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COMBINATORIAL STACKS AND THE FOUR-COLOR THEOREM

by Romain ATTAL

ABSTRACT. We interpret the number of good four-colourings of the faces of a trivalent, spherical polyhedron as the 2-holonomy of the 2-connection of a fibered category, φ, modeled on $\mathbf{Rep}_f(\mathfrak{sl}_2)$ and defined over the dual triangulation, T. We also build an $\mathfrak{sl}_2$-bundle with connection over T, that is a global, equivariant section of φ, and we prove that the four-colour theorem is equivalent to the fact that the connection of this $\mathfrak{sl}_2$-bundle vanishes nowhere. This geometric interpretation shows the cohomological nature of the four-colour problem.

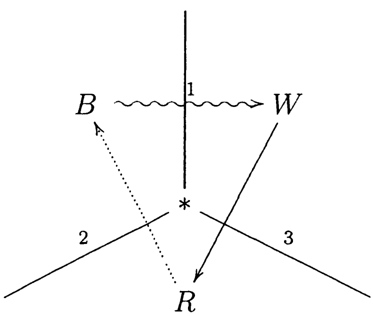
Résumé : Nous interprétons le nombre de bons quadri-coloriages des faces d'un polyèdre sphérique trivalent comme la 2-holonomie de la 2-connexion d'une catégorie fibrée, φ, modelée sur $\mathbf{Rep}_f(\mathfrak{sl}_2)$ et définie sur la triangulation duale, T. Nous construisons au-dessus de T un $\mathfrak{sl}_2$-fibré avec connexion qui est une section globale équivariante de φ, et nous prouvons que le théorème des quatre couleurs équivaut au fait que la connexion de ce $\mathfrak{sl}_2$-fibré ne s'annule nulle part. Cette interprétation géométrique montre la nature cohomologique du problème des quatre couleurs.

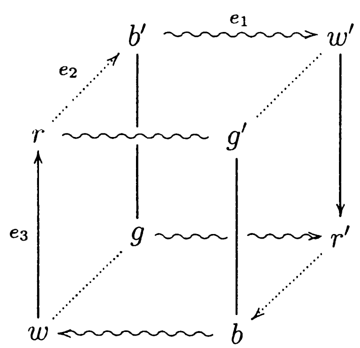
Keywords : Map colouring, iterated paths, combinatorial stacks, $\mathfrak{sl}_2$.

1. Introduction

Let us consider a finite spherical polyhedron, P, and a palette of four colours, $\{W, R, G, B\}$. We will call a good colouring of P any map which associates one of these colours to each face of P in such a way that any two adjacent faces carry distinct colours. The four-colour theorem [5, 1, 12] states that such a map exists for any P. The goal of the present work is to provide a geometric interpretation of this theorem. We obtain here two new results : the number of good colourings of a trivalent, spherical polyhedron is the 2-holonomy of a 2-connection on a fibered category over the dual triangulation, $T = P^*$ (Theorem 2) ; the four-colour theorem is equivalent to the existence of a non-vanishing, equivariant global section of this fibered category (Theorem 4).

In order to study the colourability of P, let us start by making some classical modifications. We first remark that it is sufficient to prove the colourability of trivalent polyhedra. Indeed, by cutting a little disk around each vertex of degree > 3 in P, one obtains a trivalent polyhedron and each good colouring of the latter provides a good colouring of P by shrinking this disk to the initial vertex. Henceworth, we will suppose that P is itself trivalent. Secondly, let us identify our four colours with the pairs of diametrally opposite vertices of a cube : $W = \{w, w'\}$, $R = \{r, r'\}$, $G = \{g, g'\}$ and $B = \{b, b'\}$. Then each good colouring of the three faces which surround a vertex of P defines an edge-loop in this cube such that the determinant of any triple of successive vectors be ± 1 :





A map $(u : T_1 \to \{e_1, e_2, e_3\})$ satisfying this property will be called a good numbering. Thus, the number of good numberings of the edges of T is one quarter of the number of good colourings of the faces of P, as proved by P.G. Tait [13]. We call this integer, K_T, the chromatic index of T and the four-colour theorem states that $K_T \neq 0$ for any finite, spherical triangulation, T.

Our article is organised as follows. In Section 2, we give a proof of Penrose's formula which expresses K_T as a partition function. In Section 3, we define the graph $\mathscr{P}$ of edge-paths of T. In Section 4, we collect useful results about representations of $\mathfrak{sl}_2$. In Section 5, we construct the chromatic stack, φ, which is a fibered category over T, endowed with a functorial 1-connection and with a natural 2-connection, and we prove that K_T is the 2-holonomy of this 2-connection on T. In Section 6, we define another fibered category, Φ, over $\mathscr{P}$. By integrating the functorial connection of Φ along a 2-path which sweeps each triangle of T once only, we obtain an equivariant global section of the pull-back of the chromatic stack to a triangulation $\widetilde{T}$ of the disk. This section, ζ, is an $\mathfrak{sl}_2$-bundle with connection whose holonomy on $\partial\widetilde{T}$ is K_T. Our construction is an adaptation of Stokes theorem to a case of combinatorial differential forms with values in the tensor category $\mathbf{A} = \mathbf{Rep}_f(\mathfrak{sl}_2)$ and we can write it symbolically $K_T = \int_T \varphi = \int_{\partial\widetilde{T}} \zeta$. Since K_T depends linearly on the value of ζ on each inner edge of $\widetilde{T}$, we obtain this way our second result : the four-colour theorem is equivalent to the fact that ζ vanishes nowhere.

