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## DISTRIBUTIONS AND HEAT EQUATION IN SDG

*by Anders KOCK and Gonzalo REYES*

**RESUME.** Cet article expose une théorie synthétique des distributions (qui ne sont pas nécessairement à support compact). On compare cette théorie avec la théorie classique de Schwartz. Cette comparaison s'effectue par un plongement plein de la catégorie des espaces vectoriels convenables (et leurs applications lisses) dans certains gros topos, modèles de la géométrie synthétique différentielle.

### Introduction

The simplest notion allowing a theory of function spaces to be formulated is that of cartesian closed categories.

In a cartesian closed category, containing in a suitable sense the ring  $R$  of real numbers, a notion of “distribution of compact support” on any object  $M$  can be defined, because the object of  $R$ -linear functionals on the ring  $R^M$  can be formulated, cf. e.g. [23], [20]. Thus, a “synthetic” theory of “distributions-of-compact support”, and models for it, do exist (we exploited this fact in [14]).

The content of the present note is to provide a similar theory, as well as models, for distributions which are not necessarily of compact support. This amounts to describing synthetically the notion of “test function” of compact support. The  $R$ -linear dual of the vector space of test functions then is then a synthetic version of the space of distributions.

When we say “model”, we mean more precisely a cartesian closed category, containing as full subcategories both the category of smooth manifolds, and also some suitable category of topological vector spaces, in such a way

that the synthetic constructs alluded to agree with the classical functional analytic ones.

The category of “suitable” topological vector spaces will be taken to be the category of Convenient Vector Spaces, in the sense of [5], [15]. With the smooth (not necessarily linear) maps, this category  $\underline{Con}^\infty$  is already cartesian closed, cf. loc.cit. We exhibited in 1986-1987 ([11], [13]) a full embedding of this category into a certain topos (the “Cahiers Topos” of Dubuc [3]). It is this embedding that we here shall prove is a model for a synthetic theory of distributions. The point about the Cahiers Topos is that it is also a well-adapted model for Synthetic Differential Geometry (meaning in essence that  $\mathbb{R}$  acquires sufficiently many nilpotent elements).

The functional-analytic spadework that we provide also gives, – with much less effort than what is needed for the Cahiers Topos –, a simpler model, namely Grothendieck’s “Smooth Topos”. However, a main purpose of distribution theory is to account for partial differential equations, and therefore a synthetic theory of *differentiation* should preferably be available in the model, as well, which it is in the Cahiers Topos, but not in the Smooth Topos (at least such theory has not yet been developed, and is anyway bound to be less simple).

As a pilot project for our theory, we shall finish by showing that the Cahiers topos does admit a fundamental (distributional) solution of the heat equation on the unlimited line. (Here clearly distributions of compact support will not suffice.)

Solutions of the heat equation model evolution through time of a *heat distribution*. A heat distribution is an *extensive* quantity and does not necessarily have a density *function*, which is an *intensive* quantity (cf. [17], [24]). The most important of all distributions, the point- or Dirac- distributions, do not. For the heat equation, it is well known that the evolution through time of any distribution leads ‘instantaneously’ (i.e., after any *positive* lapse of time  $t > 0$ ) to distributions that do have smooth density functions. Indeed, the evolution through time of the Dirac distribution  $\delta(0)$  is given by the map (“heat kernel”, “fundamental solution”)

$$K : \mathbb{R}_{\geq 0} \rightarrow \mathcal{D}'(\mathbb{R}) \quad (1)$$

defined by cases by the classical formula

$$K(t) = \begin{cases} e^{-x^2/4t} / \sqrt{4\pi t} & \text{if } t > 0 \\ \delta(0) & \text{if } t = 0 \end{cases} ; \quad (2)$$

here  $\mathcal{D}'(\mathbb{R})$  denotes a suitable space of distributions (in the sense of [26], [27]); notice that in the first clause we are identifying distributions with their density functions (when such density functions exist).

The fundamental mathematical object given in (2) presents a challenge to the synthetic kind of reasoning in differential geometry, where a basic tenet is “everything is smooth”; therefore, definition by cases, as in (2), has a dubious status. This challenge was one of the motivations for the present study.

One may see another lack of smoothness in (2), namely “ $\delta(0)$  is not smooth”; but this “lack of smoothness” is completely spurious, when one firmly stays in the space of distributions and their intrinsic “diffeology”, in particular avoids viewing distributions as generalized functions. We describe in Section 2 the distribution theory that is adequate for the purpose. In fact, as will be seen in Section 5 and 6, this theory is forced on us by synthetic considerations in the Smooth Topos, respectively in the Cahiers topos.

We want to thank Henrik Stetkær for useful conversations on the topic of distributions.

## 1 Diffeological spaces and convenient vector spaces

A *diffeological space* is a set  $X$  equipped with a collection of *smooth plots*, a plot  $p$  being a map from (the underlying set of) an open set  $U$  of some  $\mathbb{R}^n$  into  $X$ ,  $p : U \rightarrow X$ ; the collection should satisfy certain stability properties: a smooth plot precomposed with an ordinary smooth map  $U' \rightarrow U$  is again a smooth plot; and the property of being a smooth plot is a *local* property (local on the domain). These properties are conceptualized by considering the following site  $\underline{mf}$ : its objects are open subsets of  $\mathbb{R}^n$ , the maps are smooth maps between such sets; a covering is a jointly surjective family of local

diffeomorphisms. (This site is a site of definition of the “Smooth Topos” of Grothendieck et al., [1] p. 318; and is one of the first examples of what they call a “Gros Topos”.) Any set  $X$  gives rise to a presheaf  $c(X)$  on this site, namely  $c(X)(U) := \text{Hom}_{\text{sets}}(U, X)$ . A diffeological structure on the set  $X$  is a subsheaf  $P$  of the presheaf  $c(X)$ , the elements of  $P(U)$  are called the *smooth  $U$ -plots* on  $X$ . A set theoretic map  $f : X \rightarrow X'$  between diffeological spaces is called *(plot-) smooth* if  $f \circ p$  is a smooth plot on  $X'$  whenever  $p$  is a smooth plot on  $X$ .

Any smooth manifold  $M$  carries a canonical diffeology, namely with  $P(U)$  being the set of smooth maps  $U \rightarrow M$ . We have full inclusions of categories: smooth manifolds into diffeological spaces into the smooth topos, (= the topos of sheaves on the site  $\underline{mf}$ ),

$$\underline{Mf} \subseteq \underline{Diff} \subseteq \underline{sh}(\underline{mf}).$$

The category of diffeological spaces  $\underline{Diff}$  is cartesian closed (in fact, it is a concrete quasi-topos). Thus, if  $X$  and  $Y$  are diffeological spaces,  $Y^X$  has for its underlying set the set of smooth maps  $X \rightarrow Y$ ; and a map  $U \rightarrow Y^X$  is declared to be a smooth plot if its transpose  $U \times X \rightarrow Y$  is smooth. The inclusion of  $\underline{Diff}$  into the smooth topos preserves the cartesian closed structure.

For any smooth manifold  $M$ , we have in particular a diffeology on  $C^\infty(M) = \mathbb{R}^M$ , namely a map  $g : U \rightarrow C^\infty(M)$  is declared to be a smooth plot iff its transpose  $U \times M \rightarrow \mathbb{R}$  is smooth.

Topological vector spaces  $X$  carry a canonical diffeology: a plot  $f : U \rightarrow X$  is declared to be smooth if for every continuous linear functional  $\phi : X \rightarrow \mathbb{R}$ ,  $\phi \circ f : U \rightarrow \mathbb{R}$  is smooth in the standard sense of multivariable calculus. Note that the diffeology on a topological vector space  $X$  only depends on the dual space  $X'$ . If we call the continuous linear functionals  $\phi : X \rightarrow \mathbb{R}$  the *scalars* on  $X$ , we may express the definition of smoothness of a plot as *scalarwise* smooth. – Continuous linear functionals  $X \rightarrow \mathbb{R}$  are, almost tautologically, (plot-)smooth; on the other hand, (plot-)smooth linear functionals  $X \rightarrow \mathbb{R}$  need not be continuous.

A *convenient vector space* (cf. [5]) is a (Hausdorff) locally convex topological vector space  $X$  such that plot smooth linear functionals are continuous, and which have a completeness property. The completeness prop-

erty may be stated in several ways, cf. [5], [15]; for the purposes here, the most natural formulation is: for any smooth curve  $f : U \rightarrow X$  (where  $U$  is an open interval), there exists a smooth curve  $f' : U \rightarrow X$  which is *derivative* of  $f$  in the scalarwise sense that for any continuous linear  $\phi : X \rightarrow \mathbb{R}$ ,  $\phi \circ f' = (\phi \circ f)'$ .

More generally, if  $U \subseteq \mathbb{R}^n$  is open, and  $f : U \rightarrow X$  is a smooth plot, then partial derivatives  $f^\alpha$  of  $f$  exist, in the scalarwise sense; and they are smooth. Here  $\alpha$  is a multi-index; and to say that  $f^\alpha$  is an iterated partial derivative of  $f$ , in the *scalarwise* sense, is to say: for each  $\phi \in X'$ ,  $\phi \circ f$  has an  $\alpha$ 'th iterated derivative, and  $(\phi \circ f)^\alpha = \phi \circ f^\alpha$ .

