{aligned}
G\eta_{B,C} \circ G\eta_{A,B} \circ \chi_A &= G\eta_{B,C} \circ \chi_B \circ F'\eta_{A,B} \\
&= \chi_C \circ F'\eta_{B,C} \circ F'\eta_{A,B} \\
&= \chi_C \circ F'\eta_{A,C} = G\eta_{A,C} \circ \chi_A
\end{aligned}$$

and because $\chi_A \in \mathrm{Quot}$ we conclude that $G\eta_{B,C} \circ G\eta_{A,B} = G\eta_{A,C}$. Since $F'\eta_{A,A}$ is the identity homomorphism, from $G\eta_{A,A} \circ \chi_A = \chi_A \circ F'\eta_{A,A} = \chi_A \in \mathrm{Quot}$ it follows that $G\eta_{A,A}$ is also the identity homomorphism. Altogether, G is a functor.

We turn to (2). Note that $F'\eta_{A,\{a\}} = \pi_a \circ \tau_A = \pi_a \circ \mu_A \circ \chi_A$ and $G\eta_{A,\{a\}} \circ \chi_A = F'\eta_{A,\{a\}}$ for every $a \in A$ because $\chi_{\{a\}}$ is the identity morphism of $F'\{a\} = G\{a\}$. From $\chi_A \in \mathrm{Quot}$ we then obtain $G\eta_{A,\{a\}} = \pi_a \circ \mu_A$ for every $a \in A$. But then $\{G\eta_{A,\{a\}} \mid a \in A\}$ is a separating family because $\mu_A \in \mathrm{Inj}$ and the family $\{\pi_a \mid a \in A\}$ of projections is separating. Hence $\{G\eta_{A,B}, G\eta_{A,C}\}$ is a separating family for any $A, B, C \in \mathcal{P}(\omega)$ with $A = B \cup C$. Conversely, assume that $B \cup C \subsetneq A$ and $\{G\eta_{A,B}, G\eta_{A,C}\}$ is a separating family. Then $\{G\eta_{A,\{a\}} \mid a \in B \cup C\}$ is clearly a separating family. Set $A' = B \cup C$. Then $G\eta_{A,A'} \in \mathrm{Inj}$ and thus from the already proved (3) it follows that $G\eta_{A,A'}$ is an isomorphism. Choose $a \in A \setminus A'$. Since $G\eta_{A,\{a\}} \circ \chi_A = F'\eta_{A,\{a\}} \in \mathrm{Quot}$, we have $G\eta_{A,\{a\}} = \pi_a \circ \mu_A \in \mathrm{Quot}$. But then $\pi_a \circ \mu_A \circ (G\eta_{A,A'})^{-1} \circ \chi_{A'} : F'A' \to F'\{a\}$ is a quotient because $(G\eta_{A,A'})^{-1}, \chi_{A'} \in \mathrm{Quot}$. This is a contradiction because F' is an $\mathcal{I}(\mathbb{R})$-relatively full embedding, $F'\{a\}$ does not belong to $\mathbb{R}$ and $\{a\} \not\subseteq A'$. Hence (2) follows.

To prove that G is an $\mathcal{I}(\mathbb{R})$-relatively full embedding consider $A, B \in \mathcal{P}(\omega)$ with $B \subseteq A$. Then $\eta_{A,B}$ is a morphism of $\mathbb{N}$ and we must prove that $\mathrm{Im}(G\eta_{A,B}) \notin$