2. The chromatic index

The idea to translate the four-colour problem in terms of linear algebra is due to Roger Penrose. Let us fix a finite, spherical triangulation $T = (T_0, T_1, T_2)$. T_0 is the set of its vertices, T_1 the set of its edges and T_2 the set of its triangles. Following [10], we define the chromatic index of T as

$$(1) \qquad \boxed{K_T := \sum_u \prod_{[xyz]} i \det\left(u_{xy}, u_{yz}, u_{zx}\right)}$$

In this sum, u runs over the set of all maps from T_1 to $\{e_1, e_2, e_3\}$, the canonical basis of $\mathbb{R}^3$, and $[xyz]$ runs over the set of positively oriented triangles of T. The integrality of K_T follows from the fact that, if u is a good numbering of T_1, *i.e.* if no determinant vanishes in this product, the number of triangles where $\det = (+1)$ minus the number of triangles where $\det = (-1)$ is a multiple of 4, as proves the following lemma.

Lemma : *If u is a good numbering of T_1 and if n_+ (resp. n_-) denotes the number of triangles $[xyz]$ such that* $\det(u_{xy}, u_{yz}, u_{zx}) = (+1)$ *(resp.* (-1)*), then $n_+ \equiv n_-$ mod 4.*

Proof : 1) Starting from (T, u), we can build another triangulation, T', equipped with a good edge numbering, u', such that $n'_+ = 0$. Indeed, if two adjacent, positively oriented triangles of T, say $[xyz]$ and $[zyw]$, have $\det = (+1)$ (positive triangles), then we can flip their common edge $[yz]$ to $[xw]$ and obtain a new pair of negative triangles, $[xyw]$ and $[wzx]$, where $\det = (-1)$. During this step, $(n_+ - n_-)$ is reduced by 4. Once all these pairs of neighbour positive triangles have been eliminated this way, the remaining contributions to n_+ are triangles surrounded by three negative neighbours. By adjoining three edges and a trivalent vertex inside each isolated triangle of this kind, we change a positive triangle for three negative ones. Again, $(n_+ - n_-)$ is reduced by 4, and (T', u') is reached at the end of this process.

2) Consider all pairs of triangles, $[xyz]$ and $[zyw]$, with $u'_{yz} = e_1$ on their common edge, $[yz]$. Since $\det(u_{xy}, u_{yz}, u_{zx}) = \det(u_{zy}, u_{yw}, u_{wz}) = (-1)$, the opposite sides of the rectangle $[xywz]$ carry the same vector, say $u'_{xz} = u'_{yw} = e_2$ and $u'_{xy} = u'_{zw} = e_3$. Let us join the midpoints of two opposite edges with a simple curve. By repeating this process inside all such pairs of triangles, we obtain two simple closed curves, c_2 and c_3. If we orient these curves conveniently, their intersection number is equal to $|u'^{(-1)}(e_1)|$, the number of edges of T' marked with e_1. But, after Jordan's theorem, the intersection number of two simple closed curves in S^2 is even. Therefore, $|u'^{(-1)}(e_1)|$, the number of edges mapped to e_1 by u', is even. Similarly, $|u'^{(-1)}(e_2)|$ and $|u'^{(-1)}(e_3)|$ are also even, as well as the total number of edges of T' :

$$t'_1 = |u'^{(-1)}(e_1)| + |u'^{(-1)}(e_2)| + |u'^{(-1)}(e_3)| \in 2\,\mathbb{N}$$

3) Since T' is a triangulation of a closed surface, we have $3\,t'_2 = 2\,t'_1$. Since t'_1 is even, we obtain $t'_2 = n'_- \in 4\,\mathbb{N}$. Therefore, n_+ and n_- are congruent modulo 4 :

$$(2) \qquad \boxed{(n_+ - n_-) \in 4\,\mathbb{Z}}$$

□

Theorem 1 (R. Penrose) : *K_T is the number of good numberings of T_1.*

Proof: If u is a bad numbering, then one of the determinants is zero and the corresponding product vanishes. On the other hand, if u is a good numbering, then the corresponding product is equal to $\imath^{(n_+ - n_-)} = 1$, after the precedent lemma. Therefore, the sum of all these products equals the number of good numberings of T_1.