The category of convenient vector spaces which we deal with here is  $\underline{Con}^\infty$  (cf. [5]), whose objects are the convenient vector spaces and whose morphisms are *all* smooth maps in between them, not just the smooth *linear* ones. The category  $\underline{Con}^\infty$  is a full subcategory of the category  $\underline{Diff}$  of diffeological spaces, and is cartesian closed; the inclusion functor preserves the cartesian closed structure.

(In [15], the notion of Convenient Vector Space is taken in a slightly wider sense: it is not required that (plot-) smooth linear functionals are continuous. The resulting category of “convenient” vector spaces and smooth maps in [15] is therefore larger, but *equivalent* to the one of [5]. Every convenient vector space in the “wide” sense is smoothly (but not necessarily topologically) isomorphic to one in the “narrow” sense of [5].)

Let  $i : X \rightarrow Y$  be a (plot-) smooth linear map between convenient vector spaces. Then  $i$  preserves differentiation of smooth plots  $U \rightarrow X$ , in an obvious sense. For instance, if  $f : U \rightarrow X$  is a smooth curve, i.e.  $U \subseteq \mathbb{R}$  an open interval, then for any  $t_0 \in U$ ,

$$(i \circ f)'(t_0) = i(f'(t_0)).$$

For, it suffices to test this with the elements  $\psi \in Y'$ . If  $\psi \in Y'$ , then  $\psi \circ i \in X'$  since  $i$  is smooth and linear, and the result then follows by definition of being a scalarwise derivative in  $X$ .

## 2 The basic vector spaces of distribution theory; test plots

Let  $M$  be a smooth (paracompact) manifold  $M$ . Distribution theory starts out with the vector space  $C^\infty(M)$  of smooth real valued functions on  $M$ , and the linear subspace  $\mathcal{D}(M) \subseteq C^\infty(M)$  consisting of functions with compact support ( $\mathcal{D}(M)$  is the “space of *test functions*”). The topology relevant for distribution theory is described (in terms of convergence of sequences) in [27], p. 79 and 108, respectively. Note that topology on  $\mathcal{D}(M)$  is finer than the one induced from the (Frechet-) topology on  $C^\infty(M)$ . The sheaf semantics which we shall consider in Section 5 and 6 will justify the choice of this topology.

We shall describe the diffeological structure, arising from the topology on  $\mathcal{D}(M)$ .

**Lemma 2.1** *Let  $U \subseteq \mathbb{R}^n$  be open, and let  $f : U \rightarrow \mathcal{D}(M)$  be smooth. Then it is continuous.*

**Proof.** This is not completely evident. “Smooth” means “scalarwise smooth”, and this of course implies scalarwise continuity; now, scalarwise continuity means continuity w.r.to the *weak* topology, but the continuity assertion in the Lemma concerns the fine (inductive limit) topology. We don’t know at present whether these two topologies agree. However, since  $\mathcal{D}(M)$  is a Montel space ([7] p. 197), these topologies agree on *bounded* subsets, ([7] p. 196), which suffices here since  $U$  is locally compact.

We cover  $M$  by an increasing sequence  $K_b$  of compact subsets, each contained in the interior of the next, and with  $M = \bigcup K_b$ ; the notions that we now describe are independent of the choice of these  $K_b$ . For  $M = \mathbb{R}^n$ , we would typically take  $K_b = \{x \in \mathbb{R}^n \mid |x| \leq b\}$ ,  $b \in \mathbb{N}$ .

Consider a map  $f : U \times M \rightarrow \mathbb{R}$ , where  $U$  is an open subset of some  $\mathbb{R}^n$ . We say that it is of *uniformly bounded support* if there exists  $b$  so that

$$f(u, x) = 0 \text{ for all } u \in U \text{ and all } x \text{ with } x \notin K_b$$

We say that  $f$  is *locally* of uniformly bounded support (“l.u.b.s.”) if  $U$  can be covered by open subsets  $U_i$  such that for each  $i$ , the restriction of  $f$  to  $U_i \times M$

is of uniformly bounded support. (We may use the phrase “ $f$  is l.u.b.s., locally in the variable  $u \in U$ ”) – Alternatively, we say that  $f : U \times M \rightarrow \mathbb{R}$  is of uniformly bounded support at  $u \in U$  if there is an open neighbourhood  $U'$  around  $u$  such that the restriction of  $f$  to  $U' \times M$  is of uniformly bounded support; and  $f$  is l.u.b.s. if it is of uniformly bounded support at  $u$ , for each  $u$ . (For yet another description of the notion, see Lemma 5.2 below.)

Let  $U \subseteq \mathbb{R}^n$  be open. For  $f : U \rightarrow \mathcal{D}(M)$ , we denote by  $\check{f} : U \times M \rightarrow \mathbb{R}$  its transpose,  $\check{f}(u, m) := f(u)(m)$ . Similarly, for (suitable)  $f : U \times M \rightarrow \mathbb{R}$ , we denote its transpose  $U \rightarrow \mathcal{D}(M)$  by  $\hat{f}$ .

**Lemma 2.2** *Let  $f : U \times M \rightarrow \mathbb{R}$  be smooth, and pointwise of bounded support (so that  $\hat{f}$  factors through  $\mathcal{D}(M)$ ). Then t.f.a.e.:*

- 1)  *$f$  is locally of uniformly bounded support*
- 2)  *$\hat{f} : U \rightarrow \mathcal{D}(M)$  is continuous.*

**Proof.** We first prove that 1) implies 2). Since the question is local in  $U$ , we may assume that  $f$  is of uniformly bounded support, i.e. there exists a compact  $K \subseteq M$  so that  $f(t, x) = 0$  for  $x \notin K$  and all  $t$ . The same  $K$  applies then to all the iterated partial derivatives  $f_\alpha$  of  $f$  in the  $M$ -directions ( $\alpha$  denoting some multi-index). So  $f$  and all the  $f_\alpha$  factor through  $\mathcal{D}_K$ , the subset of  $C^\infty(M)$  of functions vanishing outside  $K$ . Now to say that  $\hat{f} : U \rightarrow \mathcal{D}_K$  is continuous is equivalent, by definition of the topology on  $\mathcal{D}_K$ , to saying that for each  $\alpha$ ,  $(f_\alpha)^\wedge$  is continuous as a map into  $\mathbb{R}^K$ , the space of continuous maps  $K \rightarrow \mathbb{R}$ , with the topology of uniform convergence. This topology is the categorical exponent (= compact open topology) (cf. [6] Ch. 7 Thm. 11), which implies that  $(f_\alpha)^\wedge : U \rightarrow \mathbb{R}^K$  is continuous iff  $f_\alpha : U \times K \rightarrow \mathbb{R}$  is continuous, iff  $f_\alpha : U \times M \rightarrow \mathbb{R}$  is continuous. But  $f_\alpha$  is indeed continuous, by the smoothness assumption on  $f$ . So  $\hat{f} : U \rightarrow \mathcal{D}(M)$  is continuous.

To prove that 2) implies 1), we show that if not 1), then not 2). Let  $f : U \times M \rightarrow \mathbb{R}$  be a function which is smooth and of pointwise bounded support, but not l.u.b.s. Then there is a  $t_0 \in U$  and a sequence  $t_k \rightarrow t_0$ , as well as a sequence  $x_k \in M \setminus K_k$  with  $c_k = f(t_k, x_k) \neq 0$ . Let  $N$  be a number so that the support of  $f(t_0, -)$  is contained in  $K_N$ . We consider the (non-linear) functional  $T : \mathcal{D}(M) \rightarrow \mathbb{R}$  given by

$$g \mapsto \sum_{n=N}^{\infty} c_n^{-2} g(x_n)^2.$$

Note that for  $g$  of compact support, this sum is finite, since the  $x_n$ 's "tend to infinity". Also, the functional  $\mathcal{D}(M) \rightarrow \mathbb{R}$  is continuous. In fact, the topology on  $\mathcal{D}(M)$  is the inductive limit of the topology  $\mathcal{D}(K_k)$ , and the restriction of  $T$  to this subspace equals a *finite* algebraic combination of the Dirac distributions. Now it is easy to see that  $T$  takes  $f(t_0, -)$  to 0, by the choice of  $N$ , whereas  $T$  applied to  $f(t_k, -)$  for  $k > N$  yields a sum of non-negative terms, one of which has value 1, namely the one with index  $k$ , which is  $c_k^{-2} f(t_k, x_k)^2 = 1$ . So  $T \circ \hat{f}$  is not continuous, hence  $\hat{f}$  is not continuous.

This proves the Lemma.

We can now characterize the diffeology on  $\mathcal{D}(M)$  arising from the fine topology on  $\mathcal{D}(M)$ :

**Theorem 2.3** *A map  $f : U \rightarrow \mathcal{D}(M)$  is a smooth if and only if  $\check{f} : U \times M \rightarrow \mathbb{R}$  is smooth and locally of uniformly bounded support.*

**Proof.** For  $\Rightarrow$ , assume that  $f : U \rightarrow \mathcal{D}(M)$  is (scalarwise) smooth. Then so is the composite  $U \rightarrow \mathcal{D}(M) \subseteq C^\infty(M)$ . By the Theorem of Lawvere-Schanuel-Zane (combined with Boman's Theorem, in case  $U$  is not 1-dimensional), we conclude that  $\check{f} : U \times M \rightarrow \mathbb{R}$  is smooth. It is also, pointwise in  $U$ , of bounded support. From Lemma 2.1, we infer that  $f$  is continuous. From the implication  $2 \Rightarrow 1$  in Lemma 2.2 it follows that  $f$  is locally of uniformly bounded support.