$\mathbb{R}$. For every $b \in B$, $G\eta_{B,\{b\}} \in$ Quot and $G\{b\} \notin \mathbb{R}$. Since $\mathbb{R}$ is closed under Quot we infer that $GB \notin \mathbb{R}$ and because $G\eta_{A,B} \in$ Quot we conclude that $\mathrm{Im}(G\eta_{A,B}) \notin \mathbb{R}$. Conversely, let $f : GA \to GB$ for $A, B \in \mathcal{P}(\omega)$ be a homomorphism such that $\mathrm{Im}(f) \notin \mathbb{R}$. To complete the proof it suffices to prove that $B \subseteq A$ and $f = G\eta_{A,B}$. Let $f' : A \to C \in$ Quot and $f'' : C \to B \in$ Inj be homomorphisms with $f = f'' \circ f'$ then C is isomorphic to $\mathrm{Im}(f)$. Since $\{G\eta_{B,\{b\}} \mid b \in B\}$ is a separating family, we infer that $\{G\eta_{B,\{b\}} \circ f'' \mid b \in B\}$ is a separating family and, by the universal property of products, the morphism $h : C \to \prod_{b\in B} \mathrm{Im}(G\eta_{B,\{b\}} \circ f'') \in$ Inj. Since $\mathrm{Im}(f) \notin \mathbb{R}$ we conclude that $\prod_{b\in B} \mathrm{Im}(G\eta_{B,\{b\}} \circ f'') \notin \mathbb{R}$, and thus there exists $b \in B$ such that $\mathrm{Im}(G\eta_{B,\{b\}} \circ f'') = \mathrm{Im}(G\eta_{B,\{b\}} \circ f) \notin \mathbb{R}$. Thus $G\eta_{B,\{b\}} \circ f \notin \mathcal{I}(\mathbb{R})$. Since $\chi_A \in$ Quot we conclude that $G\eta_{B,\{b\}} \circ f \circ \chi_A : F'A \to F'\{b\} \notin \mathcal{I}(\mathbb{R})$ and thus $b \in A$ and $G\eta_{B,\{b\}} \circ f \circ \chi_A = F'\eta_{A,\{b\}}$ because F' is an $\mathcal{I}(\mathbb{R})$-relatively full embedding. Then

$$F'\eta_{\{b\},\emptyset} \circ G\eta_{B,\{b\}} \circ f \circ \chi_A = F'\eta_{\{b\},\emptyset} \circ F'\eta_{A,\{b\}} = F'\eta_{A,\emptyset}.$$

Since for every $b' \in B$ we have

$$\begin{aligned} F'\eta_{\{b'\},\emptyset} \circ G\eta_{B,\{b'\}} \circ \chi_B &= F'\eta_{\{b'\},\emptyset} \circ F'\eta_{B,\{b'\}} \\ &= F'\eta_{B,\emptyset} = F'\eta_{\{b\},\emptyset} \circ F'_{B,\{b\}} \\ &= F'\eta_{\{b\},\emptyset} \circ G\eta_{B,\{b\}} \circ \chi_B \end{aligned}$$

we infer that $F'\eta_{\{b'\},\emptyset} \circ G\eta_{B,\{b'\}} = F'\eta_{\{b\},\emptyset} \circ G\eta_{B,\{b\}}$ for all $b' \in B$ because $\chi_B \in$ Quot. From this it follows that

$$F'\eta_{A,\emptyset} = F'\eta_{\{b\},\emptyset} \circ G\eta_{B,\{b\}} \circ f \circ \chi_A = F'\eta_{\{b'\},\emptyset} \circ G\eta_{B,\{b'\}} \circ f \circ \chi_A$$

for all $b' \in B$. Since $F'\eta_{A,\emptyset} \notin \mathcal{I}(\mathbb{R})$ we conclude that $G\eta_{B,\{b'\}} \circ f \circ \chi_A \notin \mathcal{I}(\mathbb{R})$ for all $b' \in B$ because $F'\eta_{\{b'\},\emptyset} \in$ Quot and $\mathbb{R}$ is closed under Quot. Hence $b' \in A$ and $G\eta_{B,\{b'\}} \circ f \circ \chi_A = F'\eta_{A,\{b'\}}$ for all $b' \in B$ because F' is an $\mathcal{I}(\mathbb{R})$-relatively full embedding. Thus $B \subseteq A$ and

$$\begin{aligned} G\eta_{B,\{b'\}} \circ G\eta_{A,B} \circ \chi_A &= G\eta_{B,\{b'\}} \circ \chi_B \circ F'\eta_{A,B} \\ &= F'\eta_{B,\{b'\}} \circ F'\eta_{A,B} \\ &= F'\eta_{A,\{b'\}} = G\eta_{B,\{b'\}} \circ f \circ \chi_A \end{aligned}$$

for all $b' \in B$. By (2), $\{G\eta_{B,\{b'\}} \mid b' \in B\} \in$ Inj and thus $G\eta_{A,B} \circ \chi_A = f \circ \chi_A$. But $\chi_A \in$ Quot, and this completes the proof that $f = G\eta_{A,B}$.