□

3. The graph of edge-paths

Having fixed our triangulation, T, let us define the graph $\mathscr{P}$ whose vertices are the edge-paths of T :

$$\mathscr{P}_0 = \bigcup_{\ell \geqslant 0} \{\gamma = (x_0, \cdots, x_\ell) \ : \ \{x_i, x_{i+1}\} \in T_1 \ \forall i\}$$

and whose edges, called the 2-edges of T, are the pairs of paths, with the same source and the same target, which bound a single triangle of T :

$$\mathscr{P}_1 = \{\{(x_0, \cdots, x_\ell), (x_0, \cdots, x_i, y, x_{i+1}, \cdots, x_\ell)\} \ : \ \{x_i y x_{i+1}\} \in T_2\}$$

$$\begin{tikzcd}
 & & y \arrow[dr] & & \\
x_0 \arrow[r, dotted] & x_i \arrow[ur] \arrow[rr] & & x_{i+1} \arrow[r, dotted] & x_\ell
\end{tikzcd}$$

The oriented 2-edges are the corresponding ordered pairs. A 2-path in T is an edge-path in $\mathscr{P}$, *i.e.* a family $\Gamma = (\gamma_0, \cdots, \gamma_n)$ such that $\{\gamma_i, \gamma_{i+1}\} \in \mathscr{P}_1$ for $i = 0, \cdots, n-1$. They form the set $\mathscr{P}_2$:

$$\mathscr{P}_2 = \bigcup_{n \geqslant 0} \{\Gamma = (\gamma_0, \cdots, \gamma_n) \ : \ \{\gamma_i, \gamma_{i+1}\} \in \mathscr{P}_1 \quad \text{for} \quad i = 0, \cdots, n-1$$

For each 2-path $\Gamma = (\gamma_0, \cdots, \gamma_n)$, there is a 2-path $\widetilde{\Gamma}$ going backward in time :

$$\widetilde{\Gamma} = (\gamma_n, \cdots, \gamma_0)$$

The 0-source (resp. 0-target) of Γ is the common source (resp. target) of the γ_i's. The 1-source of Γ si γ_0 and and its 1-target is γ_n. The oriented 2-cells of T are its smallest 2-paths. They have the form $\big((xz), (xyz)\big)$ or $\big((xyz), (xz)\big)$, for some triangle $\{xyz\}$.

4. Representations of $\mathfrak{sl}_2$

As we have seen above, Penrose's formula involves the determinants of triples of basis vectors of $\mathbb{R}^3$. If we endow $\mathbb{R}^3$ with its canonical euclidian structure and with the corresponding cross-product, we obtain a Lie algebra isomorphic to $\mathfrak{so}_3$. Since we will use complex coefficients and Schur's lemma, valid only for representations over an algebraically closed field, we will work with its complexification, $V = \mathfrak{sl}_2$. We will note $I = \mathrm{Id}_V$, $V^\ell = V^{\otimes \ell}$ and $I^\ell = \mathrm{Id}_{V^\ell}$, where V^ℓ carries the representation

$$\begin{aligned} \rho_\ell \ : \ V &\longrightarrow \mathrm{End}(V^\ell) \\ x &\longmapsto \rho_\ell(x) = \sum_{k=1}^{\ell} I^{k-1} \otimes \mathrm{ad}_x \otimes I^{\ell-k} \end{aligned}$$

Let $\mathbf{A} = \mathbf{Rep}_f(\mathfrak{sl}_2)$, the category of finite dimensional representations of $\mathfrak{sl}_2$ over complex vector spaces. If M and M' are two V-modules, carrying, respectively, the representations R and R', we will often identify M with $M \otimes -$, the endofunctor of $\mathbf{A}$, and write $M'M$ for $M' \otimes M$. For each $j \in \frac{1}{2}\mathbb{N}$, let $(R_j : V \to \mathrm{End}(V_j))$ be a representative of the isomorphy class of representations of spin j and dimension $2j+1$. For example, we can choose $V_0 = \mathbb{C}$, $V_{1/2} = \mathbb{C}^2$ and $V_1 = V$. After Schur's lemma, the irreducible representations are orthonormal for the bifunctor $\hom_{\mathbf{A}}$:

$$\boxed{\hom_{\mathbf{A}}(R_j, R_k) \simeq \delta_{jk} R_0} \tag{3}$$

The intertwining number between two representations R and R' is defined as the dimension of the space $\hom_{\mathbf{A}}(R, R')$:

$$c(R, R') = \dim_{\mathbb{C}} \left(\hom_{\mathbf{A}}(R, R') \right)$$

After Clebsch-Gordan's rule, we have

$$V^2 \simeq V_0 \oplus V_1 \oplus V_2$$
$$c(V, V^2) = c(V^2, V) = 1$$

The projectors onto the isotypic components of V^2, of spin 0, 1 and 2, respectively map $u \otimes v$ to

$$\begin{aligned}
T(u \otimes v) &= (u \cdot v)\, e_a \otimes e_a \\
A(u \otimes v) &= \frac{1}{2}(u_a v_b - u_b v_a)\, e_a \otimes e_b \\
S(u \otimes v) &= \frac{1}{2}(u_a v_b + u_b v_a)\, e_a \otimes e_b - (u_a v_a)\, e_a \otimes e_a
\end{aligned}$$

The line $L = \hom_{\mathbf{A}}(V, V^2)$ is spanned by the map F defined by

$$F(e_a) = i\, e_{a-1} \wedge e_{a+1}$$

and the line $\widetilde{L} = \hom_{\mathbf{A}}(V^2, V)$ is spanned by the bracket, noted $\widetilde{F}$:

$$\widetilde{F}(e_a \otimes e_b) = [e_a, e_b] = i\, \varepsilon_{abc}\, e_c$$

All these morphisms of representations satisfy the relations

$$\begin{aligned}
\widetilde{F}F &= 2I \\
F\widetilde{F} &= A \\
T+A+S &= I^2 \\
(\widetilde{F}\otimes I)(I\otimes F) &= (I\otimes \widetilde{F})(F\otimes I) \\
&= T+2A-2S \\
F &= (\widetilde{F}\otimes I)(I\otimes F)F \\
\widetilde{F} &= \widetilde{F}(I\otimes \widetilde{F})(F\otimes I)
\end{aligned}$$