Conversely, assume 1), i.e. assume  $f$  is smooth and l.u.b.s. Then we also have that  $\partial^\alpha f / \partial t^\alpha$  is smooth (iterated partial derivative in the  $U$ -directions,  $\alpha$  a multi-index) and l.u.b.s., and so its transpose is a continuous maps  $U \rightarrow \mathcal{D}(M)$ , by Lemma 2.2 ( $1 \Rightarrow 2$ ) and it serves as scalarwise iterated partial derivative (cf. the argument in [15] p. 20-21).

The vector space of distributions  $\mathcal{D}'(M)$  is, in diffeological terms, the linear subspace of the diffeological space  $\mathbb{R}^{\mathcal{D}(M)}$  consisting of the smooth *linear* maps  $\mathcal{D}(M) \rightarrow \mathbb{R}$ . They are the same as the continuous linear maps, since  $\mathcal{D}(M)$  is a convenient vector space. (So the vector space of distributions  $\mathcal{D}'(M)$ , as an abstract vector space, is the same in the diffeological and the topological context.) A map  $U \rightarrow \mathcal{D}'(M)$  is smooth iff it is smooth as a map into  $\mathbb{R}^{\mathcal{D}(M)}$ . This defines a diffeology on  $\mathcal{D}'(M)$ . With this diffeology,  $\mathcal{D}'(M)$ , too, is convenient (this is a general fact, cf. [5] Proposition 5.3.5).

### 3 Functions as distributions

Any sufficiently nice function  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  gives rise to a distribution  $i(f) \in \mathcal{D}'(\mathbb{R}^n)$  in the standard way “by integration over  $\mathbb{R}^n$ ”

$$\langle i(f), \phi \rangle := \int_{\mathbb{R}^n} f(s) \cdot \phi(s) ds.$$

This also applies if  $\mathbb{R}^n$  is replaced by another smooth manifold  $M$  equipped with a suitable measure. For simplicity of notation, we write  $M$  for  $\mathbb{R}^n$  in the following. All smooth functions  $f : M \rightarrow \mathbb{R}$  are “sufficiently nice”; so we get a map (obviously linear)

$$i : C^\infty(M) \rightarrow \mathcal{D}'(M). \quad (3)$$

It is also easy to see that this map is injective.

**Theorem 3.1** *The map  $i$  is smooth.*

**Proof.** Let  $g : V \rightarrow C^\infty(M)$  be smooth, ( $V$  an open subset of some  $\mathbb{R}^n$ ), we have to see that  $i \circ g : V \rightarrow \mathcal{D}'(M)$  is smooth, which in turn means that its transpose

$$(i \circ g)^\vee : V \times \mathcal{D}(M) \rightarrow \mathbb{R}$$

is smooth. So consider a smooth plot  $U \rightarrow V \times \mathcal{D}(M)$ , given by a pair of smooth maps  $h : U \rightarrow V$  and  $\hat{\Phi} : U \rightarrow \mathcal{D}(M)$ . Here  $U$  is again an open subset of some  $\mathbb{R}^k$ . Let us write  $\hat{F}$  for  $g \circ h : U \rightarrow C^\infty(M)$ . It is transpose of a map  $F : U \times M \rightarrow \mathbb{R}$ . Also, let us write  $\Phi$  for the transpose of  $\hat{\Phi}$ ; thus  $\Phi$  is a map

$$\Phi : U \times M \rightarrow \mathbb{R}$$

which is locally (in  $U$ ) of uniformly bounded support, by Theorem 2.3. We have to see that  $(i \circ g)^\vee \circ \langle h, \Phi \rangle$  is smooth (in the usual sense). By unravelling the transpositions, one can easily check that

$$((i \circ g)^\vee \circ \langle h, \Phi \rangle)(t) = \langle i(F(t, -)), \Phi(t, -) \rangle$$

The conclusion of the Theorem is thus the assertion that the composite map  $U \rightarrow \mathbb{R}$  given by

$$t \mapsto \int_M F(t, s) \cdot \Phi(t, s) ds \quad (4)$$

is smooth (in the standard sense of finite dimensional calculus). To prove smoothness at  $t_0 \in U$ , we may find a neighbourhood  $U'$  of  $t_0$  and a  $b$  such that

$$\Phi(t, s) = 0 \text{ if } t \in U' \text{ and } s \notin K_b,$$

because  $\Phi$  is l.u.b.s. We thus have, for any  $t \in U'$ , that the expression in (4) is  $\int_{K_b} F(t, s) \cdot \Phi(t, s) ds$ , but since  $K_b$  is compact, differentiation and other limits in the variable  $t$  may be taken inside the integration sign.

Since  $i : C^\infty(M) \rightarrow \mathcal{D}'(M)$  is smooth and linear, it preserves differentiation. In particular, if  $f : U \rightarrow C^\infty(M)$  is a smooth curve, and  $t_0 \in U$ , we have that  $(i \circ f)'(t_0) = i(f'(t_0))$ . However,  $f'$  is explicitly calculated in terms of the partial derivative of the transpose  $\hat{f} : U \times M \rightarrow \mathbb{R}$ , namely as the function  $s \mapsto \partial f(t, s) / \partial t |_{(t_0, s)}$ . This is the reason that ordinary (evolution-) differential equations for curves  $f : U \rightarrow \mathcal{D}'(M)$  manifest themselves as *partial* differential equations, as soon as the values of  $f$  are distributions represented by smooth functions.

## 4 Smoothness of the heat kernel

We consider the heat equation on the line,

$$\partial f / \partial t = \partial^2 f / \partial x^2.$$

Recall that the classical distribution solution of this equation, having  $\delta(0)$  as initial distribution, is the map

$$K : \mathbb{R}_{\geq 0} \rightarrow \mathcal{D}'(\mathbb{R})$$

whose value at  $t \geq 0$  takes a test function  $\phi$  to

$$\langle K(t), \phi \rangle = \begin{cases} \int_{-\infty}^{\infty} e^{-s^2/4t} / \sqrt{4\pi t} \phi(s) ds & \text{if } t > 0 \\ \phi(0) & \text{if } t = 0 \end{cases} \quad (5)$$

We need the smoothness of  $K$  in the diffeological sense. The diffeology on  $\mathbb{R}_{\geq 0}$  is induced by the inclusion of it into  $\mathbb{R}$ .

The following is a special case of [15] Theorem 24.5 and Proposition 24.10 (which in turn is a generalization of Seeley's Theorem, [28]).

**Theorem 4.1** *Let  $X$  be a convenient vector space, and let  $K : \mathbb{R}_{\geq 0} \rightarrow X$  be a map. Then  $K$  is smooth in the diffeological sense iff its restriction to  $\mathbb{R}_{>0}$  is smooth, and for all  $n$ ,  $\lim_{t \rightarrow 0+} K^{(n)}(t)$  exists (w.r. to the weak topology on  $X$ ). In this case,  $K$  extends to a smooth map on all of  $\mathbb{R}$ , (whose  $n$ 'th derviative at 0 then equals  $\lim_{t \rightarrow 0+} K^{(n)}(t)$ ).*

We shall apply this Theorem to the heat kernel  $K$  described in (5), so  $X$  is  $\mathcal{D}'(\mathbb{R})$ . For  $t > 0$ , the smooth two-variable function  $K(t, x)$  satisfies the heat equation as a partial differential equation  $\partial/\partial t K(t, x) = \partial^2/\partial x^2 K(t, x)$ . Since, by Section 3, the inclusion  $i : C^\infty(\mathbb{R}) \rightarrow \mathcal{D}'(\mathbb{R})$  preserves differentiation (“in the  $t$ -direction”), we get, by iteration, that for any test function  $\phi$ , and any  $t > 0$ ,

$$d^n/dt^n \langle K(t), \phi \rangle = \langle K(t), \phi^{(2n)} \rangle. \quad (6)$$

Also, it is well known that for any smooth  $\psi$ ,

$$\lim_{t \rightarrow 0+} \langle K(t), \psi \rangle = \psi(0) (= \langle \delta, \psi \rangle), \quad (7)$$

where  $\delta$  is the Dirac distribution at 0.

To prove that the conditions for smoothness in the above Theorem are satisfied, we shall prove that

$$\lim_{t \rightarrow 0+} K^{(n)}(t) = \delta^{(2n)}.$$

Now the topology on  $\mathcal{D}'(\mathbb{R})$  is the weak one and  $\mathcal{D}(\mathbb{R})$  is reflexive (in the diffeological sense – this follows from the well known topological reflexivity, together with Corollary 5.4.7 in [5]). So it suffices to prove that for each  $\phi \in \mathcal{D}(\mathbb{R})$ ,

$$\lim_{t \rightarrow 0+} d^n/dt^n \langle K(t), \phi \rangle = \phi^{(2n)}(0).$$

But this is immediate from (6) and (7).