It remains to prove that if $\mathbb{Q}$ is a quasivariety of algebras and F is synchronized then also G is synchronized. First observe that F' is also synchronized. For $A \in$

$\mathcal{P}(\omega)$ let $\rho_{F'A}$ and ρ_{GA} be the respective epireflection morphisms of $F'A$ and GA. Let S be an algebra and for $A \in \mathcal{P}(\omega)$ let $\nu_A : \mathbf{S} \to RF'A$ witness the fact that F' is synchronized. Since for every $a \in A$ we have $F'\eta_{A,\{a\}} = G\eta_{A,\{a\}} \circ \chi_A$ we conclude that $RF'\eta_{A,\{a\}} = R(G\eta_{A,\{a\}} \circ \chi_A)$. Set $\zeta_A = R\chi_A \circ \nu_A : \mathbf{S} \to RGA$, then the property that for every $s \in S$ and $A, B \in \mathcal{P}(\omega)$ with $B \subseteq A$ we have $RF'\eta_{A,B}(\nu_A(s)) = \nu_B(s)$ implies $RG\eta_{A,B}(\zeta_A(s)) = \zeta_B(s)$ and the fact that ν_A is injective for every $A \in \mathcal{P}(\omega)$ and $\chi_{\{a\}}$ is the identity mapping for every $a \in \omega$ imply that ζ_A is injective for all $A \in \mathcal{P}(\omega)$. The validity of (s1) and (s2) for F' implies that G also satisfies (s1) and (s2). From the facts that F' satisfies (s4) and $\zeta_{\{a\}} = \nu_{\{a\}}$ for all $a \in \omega$ and $\{G\eta_{A,\{a\}} \mid a \in A\}$ is a separating family for all $A \in \mathcal{P}(\omega)$ it follows that (s4) holds for G. Indeed, if u and v are distinct elements of RGA with $\rho_{GA}(u), \rho_{GA}(v) \notin \operatorname{Im}(\zeta_A)$ then there exists $a \in A$ with $F'\eta_{A,\{a\}}(u) \neq F'\eta_{A,\{a\}}(v)$ and hence $\rho_{G\{a\}} \circ F'\eta_{A,\{a\}}(u) \neq \rho_{G\{a\}} \circ F'\eta_{A,\{a\}}(v)$. Then $\rho_{G\{a\}} \circ F'\eta_{A,\{a\}} = RF'\eta_{A,\{a\}} \circ \rho_{GA}$ implies that $\rho_{GA}(u) \neq \rho_{GA}(v)$. If u and v are elements of RGA with $\rho_{GA}(u) \notin \operatorname{Im}(\zeta_A)$ and $v \in \operatorname{Im}(\zeta_A)$ then, by the same argument, we obtain that $\rho_{GA}(u) \neq \rho_{GA}(v)$ and hence GA satisfies (s4). To prove (s3) consider $A, B \in \mathcal{P}(\omega)$ with $B \subseteq A$. Choose $b \in B$. Since F' satisfies (s3), the condition (•) from Proposition 3.1 is satisfied for $F'\eta_{A,B}$ and $F'\eta_{B,\{b\}}$. Since χ is a surjective natural transformation from F' onto G and since $F'\{b\} = G\{b\}$ we conclude, by Proposition 3.1, that every mapping h from the underlying set of RGB into RGA such that $RG\eta_{A,B} \circ h$ is the identity mapping is a homomorphism from RGA into RGB. Thus G satisfies (s3) and whence G is synchronized. □

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