5. The chromatic stack, φ

The notion of combinatorial stack appeared in [6] and we used it in [2] to give a construction of non-abelian G-gerbes over a simplicial complex. Dually, we can also use coefficients in a category of representation. Thus, we define the chromatic stack, φ, as a 2-functor which represents the simplicial homotopy groupoid $\Pi_1(\mathscr{P})$ into the 2-category of $\mathbf{A}$-modules. φ is generated by pasting the following data :

$$\begin{aligned}
\varphi_x &= \mathbf{A} \\
\varphi_{xy} &= (V\otimes - : \varphi_y \longrightarrow \varphi_x) \\
\varphi_{(x_0,\cdots,x_\ell)} &= (V^\ell \otimes - : \varphi_{x_\ell} \longrightarrow \varphi_{x_0}) \\
\varphi_\sigma &= F \quad \text{if} \quad \sigma = ((xyz),(xz)) \\
&= \widetilde{F} \quad \text{if} \quad \sigma = ((xz),(xyz)) \\
\varphi_{\gamma\gamma'} &= (I^k \otimes \varphi_\sigma \otimes I^{\ell-k-1} : \varphi_{\gamma'} \longrightarrow \varphi_\gamma) \\
\varphi_{(\gamma_0,\cdots,\gamma_n)} &= (\varphi_{\gamma_0\gamma_1}\circ\cdots\circ\varphi_{\gamma_{n-1}\gamma_n} : \varphi_{\gamma_n} \longrightarrow \varphi_{\gamma_0})
\end{aligned}$$

The 1-connection of φ is the family of functors $(\varphi_\gamma)_{\gamma\in\mathscr{P}_0}$. The 2-connection of φ is the family of natural transformations $(\varphi_\Gamma)_{\Gamma\in\mathscr{P}_2}$. In order to compute the chromatic index, we choose a 2-loop, $\Gamma = (\gamma_0,\cdots,\gamma_n)$, based at $(a,b) = \gamma_0 = \gamma_n$, and sweeping each triangle of T once only. To each path $\gamma_p = (a, x_{p1},\cdots, x_{p\ell_{p-1}}, b)$, of length $|\gamma_p| = \ell_p$, φ associates a copy of V^{ℓ_p}. For each $p\in\{2,\cdots,n\}$, the loop γ_p differs from γ_{p-1} either by the insertion of a vertex $y \in T_0$ between x_{p-1,k_p} and x_{p-1,k_p+1} or by the deletion of x_{p-1,k_p}, where x_{p-1,k_p-1} and x_{p-1,k_p+1} are supposed to be adjacent. Each such move is represented by a linear map of the form

$$\varphi_{\gamma_{p-1}\gamma_p} = F_{k_p\ell_p} = (I_{V^{k_p-1}} \otimes F \otimes I_{V^{\ell_p-k_p}} \ : \ V^{\ell_p} \longrightarrow V^{\ell_p+1})$$

if $\ell_{p-1} = \ell_p + 1$ and

$$\varphi_{\gamma_{p-1}\gamma_p} = \widetilde{F}_{k_p\ell_p} = (I_{V^{k_p-1}} \otimes \widetilde{F} \otimes I_{V^{\ell_p-k_p-1}} \ : \ V^{\ell_p} \longrightarrow V^{\ell_p-1})$$

if $\ell_{p-1} = \ell_p - 1$.

Since Penrose's formula looks like the partition function of a statistical model, it is natural to express K_T as the trace of a product of transfer matrices which represent linear maps between tensor powers of V. This approach will give us an efficient way to compute it, because the bad numberings are eliminated progressively during the sweeping process. Geometrically, the construction of the chromatic stack allows us to reinterpret K_T as a 2-holonomy, which is the categorical analogue of a holonomy in a fiber bundle.

Definition : *The 2-holonomy of φ on a 2-loop $\Gamma = (\gamma_0, \gamma_1, \cdots, \gamma_{n-1}, \gamma_0)$ based at γ_0, is the natural transformation*

$$\varphi_\Gamma = \varphi_{\gamma_0\gamma_1} \circ \cdots \circ \varphi_{\gamma_{n-1}\gamma_0} \quad : \quad \varphi_{\gamma_0} \longrightarrow \varphi_{\gamma_0}$$

When $\gamma_0 = (a)$, φ_Γ is an endomorphism of $\varphi_{\gamma_0} = \mathrm{Id}_{\mathbf{A}}$ so that φ_Γ defines canonically a complex number. Moreover, after the following theorem, which illustrates the pasting lemma [11] in the 2-category of A-modules, the trace of $\varphi_\Gamma \in \mathrm{End}(\varphi_{\gamma_0})$ depends only on T and not on the 2-path Γ.