## 5 Distributions in the Smooth Topos

Recall from section 1 that the Smooth Topos is the topos  $\mathcal{S} = \underline{sh}(\underline{mf})$  of sheaves on the site  $\underline{mf}$  of open subsets of coordinate vector spaces  $\mathbb{R}^n$ . It contains the category of diffeological spaces (and hence also  $\underline{Con}^\infty$ ) as a

full subcategory, and the inclusion preserves exponentials. We let  $h$  be the embedding  $\underline{Diff} \subseteq \mathcal{S}$ , but write  $R$  instead of  $h(\mathbb{R})$ .

We want to give a synthetic status to  $h(\mathcal{D}(M))$  and to  $h(\mathcal{D}(M)')$ . Here  $M$  is any paracompact smooth manifold, and for the synthetic description, one needs to cover  $M$  by an increasing sequence of compacts  $K_b$ , as in Section 2. The predicate of “belonging to  $K_b \subseteq M$ ” will have to be part of the language. In order not to load the exposition too heavily, we shall consider the case of  $M = \mathbb{R}$  only, with  $K_b$  the closed interval from  $-b$  to  $b$  ( $b \in \mathbb{N}$ ).

Because  $h$  preserves exponentials, and  $R = h(\mathbb{R})$ ,  $R^R$  is  $h(C^\infty(\mathbb{R}))$ . (For,  $C^\infty(\mathbb{R})$  with its standard Frechet topology is the exponential in  $Conv^\infty$ , by [15], Theorem 3.2.)

The following is a formula with a free variable  $f$  that ranges over  $R^R$ :

$$\exists b > 0 [\forall x (x < -b \vee x > b) \Rightarrow f(x) = 0]. \quad (8)$$

Let us write  $|x| > b$  as shorthand for the formula  $x < -b \vee x > b$  (so, in spite of the notation, we don’t assume an “absolute value” function). Then the formula (8) gets the more readable appearance:

$$\exists b > 0 [\forall x (|x| > b \Rightarrow f(x) = 0)]. \quad (9)$$

(verbally: “ $f$  is a function  $R \rightarrow R$  of bounded support” (namely support contained in the interval  $[-b, b]$ ). Its extension is a subobject  $\mathcal{D}(R) \subseteq R^R$ ).

**Theorem 5.1 (Test functions in the Smooth Topos)** *The inclusion  $\mathcal{D}(\mathbb{R}) \subseteq C^\infty(\mathbb{R})$  goes by  $h : \underline{Con}^\infty \rightarrow \mathcal{S}$  to the inclusion  $\mathcal{D}(R) \subseteq R^R$ .*

**Proof.** We shall freely use sheaf semantics, cf. e.g. [9], [20], and thus consider “generalized elements” or “elements defined at different stages”, the stages being the objects of the site  $\underline{mf}$ .

Consider an element  $f \in_U R^R$  (a generalized element at stage  $U$ ). This means a map  $h(U) \rightarrow R^R$  in  $\mathcal{C}$ , and this in turn corresponds, by transposition, and by fullness of the embedding  $h$ , to a smooth map

$$\check{f} : U \times \mathbb{R} \rightarrow \mathbb{R}.$$

Now we have that

$$\vdash_U \exists b > 0 [\forall x (|x| > b \Rightarrow f(x) = 0)]$$

if and only if there is a covering  $U_i$  of  $U$  ( $i \in I$ ) and witnesses  $b_i \in U_i R_{>0}$ , so that for each  $i$

$$\vdash_{U_i} \forall x (|x| > b_i \Rightarrow f(x) = 0)$$

Externally, this implies that  $b_i : U_i \rightarrow \mathbb{R}$  is a smooth function with positive values, with the property that for all  $t \in U_i$ , if  $x$  has  $x > b_i(t)$ , then  $\check{f}(t, x) = 0$ . The following Lemma then implies that  $\check{f}$  is of l.u.b.s. on  $U_i$ , and since the  $U_i$ 's cover  $K$ ,  $\check{f}$  is of l.u.b.s. on  $K$ .

**Lemma 5.2** *Let  $g : U \times \mathbb{R} \rightarrow \mathbb{R}$  have the property that there exists a smooth (or just continuous)  $b : U \rightarrow R_{>0}$  so that for all  $t \in U$   $|x| > b(t)$  implies  $g(t, x) = 0$ . Then  $g$  is l.u.b.s.*

**Proof.** For each  $t \in U$ , let  $c_t$  denote  $b(t) + 1$ . There is a neighbourhood  $V_t$  around  $t$  such that  $b(y) < c_t$  for all  $y \in V_t$ . The family of  $V_t$ 's, together with the constants  $c_t$  now witness that  $g$  is l.u.b.s. For, for all  $y \in V_t$  and any  $x$  with  $|x| > c_t$ , we have  $|x| > c_t > b(y)$ , so  $g(y, x) = 0$ .

Conversely, if  $\check{f}$  is l.u.b.s., it is easy to see that the element  $f \in_K U^R$  satisfies the formula (reduce to the uniformly bounded case, and write the condition as existence of a commutative square).

So we conclude that for  $f \in_U R^R$ ,  $f \in_U \mathcal{D}(R)$  iff the external function  $\check{f} : U \times \mathbb{R} \rightarrow \mathbb{R}$  is l.u.b.s., i.e., by Theorem 2.3, iff  $\hat{f} : U \rightarrow C^\infty(\mathbb{R})$  factors by a (diffeologically!) smooth map through the inclusion  $\mathcal{D}(\mathbb{R}) \subseteq C^\infty(\mathbb{R})$ , i.e. belongs to  $C^\infty(U, \mathcal{D}(\mathbb{R})) = h(\mathcal{D}(\mathbb{R}))(C^\infty(U))$ . This proves that  $h(\mathcal{D}(\mathbb{R})) = \mathcal{D}(R)$ .

We next consider the synthetic status in  $\mathcal{S}$  of the space of distributions  $\mathcal{D}(\mathbb{R})'$ .

If  $R$  is a commutative ring object in a topos, and it is equipped with a compatible preorder  $\leq$ , we have already described the  $R$ -module  $\mathcal{D}(R)$ , (space of test functions). For any  $R$ -module object  $Y$ , we may then form its  $R$ -linear dual object  $Y' = \text{Lin}_R(Y, R)$  as a subobject of  $R^Y$ ; in particular, we may form  $(\mathcal{D}(R))'$  which is then the internal object of distributions on  $R$ , as alluded to in the Introduction.

**Theorem 5.3 (Distributions in the Smooth Topos)** *The convenient vector space  $(\mathcal{D}(\mathbb{R}))'$  goes by  $h : \underline{\text{Con}}^\infty \rightarrow \mathcal{S}$  to the internal object of distributions  $(\mathcal{D}(R))'$ .*

We first make an analysis of  $h(Y')$  for a general convenient vector space  $Y$ . (Here,  $Y'$  denotes the diffeological dual consisting of smooth linear functionals.) Recall that the diffeology on  $Y'$  is inherited from that of  $C^\infty(Y, \mathbb{R})$ , so that (for an open  $U \subseteq \mathbb{R}^k$ ), the smooth plots  $U \rightarrow Y'$  are in bijective correspondence with smooth maps  $U \times Y \rightarrow \mathbb{R}$ , which are  $\mathbb{R}$ -linear in the second variable  $y \in Y$ . It follows that the elements at stage  $U$  are in bijective correspondence with smooth maps  $U \times Y \rightarrow \mathbb{R}$ ,  $\mathbb{R}$ -linear in the second variable, or equivalently, with smooth  $\mathbb{R}$ -linear maps  $Y \rightarrow C^\infty(U, \mathbb{R})$ .

On the other hand, an element of  $R^{h(Y)}$  defined at stage  $U$  is a morphism  $h(U) \rightarrow R^{h(Y)}$ , hence by double transposition it corresponds to a map  $h(Y) \rightarrow R^{h(U)}$ ; and it belongs to the subobject  $\text{Lin}_R(h(Y), R)$  iff its double transpose is  $R$ -linear. Since  $h$  is full and faithful, and preserves the cartesian closed structure (hence the transpositions), this double transpose corresponds bijectively to a smooth map  $Y \rightarrow C^\infty(U, \mathbb{R}) = C^\infty(U)$ , and  $R$ -linearity is equivalent to  $\mathbb{R}$ -linearity, by the following general

**Lemma 5.4** *Let  $X$  and  $Y$  be convenient vector spaces. Then a smooth map  $f : Y \rightarrow X$  is  $\mathbb{R}$ -linear iff  $h(f) : h(Y) \rightarrow h(X)$  is  $R$ -linear.*

**Proof.** The implication  $\Rightarrow$  is a consequence of the fact that  $h$  preserves binary cartesian products (and  $h(\mathbf{R}) = R$ ). For the implication  $\Leftarrow$ , we just apply the global sections functor  $\Gamma$ ; note that  $\Gamma(Y)$  is the underlying set of the vector space  $Y$ , and similar for  $X$ ; and  $\Gamma(R) = \mathbb{R}$ .

The Theorem now follows from Theorem 5.1.