Theorem 2 : *If Γ is a 2-loop which sweeps each triangle of T once only, then the trace of the 2-holonomy of φ along Γ, evaluated in the representation associated to the base path of Γ, is the chromatic index of T :*

$$\boxed{\mathrm{tr}_{\varphi_{\gamma_0}}(\varphi_\Gamma) = K_T} \tag{4}$$

Proof: Let $\Gamma = (\gamma_0, \gamma_1, \cdots, \gamma_{n-1}, \gamma_0)$ be a 2-loop. Let $p \in \{0, \cdots, n-1\}$ and suppose that γ_{p+1} is obtained from γ_p by inserting y between x_j and x_{j+1}, with $x_j \neq y \neq x_{j+1} \neq x_j$:

$$\begin{aligned} \gamma_p &= (x_0, \cdots, x_\ell) \\ \gamma_{p+1} &= (x_0, \cdots, x_j, y, x_{j+1}, \cdots, x_\ell) \end{aligned}$$

Then the 2-arrow $\varphi_{\gamma_p\gamma_{p+1}}$ is the intertwiner

$$\varphi_{\gamma_p\gamma_{p+1}} = I_{\varphi_{x_0x_1}} \otimes \cdots \otimes I_{\varphi_{x_{j-1}x_j}} \otimes \widetilde{F} \otimes I_{\varphi_{x_{j+1,j+2}}} \otimes \cdots \otimes I_{\varphi_{x_{\ell-1}x_\ell}} = \widetilde{F}_{\ell k}$$

which is represented by the matrix M_p whose entries are given by

$$M_{p,ab} = \delta_{a_0b_0} \cdots \delta_{a_{j-1}b_{j-1}} \left(i\, \varepsilon_{a_jb_jb_{j+1}}\right) \delta_{a_{j+1}b_{j+2}} \cdots \delta_{a_{\ell-1}b_\ell}$$

If γ_{q+1} is obtained from γ_q by deleting a vertex between y_k and y_{k+1}, then $\varphi_{\gamma_q\gamma_{q+1}}$ is the intertwiner going backwards

$$\varphi_{\gamma_q\gamma_{q+1}} = I_{\varphi_{y_0y_1}} \otimes \cdots \otimes I_{\varphi_{y_{k-1}y_k}} \otimes F \otimes I_{\varphi_{y_{k+1,k+2}}} \otimes \cdots \otimes I_{\varphi_{y_{\ell-1}y_\ell}} = F_{\ell+1,k}$$

and is represented by the matrix whose entries are

$$M_{q,ab} = \delta_{a_0b_0} \cdots \delta_{a_{k-1}b_{k-1}} \left((-i)\, \varepsilon_{a_kb_{k+1}b_k}\right) \delta_{a_{k+2}b_{k+1}} \cdots \delta_{a_\ell b_{\ell-1}}$$

Now, let $a^p = \left(a_1^p, \cdots, a_{\ell_p}^p\right)$ be a generic multi-index for the basis vectors of the representation φ_{γ_p}, with $a_j^p \in \{1, 2, 3\}$ for $j = 1, \cdots, \ell_p$. The number $\mathrm{tr}_{\varphi_{\gamma_0}}(\varphi_\Gamma)$ is the trace of the product of these matrices :

$$\mathrm{tr}_{\varphi_{\gamma_0}}(\varphi_\Gamma) = \sum_a \prod_{p=0}^{n-1} M_{p,a^pa^{p+1}}$$

In this sum, a runs over the set of families $(a^0, \cdots, a^n)$ of multi-indices $a^p = (a_1^p, \cdots, a_{\ell_p}^p)$ with $a_i^p \in \{1, 2, 3\}$. To each edge of T are associated as many indices as there are paths γ_p which contain it. Let N_{xy} be the number of indices associated to (xy). Among them, $(N_{xy} - 2)$ indices are constrained by the δ's to be equal. Similarly, the two ε's associated to the two triangles which contain (xy) force the two remaining indices to take the same value. Since the δ's are sandwiched between these two ε's, these two indices are in fact equal and there is one and only one free index a_{xy} associated to each edge (xy). The various factors of the product are equal to one except for the ε's which can be indexed by the positively oriented triangles of T. Therefore, the precedent formula becomes

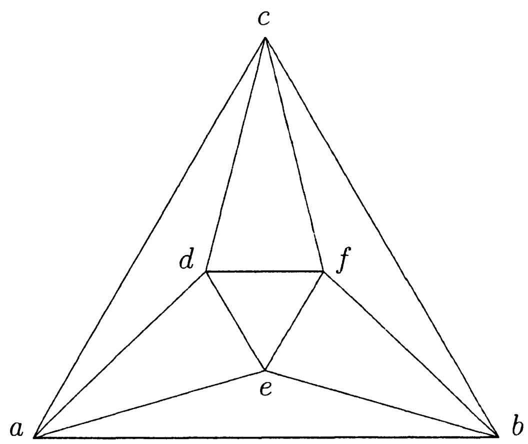
$$\begin{aligned}
\mathrm{tr}_{\varphi_{\gamma_0}}(\varphi_\Gamma) &= \sum_{(xy)\in T_1} \sum_{a_{xy}\in\{1,2,3\}} \Big(\prod_{[xyz]\in T_2} i\, \varepsilon_{a_{xy}a_{yz}a_{zx}} \Big) \\
&= \sum_u \prod_{[xyz]} i \det(u_{xy}, u_{yz}, u_{zx})
\end{aligned}$$

where u describes the set of all maps from T_1 to $\{e_1, e_2, e_3\}$ and the triangles $[xyz]$ all have the same orientation.