We have in particular:

**Proposition 5.5** *There is a natural one-to-one correspondence between distributions on  $\mathbb{R}$ , and  $R$ -linear maps  $\mathcal{D}(R) \rightarrow R$*

**Proof.** This follows from fullness of the embedding  $h$ .

This result should be compared to the Theorem of [23], or Proposition II.3.6 in [20], where a related assertion is made for distributions-with-compact-support, i.e. where  $\mathcal{D}(R)$  is replaced by the whole of  $R^R$ , (or even with  $R^M$ , with  $M$  an arbitrary smooth manifold; the generalization of our theory is straightforward). Distributions with compact support are generally easier to deal with synthetically (as we did in [14]).

## 6 Cahiers Topos

This topos was constructed by Dubuc [3] in order to get what he called a well-adapted model for Synthetic Differential Geometry (SDG). The site of definition contains not only a suitable representative category of smooth manifolds, but also objects which represent the infinitesimal objects (of “nilpotent elements”), like  $D$ , which are crucial in SDG. The category of infinitesimal objects is taken to be the dual of the category of Weil-algebras (i.e. finite dimensional commutative real algebras, where the nilpotent elements form an ideal of codimension 1). This prompts us to replace also the representative category of smooth manifolds with the dual of a category of  $(C^\infty\text{-})$  algebras, capitalizing on the fact that smooth maps  $U \rightarrow V$  correspond bijectively to  $C^\infty$ -algebra maps  $C^\infty(V) \rightarrow C^\infty(U)$ .

To conform with our exposition in [13], we take the representative smooth manifolds just to be the coordinate vector spaces  $\mathbb{R}^k$ , rather than all open subsets  $U$  of such. (We could, by suitable comparison theorem of site theory, have used the category of just these  $\mathbb{R}^k$  for the Smooth Topos also.)

We recall the site of definition  $\underline{D}$  for the Cahiers Topos  $\mathcal{C}$ . The underlying category is the dual of a certain category of  $C^\infty$ -rings, namely those that are of the form  $C^\infty(\mathbb{R}^{l+k})/J$  where  $J$  is a semi-Weil ideal; we explain this notion. Let  $\mathcal{M} \subseteq C^\infty(\mathbb{R}^l)$  be the maximal ideal of functions vanishing at 0. A *Weil ideal*  $I \subseteq C^\infty(\mathbb{R}^l)$  is an ideal  $I$  containing some power of this maximal ideal; in particular,  $I$  is of finite codimension). A *semi-Weil ideal*  $J \subseteq C^\infty(\mathbb{R}^{l+k})$  is an ideal which comes about from a Weil ideal  $I$  in  $C^\infty(\mathbb{R}^l)$  as  $I^*$ , where  $I^*$  is the ideal of functions of the form  $\sum f_i(x, y) \cdot g_i(x)$  with  $g_i \in I$ .

To describe and analyze the embedding  $h$  of  $\underline{Con}^\infty$  into  $\mathcal{C}$ , we need a more elaborate account of the relationship between semi-Weil ideals and convenient vector spaces:

For any ideal  $J \subseteq C^\infty\mathbb{R}^n$ , and any CVS  $X$ , we define a linear subspace  $J(X)$  of  $C^\infty(\mathbb{R}^n, X)$  as the set of those  $f : \mathbb{R}^n \rightarrow X$  such that for every  $\phi \in X'$ ,  $\phi \circ f \in J$ . (There is also a, usually smaller linear subspace,  $J_s(X)$  consisting of linear combinations of functions  $i(t) \cdot g(t)$ , where  $i : \mathbb{R}^n \rightarrow \mathbb{R}$  belongs to  $J$  and  $g : \mathbb{R}^n \rightarrow X$  is arbitrary. For  $J$  a semi-Weil ideal,  $J(X) = J_s(X)$ .)

Two smooth maps  $g_1, g_2 : \mathbb{R}^n \rightarrow X$  are called *congruent mod  $J$*  if  $g_1 - g_2 \in J(X)$ .

Let  $I \subseteq \mathbb{R}^l$  be a Weil ideal,  $I \supseteq \mathcal{M}^r$ . Let  $\{D^\beta \mid \beta \in B\}$  be a family of

differential operators at 0, of degree  $< r$ , forming a basis for  $(C^\infty(\mathbb{R}^l)/I)^*$ . Note that  $B$  is a finite set. Let the dual basis for  $C^\infty(\mathbb{R}^l)/I$  be represented by polynomials of degree  $< r$ ,  $\{p_\beta(s) \mid \beta \in B\}$ . Then we can construct a linear isomorphism

$$C^\infty(\mathbb{R}^l, Y)/I(Y) \rightarrow \prod_B Y, \quad (10)$$

by sending the class of  $f : \mathbb{R}^l \rightarrow Y$  into the  $B$ -tuple  $D_Y^\beta(f)$ . Its inverse is given by sending a  $B$ -tuple  $y_\beta \in Y$  to the map  $s \mapsto \sum_B p_\beta(s) \cdot y_\beta$ .

It follows that for a semi-Weil ideal  $J = p^*(I) \subseteq \mathbb{R}^{l+k}$ , as above,

$$C^\infty(\mathbb{R}^{k+l}, Y)/J(Y) \cong \prod_B C^\infty(\mathbb{R}^k, Y). \quad (11)$$

(The isomorphisms (10) and (11) are not canonical but depend on the choice of a linear basis  $p_\beta(s)$  for the Weil algebra  $C^\infty(\mathbb{R}^l)/I$ .)

The isomorphisms here are the “external” version of the validity of the general K-L-axiom for convenient vector spaces in the Cahiers topos.

The full embedding  $h$ , described in [13], of  $\underline{\text{Con}}^\infty$  into  $\mathcal{C}$  is, on objects, given by sending a convenient vector space  $X$  into the presheaf on  $\underline{D}$  given by

$$C^\infty(\mathbb{R}^{l+k})/J \mapsto C^\infty(\mathbb{R}^{l+k}, X)/J(X).$$

Note that if  $X = \mathbb{R}$ , this presheaf is the “undelying set” functor  $R$ . – To describe  $h(f)$  for  $f$  a smooth map between CVS’s, we send the congruence class of  $g : \mathbb{R}^n \rightarrow X$  into the congruence class of  $f \circ g$ ; this is well defined, by the fundamental observation in [13] that for smooth maps  $f : X \rightarrow Y$ , composing with  $f$  preserves the property of “being congruent mod  $J$ ”, provided that  $J$  is a semi-Weil ideal, cf. Coroll. 2 in [13]. (This fundamental observation, in turn, is a generalization of the theory from [11] that the category of Weil algebras acts “by Weil prolongation” on the category  $\underline{\text{Con}}^\infty$ ; this prolongation construction is expounded also in [15] Section 31.)

The embedding  $h$  is full. It preserves the exponentials in  $\underline{\text{Con}}^\infty$ , and furthermore, if  $X$  is a convenient vector space, the  $R$ -module  $h(X)$  in  $\mathcal{C}$  “satisfies the vector form of Axiom 1” (generalized Kock-Lawvere Axiom), so that in particular synthetic calculus for curves  $R \rightarrow h(X)$  is available; cf. the final remark in [11]. From this, one may deduce that the embedding  $h$  preserves differentiation, i.e. for  $f : \mathbb{R} \rightarrow X$  a smooth curve, its derivative

$f' : \mathbb{R} \rightarrow X$  goes by  $h$  to the synthetically defined derivative of the curve  $h(f) : R = h(\mathbb{R}) \rightarrow h(X)$ . This follows by repeating the argument for Theorem 1 in [8] (the Theorem there deals with the case where the codomain of  $f$  is  $\mathbb{R}$ , but it is valid for  $X$  as well because  $h(X)$  satisfies the vector form of Axiom 1).

We note the following aspect of the embedding  $h$ . Let  $X$  be a convenient vector space. Each  $\phi \in X'$  is smooth linear  $X \rightarrow \mathbb{R}$  and hence defines a map  $h(\phi) : h(X) \rightarrow h(\mathbb{R}) = R$  in  $\mathcal{C}$ . This map is  $R$ -linear.

**Proposition 6.1** *The maps  $h(\phi) : h(X) \rightarrow R$ , as  $\phi$  ranges over  $X'$ , form a jointly monic family.*

**Proof.** The assertion can also be formulated: the natural map

$$e : h(X) \rightarrow \prod_{\phi \in X'} R$$

is monic (where  $\text{proj}_{\phi} \circ e := h(\phi)$ ). To prove that this (linear) map is monic, consider an element  $a$  of the domain, defined at stage  $C^\infty(\mathbb{R}^{l+k})/J$ , where  $J$  is a semi-Weil ideal. So  $a \in C^\infty(\mathbb{R}^{l+k}, X)/J(X)$ . Let  $\alpha \in C^\infty(\mathbb{R}^{l+k}, X)$  be a smooth map representing the class  $a$ ,  $a = \alpha + J(X)$ . The element  $e(a)$  is the  $X'$  tuple  $a_\phi + J$ , where  $a_\phi \in C^\infty(\mathbb{R}^{l+k})/J$  is represented by the smooth map  $\phi \circ \alpha : \mathbb{R}^{l+k} \rightarrow \mathbb{R}$ . To say that  $a$  maps to 0 by  $e$  is thus to say that for each  $\phi \in X'$ ,  $\phi \circ \alpha \in J$ . But this is precisely the defining property for  $\alpha$  itself to be in  $J(X)$ , i.e. for  $a$  to be the zero as an element of  $h(X)$  (at the given stage  $C^\infty(\mathbb{R}^{l+k})/J$ ).