□

Initially, K_T is defined as a sum of 3^{t_1} terms and most of them vanish. By working in the tensor algebra, $T(V)$, the bad bumberings are eliminated during the sweeping process and the computation is much quicker if we use formula (4). Moreover, this method provides explicitely all good numberings.

Example : Let us apply the relation (4) to the computation of the chromatic index of the octahedron.



We sweep this triangulation with the 2-path

$$\Gamma = \big((ab), (aeb), (adeb), (adefb), (adfb), (adcfb), (acfb), (acb), (ab)\big)$$

For simplicity, we will write $\mathbf{a}_1 \cdots \mathbf{a}_\ell$ for $e_{a_1} \otimes \cdots \otimes e_{a_\ell}$ with $\mathbf{a}_i \in \{\mathbf{1}, \mathbf{2}, \mathbf{3}\}$. The successive images of $\mathbf{1}$ via the maps $\varphi_{\gamma\gamma'}$ are :

$$\begin{aligned}
\mathbf{1} &\mapsto i(\mathbf{23}-\mathbf{32})\\
&\mapsto i^2(\mathbf{313}-\mathbf{133}-\mathbf{122}+\mathbf{212})\\
&\mapsto i^3(\mathbf{3112}-\mathbf{3121}-\mathbf{1312}+\mathbf{1321}\\
&\qquad -\mathbf{1231}+\mathbf{1213}+\mathbf{2131}-\mathbf{2113})\\
&\mapsto i^4(-\mathbf{331}-\mathbf{122}-\mathbf{111}-\mathbf{111}-\mathbf{133}-\mathbf{221})\\
&\mapsto i^5(-\mathbf{3121}+\mathbf{3211}-\mathbf{1312}+\mathbf{1132}-\mathbf{1231}+\mathbf{1321}\\
&\qquad -\mathbf{1231}+\mathbf{1321}-\mathbf{1123}+\mathbf{1213}-\mathbf{2311}+\mathbf{2131})\\
&\mapsto i^6(-\mathbf{221}-\mathbf{111}+\mathbf{212}-\mathbf{331}-\mathbf{221}\\
&\qquad -\mathbf{331}-\mathbf{221}+\mathbf{313}-\mathbf{111}-\mathbf{331})\\
&\mapsto i^7(\mathbf{23}+\mathbf{23}-\mathbf{32}+\mathbf{23}-\mathbf{32}+\mathbf{23}-\mathbf{32}-\mathbf{32})\\
&\mapsto i^8(\mathbf{1}+\mathbf{1}+\mathbf{1}+\mathbf{1}) \;=\; 4\cdot\mathbf{1}
\end{aligned}$$

Consequently, $K_{octa.} = 3!\cdot 4 = 24$ and there exist $4\cdot 24 = 96$ good colourings of the dual cube. We have made 64 operations instead of $3^{12} = 531441$. It would be interesting to evaluate the complexity of this method for generic triangulations. Using the same method, one can compute the chromatic index of the icosahedron and one finds $K_{ico.} = 60$, proving this way that there exist 240 good colourings of the faces of the dual dodecahedron.

6. A global section of φ

φ induces over $\mathscr{P}$ another fibered category, Φ, defined as follows. To each path $\alpha = (a_0, \cdots, a_\ell)$, we associate the category Φ_α whose objects are the sections of φ over α. These are the families of V-modules, $\zeta_{a_i} \in \mathrm{Ob}\,(\varphi_{a_i})$, connected by intertwiners :

$$\zeta = \left(\begin{tikzcd} \zeta_{a_{i-1}} \arrow[r, bend right, "\zeta_{a_i a_{i-1}}"'] & V\zeta_{a_i} \arrow[l, bend right, "\zeta_{a_{i-1}a_i}"'] \end{tikzcd} \right)_{1\leqslant i\leqslant \ell} \in \mathrm{Ob}\,(\Phi_\alpha)$$

If $\zeta, \omega \in \mathrm{Ob}\,(\Phi_\alpha)$, then $\hom_{\Phi_\alpha}(\zeta,\omega)$ is the vector space whose elements are the families $(u_i : \zeta_{a_i} \longrightarrow \omega_{a_i})_{0\leqslant i\leqslant \ell}$ of linear maps such that the following diagrams commute :

$$\begin{tikzcd}
\zeta_{a_{i-1}} \arrow[r, "\zeta_{a_i a_{i-1}}"] \arrow[d, "u_{i-1}"'] & V\zeta_{a_i} \arrow[d, "I\otimes u_i"] \\
\omega_{a_{i-1}} \arrow[r, "\omega_{a_i a_{i-1}}"'] & V\omega_{a_i}
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
\zeta_{a_{i-1}} \arrow[d, "u_{i-1}"'] & V\zeta_{a_i} \arrow[l, "\zeta_{a_{i-1} a_i}"'] \arrow[d, "I\otimes u_i"] \\
\omega_{a_{i-1}} & V\omega_{a_i} \arrow[l, "\omega_{a_{i-1} a_i}"]
\end{tikzcd}$$

that is to say :

$$\begin{aligned}
\omega_{a_i a_{i-1}} \circ u_{i-1} &= (I \otimes u_i) \circ \zeta_{a_i a_{i-1}} \\
u_{i-1} \circ \zeta_{a_{i-1} a_i} &= \omega_{a_{i-1} a_i} \circ (I \otimes u_i)
\end{aligned}$$