We now analyze the object of test functions. We shall prove the analogue of Theorem 5.1, now for the embedding  $h : \underline{\text{Con}}^\infty \rightarrow \mathcal{C}$ . The object  $\mathcal{D}(R)$  is defined synthetically by the same formula (8) as in Section 5. Part of the proof of the Theorem 5.1 there can be “recycled”. In fact, letting  $U$  be  $\mathbb{R}^k$ , the proof recycles to give information about the elements of  $\mathcal{D}(R)$  defined at stage  $C^\infty(\mathbb{R}^k)$ ; they are the same as the elements of  $h(\mathcal{D})(\mathbb{R})$ , more precisely,

$$h(\mathcal{D}(\mathbb{R}))(C^\infty(\mathbb{R}^k)) = \mathcal{D}(R)(C^\infty(\mathbb{R}^k)). \quad (12)$$

To get a similar conclusion for elements of  $\mathcal{D}(R)$  (as synthetically defined by (9)), defined at stage  $C^\infty(\mathbb{R}^{l+k})/J$  (where  $J$  is a semi-Weil ideal),

we shall prove that such an element can be represented by a  $B$ -tuple of elements defined at stage  $C^\infty(\mathbb{R}^k)$ ; we shall prove that such a  $B$ -tuple defines an element of  $\mathcal{D}(R)$  precisely if each of these  $B$  elements is an element in  $\mathcal{D}(R)$ . This proof is a piece of purely synthetic reasoning:

We consider an  $\mathbb{R}$ -algebra object  $R$  in a topos  $\mathcal{C}$ , and assume that  $R$  satisfies the general “Kock-Lawvere” (K-L) axiom (recalled below), and is equipped with a strict order relation  $<$ . Because the reasoning is purely synthetic, we don’t have to think in terms of sheaf semantics, so for instance we don’t have to be specific at what “stages”, the “elements” in question are defined; we reason *as if* all elements are global elements. For  $b > 0$ , we write  $|x| > b$  as shorthand for  $x < -b \vee x > b$  as before; and we stress again that we don’t assume any absolute-value function (it does not exist in the Cahiers topos). We argue in  $\mathcal{C}$  as if it were the category of sets, making sure to use only intuitionistically valid reasoning.

A Weil algebra  $C^\infty(\mathbb{R}^l)/I$ , as above, gives rise to an “infinitesimal” sub-object  $W \subseteq R^l$ : pick a (finite) set of differential operators  $D_\beta$  ( $\beta \in B$ ) forming a basis for  $(C^\infty(\mathbb{R}^l)/I)^*$ , and take the dual basis for  $C^\infty(\mathbb{R}^l)/I$ , whose elements are represented mod  $I$  by polynomials  $p_\beta(s)$  in  $l$  variables. Then  $W \subseteq R^l$  is the extension of the formulas  $p_\beta(s) = 0$ ,  $s$  being a variable ranging over  $R^l$  (note that real polynomials in  $l$  variables define functions  $R^l \rightarrow R$  in  $\mathcal{C}$ ).

We assume that such  $W$ ’s are internal atoms, in a sense we partially recall below; this is so for all interesting models  $\mathcal{C}$  of SDG, including the Cahiers Topos.

To say that an  $R$ -module object  $Y$  in  $\mathcal{C}$  satisfies the general K-L axiom is to say that for each such Weil algebra, the map

$$\prod_B Y \rightarrow Y^W$$

given by

$$(y_\beta)_{\beta \in B} \mapsto [s \mapsto \sum_B p_\beta(s) \cdot y_\beta]$$

is an isomorphism.

We assume that  $R$  itself satisfies K-L. This immediately implies that  $R^M$ , as an  $R$ -module, does so for any  $M \in \mathcal{C}$ . We shall consider  $R^R$ .

Now recall that  $\mathcal{D}(R) \subseteq R^R$  was the subobject which is the extension of the formula (9) (with free variable  $f$  ranging over  $R^R$ )  $\exists b > 0(|x| > b \Rightarrow f(x) = 0)$ .

**Proposition 6.2** *Let a  $B$ -tuple of elements  $f_\beta$  in  $R^R$  represent an element in  $(R^R)^W$ . Then it defines an element in the sub“set”  $(\mathcal{D}(R))^W$  if and only if each  $f_\beta$  is in  $\mathcal{D}(R)$ .*

**Proof.** Assume first that all  $f_\beta$  are in  $\mathcal{D}(R)$ . For each  $\beta$  there exists a witness  $b_\beta > 0$  witnessing the fact that the formula (9) holds for  $f_\beta$ , but since there are only finitely many  $\beta$ 's, we may assume one common witness  $b > 0$ . So for all  $\beta$ , and for all  $x$  with  $|x| > b$ ,  $f_\beta(x) = 0$ . But then for each such  $x$ , the function of  $s \in W$  given by

$$s \mapsto \sum_{\beta} p_{\beta}(s) \cdot f_{\beta}(x)$$

is the zero function. The sum here, as a function of  $s$  and  $x$ , is the element of  $(R^R)^W$  corresponding to the  $B$ -tuple  $f_\beta$ , and for  $|x| > b$ , it is the zero. So for each  $s$ , the given fixed  $b$  witnesses that the sum, as a function of  $x$ , is in  $\mathcal{D}(R)$ .

Conversely, assume that the  $f_\beta$ 's are such that the corresponding function  $W \rightarrow R^R$  factors through  $\mathcal{D}(R)$ . So for each  $s \in W$ , the function

$$x \mapsto \sum_{\beta} p_{\beta}(s) \cdot f_{\beta}(x)$$

belongs to  $\mathcal{D}(R)$ . So

$$\forall s \in W \exists b > 0(|x| > b \Rightarrow \sum_{\beta} p_{\beta}(s) \cdot f_{\beta}(x) = 0). \quad (13)$$

We would like to *pick* for each  $s \in W$  a  $\tilde{b}(s)$  such that

$$\forall s \in W(|x| > \tilde{b}(s) \Rightarrow \sum_{\beta} p_{\beta}(s) \cdot f_{\beta}(x) = 0);$$

the existence of such a *function*  $\tilde{b}$  follows from (13) by a use of the Axiom of Choice, so in general is not possible in a topos. But since  $W$  is an internal atom, and  $s$  ranges over  $W$ , such a function  $\tilde{b}$  exists after all. (See the Appendix for a general formulation and proof of this principle.)

But now  $|x| > \tilde{b}(0) \Rightarrow |x| > \tilde{b}(s)$  for all  $s \in W$ , because  $\tilde{b}$ , as does any function, preserves infinitesimals, and because strict inequality is unaffected by infinitesimals. So we have a  $b$ , namely  $\tilde{b}(0)$ , so that

$$\forall s \in W (|x| > b \Rightarrow \sum p_\beta(s) \cdot f_\beta(x) = 0).$$

So for  $|x| > b$ ,

$$\forall s \in W (\sum p_\beta(s) \cdot f_\beta(x) = 0).$$

Thus, for fixed  $x$  with  $|x| > b$ , the function of  $s$  here is constantly 0. But functions  $W \rightarrow R$  can uniquely be described as linear combinations of the  $p_\beta(s)$ 's (this is a verbal rendering of the K-L axiom for  $R$ ). So for such  $x$  each  $f_\beta(x)$  is 0. So  $b$  witnesses, for each  $\beta$ , that  $f_\beta \in \mathcal{D}(R)$ . This proves the Proposition.

Combining (11) (with  $\mathcal{D}(\mathbb{R})$  for  $Y$ ) with (12) and Proposition 6.2, we get

**Theorem 6.3 (Test functions in the Cahiers Topos)** *The inclusion  $\mathcal{D}(\mathbb{R})$  goes by  $h : \underline{\text{Con}}^\infty \rightarrow \mathcal{C}$  to the inclusion  $\mathcal{D}(R) \subseteq R^R$*

We proceed to use this result as a tool to analyze the object of distributions  $\mathcal{D}(R)'$ . We could proceed along the lines of the proof of Theorem 5.1, but a more elegant argument is available. For any convenient vector space  $Y$ , the dual  $Y'$  is not only a subspace of  $\mathbb{R}^Y$ , it is even a retract, namely the fixpoint set of the smooth linear endomap  $d_0$  on  $\mathbb{R}^Y$  given by  $f \mapsto d_0 f$ , the differential of  $f$  at  $0 \in Y$ . In the Cahiers topos  $\mathcal{C}$ , synthetic differential calculus is available, and there is a similar retraction operator  $d_0$  on  $R^Z$ , for any vector space ( $R$ -module)  $Z$ , and in fact, the object  $\text{Lin}_R(Z, R) \subseteq R^Z$  is the fixpoint object for this operator (this follows from elementary synthetic differential calculus, cf. [16] 1.2.3 and 1.2.4). But the embedding  $h$  takes the “external”  $d_0$  to the internal one, and any functor preserves fixpoint objects for idempotent endomaps. Thus  $h$  takes the subobject  $Y' \subseteq \mathbb{R}^Y$  to the subobject  $\text{Lin}_R(h(Y), R)$ . If we apply this observation to the case of  $Y = \mathcal{D}(\mathbb{R})$ , and use the Theorem 6.3 above, we get

**Theorem 6.4 (Distributions in the Cahiers Topos)** *The embedding  $h : \underline{\text{Con}}^\infty \rightarrow \mathcal{C}$  takes the convenient vector space  $\mathcal{D}(\mathbb{R})'$  of distributions on  $\mathbb{R}$  into the internal object of distributions  $\mathcal{D}(R)'$ .*

We get in particular

**Proposition 6.5** *There is a bijective correspondence between distributions on  $\mathbb{R}$ , and  $R$ -linear maps in  $\mathcal{C}$ ,  $\mathcal{D}(R) \rightarrow R$ .*

(The analogous result for distributions of compact support may be found in [23].)

## 7 Half line in $\mathcal{C}$ , and the heat equation

By Theorem 4.1, the two  $C^\infty$ -rings  $C^\infty(\mathbb{R})/\mathcal{M}_{\geq 0}^\infty$  and  $C^\infty(\mathbb{R}_{\geq 0})$  are isomorphic, where  $\mathcal{M}_{\geq 0}^\infty$  is the ideal of smooth functions vanishing on the non-negative half line, and  $C^\infty(\mathbb{R}_{\geq 0})$  is the ring of smooth functions  $\mathbb{R}_{\geq 0} \rightarrow \mathbb{R}$ . Being a quotient of the ring  $C^\infty(\mathbb{R})$  which represents  $R \in \mathcal{C}$ , it defines a sub-object of  $R$ , which we denote  $R_{\geq 0}$  (also considered in [12]<sup>1</sup>). – Thus,  $R_{\geq 0}$  is “represented from the outside” by the  $C^\infty$ -ring  $C^\infty(\mathbb{R})/\mathcal{M}_{\geq 0}^\infty \cong C^\infty(\mathbb{R}_{\geq 0})$ .

**Proposition 7.1** *Let  $I \subseteq C^\infty(\mathbb{R}^l)$  be a Weil ideal and let  $f : \mathbb{R}^l \times \mathbb{R}^k \rightarrow \mathbb{R}$  be a smooth function. Then the following are equivalent:*

1.  $f(\underline{0}, x) \geq 0$  for all  $x \in \mathbb{R}^k$
2.  $\rho(f(\underline{w}, x)) \in I^*$  for all  $\rho \in \mathcal{M}_{\mathbb{R}_{\geq 0}}^\infty$ .

**Proof.** “not 1” implies “not 2”; for, if  $f(0, x) < 0$ , we may find a function  $\rho$  vanishing on  $\mathbb{R}_{\geq 0}$  and with value 1 at  $f(0, x)$ . Then  $f \notin I^*$  (recall that any Weil ideal  $I$  consists of functions vanishing at 0). – On the other hand, “1” implies “2”: For, by Taylor expansion,

$$(\rho \circ f)(\underline{w}, x) = (\rho \circ f)(\underline{0}, x) + \sum_i w_i (\rho \circ f)'_i(\underline{0}, x) + \sum_{i,j} w_i w_j (\rho \circ f)'_{i,j}(\underline{0}, x) + \dots$$

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<sup>1</sup>The ring representing  $R_{\geq 0}$ , was in loc.cit. defined using the ideal  $\mathcal{M}_{\geq 0}^g$  of functions vanishing on an open neighbourhood of  $\mathbb{R}_{\geq 0}$ , rather than the ideal  $\mathcal{M}_{\geq 0}^\infty$  considered here. But it can be proved that they represent (from the outside) the same object in the Cahiers topos.

where  $(-)_i = \partial/\partial x_i$ ,  $(-)_{i,j} = \partial^2/\partial x_i \partial x_j$  etc. This series finishes after finitely many terms modulo  $I^*$ , since a product of powers of  $w_i$ 's belong to the ideal  $I$ . But each of its terms is 0: Indeed, so is the term without derivatives, by hypothesis. But so are the others. For instance.  $(\rho \circ f)'_i(\underline{0}, x) = \rho'(f(\underline{0}, x)) \partial f / \partial x_i(\underline{0}, x)$  is 0, since the derivative of  $\rho$  is zero on non-negative reals (by definition of  $m_{\mathbb{R}_{\geq 0}}^\infty$ ).

Let  $J$  denote  $I^*$ . Then an element  $F$  of  $R_{\geq 0}$  defined at stage  $C^\infty(\mathbb{R}^{l+k})/J$  is represented by a function  $f$  satisfying the conditions of the Proposition.

**Proposition 7.2** *There is a bijection between the set of smooth maps  $K : \mathbb{R}_{\geq 0} \rightarrow X$  and the set of maps  $\bar{K} : R_{\geq 0} \rightarrow h(X)$  in  $\mathcal{C}$ .*

**Proof/Construction.** Passing from  $\bar{K}$  to  $K$  is just by taking global sections. – Conversely, given  $K$ , we extend it (using Theorem 4.1) to a smooth map  $K_1 : \mathbb{R} \rightarrow X$ , and apply the embedding  $h$  to get a map  $h(K_1) : R \rightarrow h(X)$  in  $\mathcal{C}$ ; its restriction to  $R_{\geq 0}$  is the desired  $\bar{K}$ . We have to see that this  $\bar{K}$  does not depend on the choice of the extension  $K_1$ . Given some other extension  $K_2$ , we should prove that for any generalized element  $F$  of  $R_{\geq 0}$ ,  $h(K_1)(F) = h(K_2)(F)$ . Suppose  $F$  is an element of stage  $C^\infty(\mathbb{R}^{l+k})/J$ , where  $J$  is the semi-Weil ideal  $I^*$  considered in the Proposition above. Thus, as a generalized element of  $R$ , it is identified with  $f + J$ , where  $f \in C^\infty(\mathbb{R}^{l+k})$ , and it satisfies condition 2. of the Proposition, being an element of  $R_{\geq 0}$ .

We should prove that  $K_1 \circ f \cong K_2 \circ f$  modulo  $J(X)$ . This means to prove, for any  $\phi \in Y'$  that

$$\phi \circ K_1 \circ f \equiv \phi \circ K_2 \circ f$$

modulo  $J$ . But subtracting the two entries to be compared yields, by linearity of  $\phi$  the map

$$\phi \circ (K_1 - K_2) \circ f,$$

and since  $K_1 - K_2$  vanishes on  $\mathbb{R}_{\geq 0}$ , then so does  $\phi \circ (K_1 - K_2)$ . We may thus take  $\rho = \phi \circ (K_1 - K_2)$  in the condition 2. in the Proposition, and conclude that  $\phi \circ (K_1 - K_2) \circ f$  is in  $I^* = J$ , as desired.

– Uniqueness is easy, using Proposition 6.1, together with the fullness result from [22] on manifolds with boundary (see also [9] and [25]).

The Proposition is a “mixed fullness” result; we have that  $Con^\infty$  and  $\underline{Mf}$  (= smooth manifolds), (even the category of smooth manifolds with boundary), embed fully in The Cahiers Topos; but at present we do not have a

general result on what can be said about  $C^\infty(M, X)$ , for  $M$  a manifold with boundary and  $X$  a convenient vector space.

For any topos  $\mathcal{C}$  with a ring object  $R$  with a preorder  $\leq$ , we may form the  $R$ -module  $\mathcal{D}'(R^n)$  of distributions on  $R^n$ , as explained in Section 5 and 6. If  $\mathcal{C}, R$  is a model of SDG, then  $\mathcal{D}'(R^n)$  automatically satisfies the “vector form” of the general Kock-Lawvere axiom, so that (synthetic) differentiation of functions  $K : R \rightarrow \mathcal{D}'(R^n)$  is possible – it is even enough that  $K$  be defined on suitable (“formally étale”) subobjects of  $R$ , like  $R_{\geq 0}$ . We think of the domain  $R$  or  $R_{\geq 0}$  as “time”, and denote the differentiation of curves  $K$  w.r. to time by the Newton dot,  $\dot{K}$ . On the other hand, we think of  $R^n$  as a space, and the various partial derivatives  $\partial/\partial x_i$  ( $i = 1, \dots, n$ ), as well as their iterates, we call spatial derivatives; in case  $n = 1$ , they are just denoted  $(-)'$ ,  $(-)^{''}$ , etc. They live on  $\mathcal{D}'(R^n)$  as well, by the standard way of differentiating distributions (which immediately translates into the synthetic context, cf. e.g. [14]). The heat equation for (Euclidean) space in  $n$  dimensions says  $\dot{K} = \Delta \circ K$ , where  $\Delta$  is the Laplace operator; in one dimension it is thus the equation

$$\dot{K} = K''.$$

We can summarize the constructions into an general existence theorem about models for SDG:

**Theorem 7.3** *There exists a well-adapted model for SDG (with a preorder  $\leq$  on  $R$ ), in which the heat equation on the (unlimited) line  $R$  has a unique solution  $k : R_{\geq 0} \rightarrow \mathcal{D}'(R)$  with initial value  $k(0) = \delta(0)$  (the Dirac distribution).*