Definition : *Let* $\alpha = (a_0, \cdots, a_\ell)$ *be a path of length* ℓ *and let* $\zeta \in \mathrm{Ob}\,(\Phi_\alpha)$ *be a section of* φ *over* α. *The direct transport operator of* ζ *along* α *is the morphism*

$$\begin{aligned}
T_\alpha(\zeta) &= (I^{\ell-1} \otimes \zeta_{a_\ell a_{\ell-1}}) \circ (I^{\ell-2} \otimes \zeta_{a_{\ell-1} a_{\ell-2}}) \circ \cdots \circ (I \otimes \zeta_{a_2 a_1}) \circ \zeta_{a_1 a_0} \\
&: \ \zeta_{a_0} \longrightarrow V^\ell \zeta_{a_\ell}
\end{aligned}$$

and the inverse transport operator of ζ *is the morphism*

$$\begin{aligned}
\overline{T}_\alpha(\zeta) &= \zeta_{a_0 a_1} \circ (I \otimes \zeta_{a_1 a_2}) \circ \cdots \circ (I^{\ell-2} \otimes \zeta_{a_{\ell-2} a_{\ell-1}}) \circ (I^{\ell-1} \otimes \zeta_{a_{\ell-1} a_\ell}) \\
&: \ V^\ell \zeta_{a_\ell} \longrightarrow \zeta_{a_0}
\end{aligned}$$

Let us note that $T_{\overline{\alpha}} \neq \overline{T}_\alpha$. $\Phi_{|\mathscr{P}_1}$ is generated by its restriction to the oriented 2-cells of T. If $\sigma = \big((xz), (xyz)\big)$ and if $\zeta \in \mathrm{Ob}\,(\Phi_{(xyz)})$, then we define $\xi = \Phi_\sigma(\zeta) \in \mathrm{Ob}\,(\Phi_{(xz)})$ by

$$\begin{aligned}
\xi_x &= \zeta_x \\
\xi_z &= \zeta_z \\
\xi_{zx} &= (\widetilde{F} \otimes I_{\zeta_z}) \circ (I \otimes \zeta_{zy}) \circ \zeta_{yx} \\
\xi_{xz} &= \zeta_{xy} \circ (I \otimes \zeta_{yz}) \circ (F \otimes I_{\zeta_z})
\end{aligned}$$

Similarly, if $\xi \in \mathrm{Ob}\,(\Phi_{(xz)})$, we define $\zeta = \Phi_{\overline{\sigma}}(\xi) \in \mathrm{Ob}\,(\Phi_{(xyz)})$ by

$$\begin{aligned}
\zeta_x &= \xi_x \\
\zeta_y &= V\xi_z \\
\zeta_z &= \xi_z \\
\zeta_{yx} &= (F \otimes I_{\xi_z}) \circ \xi_{zx} \\
\zeta_{xy} &= \xi_{xz} \circ (\widetilde{F} \otimes I_{\xi_z}) \\
\zeta_{yz} &= I \otimes I_{\xi_z} \;=\; \zeta_{zy}
\end{aligned}$$

If $u \in \hom_{\Phi_{(xyz)}}(\zeta, \omega)$, then we have the commutative diagrams

$$\begin{tikzcd}
\zeta_x \arrow[r, "\zeta_{yx}"] \arrow[d, "u_x"'] & V\zeta_y \arrow[r, "I\otimes\zeta_{zy}"] \arrow[d, "I\otimes u_y"] & V^2\zeta_z \arrow[r, "\widetilde{F}\otimes I_{\zeta_z}"] \arrow[d, "I^2\otimes u_z"] & V\zeta_z \arrow[d, "I\otimes u_z"] \\
\omega_x \arrow[r, "\omega_{yx}"'] & V\omega_y \arrow[r, "I\otimes\omega_{zy}"'] & V^2\omega_z \arrow[r, "\widetilde{F}\otimes I_{\omega_z}"'] & V\omega_z
\end{tikzcd}$$

$$
\begin{tikzcd}
\zeta_x \arrow[d, "u_x"'] & V\zeta_y \arrow[l, "\zeta_{xy}"'] \arrow[d, "I\otimes u_y"] & V^2\zeta_z \arrow[l, "I\otimes\zeta_{yz}"'] \arrow[d, "I^2\otimes u_z"] & V\zeta_z \arrow[l, "F\otimes I_{\zeta_z}"'] \arrow[d, "I\otimes u_z"] \\
\omega_x & V\omega_y \arrow[l, "\omega_{xy}"] & V^2\omega_z \arrow[l, "I\otimes\omega_{yz}"] & V\omega_z \arrow[l, "F\otimes I_{\omega_z}"]
\end{tikzcd}
$$

and we can define the action of Φ_σ and of $\Phi_{\overline{\sigma}}$ on the arrows by

$$
\begin{aligned}
\Phi_\sigma(u_x, u_y, u_z) &= (u_x, u_z) \\
\Phi_{\overline{\sigma}}(v_x, v_z) &= (v_x, I \otimes v_z, v_z)
\end{aligned}
$$