**Proof.** The well adapted model witnessing the validity of the Theorem is the Cahiers Topos  $\mathcal{C}$ . Consider the classical heat kernel, viewed, as we did in Section 4, as a map  $\mathbb{R}_{\geq 0} \rightarrow \mathcal{D}'(\mathbb{R})$ . By Section 4, this map is smooth, hence by Proposition 7.2, it defines a morphism in  $\mathcal{C}$ ,  $\bar{K} : R_{\geq 0} \rightarrow h(\mathcal{D}'(\mathbb{R}))$ . This  $\bar{K}$  is going to be our  $k$ . By Theorem 6.4, its codomain is the desired  $\mathcal{D}'(R)$ . We prove that this  $k$  satisfies the heat equation  $\dot{k} = \Delta \circ k$ . This is a purely formal argument, given below, from the fact that  $K$  does, and the fact that  $h$  takes “analytic” differentiation into the “synthetic” differentiation in  $\mathcal{C}$ . Synthetically, we want to prove that for all  $x \in \mathbb{R}_{\geq 0}$  and  $d \in D$

$$k(x+d) = k(x) + d \cdot \Delta(k(x)).$$

Universal validity of this equation means that a certain diagram, with domain  $R_{\geq 0} \times D$  and codomain  $\mathcal{D}'(R)$ , commutes. Taking the transpose of this diagram, we get a diagram with domain  $R_{\geq 0}$  and codomain  $(\mathcal{D}'(R))^D \cong \mathcal{D}'(R) \times \mathcal{D}'(R)$  (by K-L for  $\mathcal{D}'(R)$ ):

$$\begin{array}{ccc}
 R_{\geq 0} & \xrightarrow{\hat{\dagger}} & (R_{\geq 0})^D \\
 \downarrow k & & \downarrow k^D \\
 \mathcal{D}'(R) & & (\mathcal{D}'(R))^D \\
 \downarrow (1, \Delta) & & \downarrow \\
 \mathcal{D}'(R) \times \mathcal{D}'(R) & \xleftarrow{\cong} & (\mathcal{D}'(R))^D
 \end{array}$$

When the global sections functor  $\Gamma$  is applied to this diagram, the left hand column yields  $(K, \Delta \circ K)$ , because  $\Gamma(k) = K$ ; the composite of the other maps is  $(K, \dot{K})$  because  $\Gamma$  takes synthetic differentiation into usual differentiation. Since  $K$  satisfies  $\dot{K} = \Delta \circ K$ , we conclude that  $\Gamma$  applied to the exhibited diagram commutes. Now  $\Gamma$  is not faithful, but because of the special form of the domain and codomain of the two maps to be compared, we may still get the conclusion, by virtue of the following

**Proposition 7.4** *Given a map  $a : R_{\geq 0} \rightarrow h(X)$ , where  $X$  is a convenient vector space. If  $\Gamma(a) = 0$ , then  $a = 0$ .*

**Proof.** Since the  $h(\phi) : h(X) \rightarrow R$  are jointly monic as  $\phi$  ranges over  $X'$ , by Proposition 6.1, it suffices to see that each  $h(\phi) \circ a$  is 0. Since  $\Gamma(h(\phi) \circ a) = \phi \circ \Gamma(a)$ , this reduces the question to the case where  $X = \mathbb{R}$ . A map  $a : R_{\geq 0} \rightarrow R$  is tantamount to an element in  $\bar{a} : C^\infty(\mathbb{R}_{\geq 0})$ , and the assumption  $\Gamma(a) = 0$  is tantamount to  $\bar{a}(t) = 0$  for all  $t \in \mathbb{R}_{\geq 0}$ . But this clearly implies that  $\bar{a}$ , and hence  $a$ , is 0.

The uniqueness assertion in the Theorem is likewise an easy consequence of this Proposition.

## Appendix

Recall that an *atom*  $A$  in a cartesian closed category  $\mathcal{C}$  is an object so that the exponential functor  $(-)^A$  has a right adjoint; in particular, it takes epimorphisms to epimorphisms. The following says that “axiom of choice” holds for “ $A$ ”-tuples sets:

**Proposition 7.5** *Assume that  $A$  is an atom,  $B$  an arbitrary object, and  $R \subseteq A \times B$ . Then*

$$(\forall a \in A)(\exists b \in B) R(a, b) \implies (\exists \tilde{b} \in B^A)(\forall a \in A) R(a, \tilde{b}(a))$$

**Proof.** The hypothesis means that the composite  $R \rightarrow A \times B \xrightarrow{\pi_1} A$  is surjective. By exponentiation, and the assumption that  $A$  is an atom, the composite  $R^A \rightarrow A^A \times B^A \xrightarrow{\pi_1} A^A$  is surjective. In particular,  $1_A \in A^A$  must have a pre-image  $(1_A, \tilde{b})$ . This  $\tilde{b}$  obviously does the job.

## References

- [1] Artin, M., Grothendieck, A. and Verdier, J.-L., Théorie des Topos (SGA 4), Springer LNM 269 (1972).
- [2] Chen, K.T. Iterated integrals of differential forms and loop space homology, Annals of Math. (2) 97 (1972), 217-146.
- [3] Dubuc, E., Sur les modèles de la géométrie différentielle synthétique, Cahiers de Topologie et Géométrie Diff. 20 (1979), 231-279.
- [4] Frölicher, A., Cartesian Closed Categories and Analysis of Smooth Maps, in “Categories in Continuum Physics”, Buffalo 1982 (ed. F.W. Lawvere and S.H. Schanuel), Springer Lecture Notes in Math. 1174 (1986).
- [5] Frölicher, A., Kriegl, A., Linear Spaces and Differentiation Theory, Wiley 1988.
- [6] Kelley, J.L., General Topology, Van Nostrand 1955.

- [7] Kelley, J.L., Namioka, I., and co-authors, *Linear Topological Spaces*, Van Nostrand 1963.
- [8] Kock, A. Properties of well-adapted models for synthetic differential geometry, *Journ. Pure Appl. Alg.* 20 (1981), 55-70.
- [9] Kock, A., *Synthetic Differential Geometry*, Cambridge University Press 1981.
- [10] Kock, A., Calculus of smooth functions between convenient vector spaces, Aarhus Preprint Series 1984/85 No. 18. Retyped in <http://home.imf.au.dk/kock/CSF.pdf>
- [11] Kock, A., Convenient vector spaces embed into the Cahiers topos, *Cahiers de Topologie et Géométrie Diff. Catégoriques* 27 (1986), 3-17.
- [12] Kock, A. and Reyes, G.E., Models for synthetic integration theory, *Math. Scand.* 48 (1981), 145-152.
- [13] Kock, A. and Reyes, G.E., Corrigendum and addenda to “Convenient vector spaces embed”, *Cahiers de Topologie et Géométrie Diff. Catégoriques* 28 (1987), 99-110.
- [14] Kock, A. and Reyes, G.E., Some calculus with extensive quantities: wave equation, *Theory and Applications of Categories*, Vol. 11 (2003), No. 14.
- [15] Kriegl, A. and Michor, P., *The Convenient Setting of Global Analysis*, Amer. Math. Soc. 1997.
- [16] Lavendhomme, R., *Basic Concepts of Synthetic Differential Geometry*, Kluwer Acad. Publishers 1996.
- [17] Lawvere, F.W., Categories of Space and of Quantity, in “The Space of Mathematics. Philosophical, Epistemological and Historical Explorations”, DeGruyter, Berlin 1992, 14-30.
- [18] Lawvere, F.W., Volterra’s Functionals and Covariant Cohesion of Space, *Rendiconti del Circolo Mat. di Palermo Serie II, Suppl.* 64 (2000), 201-214.

- [19] Losik, M.V. Frechet manifolds as diffeological spaces, Soviet Math. (Iz. vuz) 5 (1992), 36-42.
- [20] Moerdijk, I. and Reyes, G.E., Models for Smooth Infinitesimal Analysis, Springer 1991.
- [21] Nel, L.D., Enriched locally convex structures, Differential Calculus and Riesz representations, Journ. Pure Appl. Alg. 42 (1986), 165-184.
- [22] Porta, H. and Reyes, G.E., Variétés à bord et topos lisse, in “Analyse dans les topos lisses” (ed. G.E. Reyes), Rapport de Recherches D.M.S. 80-12, Université de Montréal 1980.
- [23] Quê, N.V. and Reyes, G.E. Théorie des distribution et théorème d’extension de Whitney, in “Analyse dans les topos lisses” (ed. G.E. Reyes), Rapport de Recherches D.M.S. 80-12, Université de Montréal 1980.
- [24] Reyes, G.E., A model of SDG in which only trivial distributions with compact support have a density, in [http://reyes-reyes.com/gonzalo/recent\\_work/syntheticdifferential/](http://reyes-reyes.com/gonzalo/recent_work/syntheticdifferential/)
- [25] Reyes, G.E., Embedding manifolds with boundary in smooth toposes, in [http://reyes-reyes.com/gonzalo/recent\\_work/syntheticdifferential/](http://reyes-reyes.com/gonzalo/recent_work/syntheticdifferential/)
- [26] Schwartz, L., Théorie des distributions, Tome 1, Hermann Paris 1957.
- [27] Schwartz, L., Méthodes mathématiques pour les sciences physiques, Hermann Paris 1961.
- [28] Seeley, R.T. , Extension of  $C^\infty$  functions defined in a half space, Proc. Amer. Math. Soc. 15 (1964), 625-626.

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