These functors satisfy the relations :

$$
\begin{aligned}
\Phi_\sigma\Phi_{\overline{\sigma}}(\xi_x, \xi_{zx}, \xi_z) &= (\xi_x, 2\xi_{zx}, \xi_z) \\
\Phi_{\overline{\sigma}}\Phi_\sigma(\zeta_x, \zeta_{yx}, \zeta_y, \zeta_{zy}, \zeta_z) &= (\zeta_x, \zeta_{yx} \circ (A \otimes \zeta_{zy}), V\zeta_z, I \otimes I_{\zeta_z}, \zeta_z)
\end{aligned}
$$

If $(\alpha, \beta) \in \mathscr{P}_1$ is a generic 2-edge, then $\Phi_{\alpha\beta}$ acts locally as above without modifying the other entries.

For $p = 0, \cdots, n$, let $\Gamma_p = (\gamma_p, \cdots, \gamma_n)$ be the partial 2-path made of the last $(n - p + 1)$ entries of Γ and let

$$
\Phi_{\Gamma_p} = \Phi_{\gamma_p\gamma_{p+1}} \circ \cdots \circ \Phi_{\gamma_{n-1}\gamma_n} \quad : \quad \Phi_{\gamma_n} \longrightarrow \Phi_{\gamma_p}
$$

be the functor which maps the sections of φ over γ_n to sections over γ_p. For example, we can choose $\gamma_n = (ab)$. Let us apply Φ_{Γ_p} to the section $\zeta^n \in \mathrm{Ob}\,(\Phi_{(ab)})$ defined by

$$
\begin{aligned}
\zeta^n_a &= \zeta^n_b = V \\
\zeta^n_{ba} &= F \\
\zeta^n_{ab} &= \widetilde{F}
\end{aligned}
$$

$$
\zeta^n = \left(\begin{tikzcd} V \arrow[r, bend right, "F"'] & V^2 \arrow[l, bend right, "\widetilde{F}"'] \end{tikzcd} \right)
$$

Theorem 3 : Φ_Γ *multiplies the arrows of* ζ^n *by* K_T *:*

$$
\boxed{\Phi_\Gamma(\zeta^n) = \left(\begin{tikzcd} V \arrow[r, bend right, "K_T\,F"'] & V^2 \arrow[l, bend right, "K_T\,\widetilde{F}"'] \end{tikzcd} \right) \in \mathrm{Ob}\,(\Phi_{(ab)})}
$$

Proof: The inverse transport operator of $\zeta^p := \Phi_{\Gamma_p}(\zeta^n) \in \mathrm{Ob}\,(\Phi_{\gamma_p})$ is

$$
\overline{T}_{\gamma_p}(\zeta^p) = \overline{T}_{\gamma_p}\left(\Phi_{\gamma_p\gamma_{p+1}} \circ \cdots \circ \Phi_{\gamma_{n-1}\gamma_n}(\zeta^n)\right) : V^{\ell_p+1} \longrightarrow V
$$

By a decreasing induction on p, we have :

$$
\overline{T}_{\gamma_p}(\zeta^p) = \zeta^n_{ab} \circ (\varphi_{\Gamma_p} \otimes I) = \widetilde{F} \circ (\varphi_{\Gamma_p} \otimes I)
$$

$$
\begin{tikzcd}
V^{\ell_p+1} \arrow[r, bend left, "\varphi_{\Gamma_p}\otimes I"] \arrow[rr, bend right, "\overline{T}_{\gamma_p}(\zeta^p)"'] & V^2 \arrow[r, bend left, "\zeta^n_{ab}"] & V
\end{tikzcd}
$$

For $p = 0$:

$$\zeta^0_{ab} = \overline{T}_{\gamma_0}(\zeta^0) = \widetilde{F} \circ (\varphi_\Gamma \otimes I) = K_T \, \widetilde{F}$$

Similarly, by using the direct transport operator, we obtain

$$\begin{tikzcd} V \arrow[r, bend left, "\zeta^n_{ba}"] \arrow[rr, bend right, "T_{\gamma_p}(\zeta^p)"'] & V^2 \arrow[r, bend left, "\varphi_{\widetilde{\Gamma}_p} \otimes I"] & V^{\ell_p+1} \end{tikzcd}$$

$$\zeta^0_{ba} = T_{\gamma_0}(\zeta^0) = (\varphi_{\widetilde{\Gamma}} \otimes I) \circ F = K_T \, F$$

$\square$

Once $\Gamma = (\gamma_0, \cdots, \gamma_n)$ has been chosen, the sweeping process constructs a V-module, $\zeta_x = V^{n_x}$, for each $x \in T_0$, and a morphism, ζ_{xy}, for each oriented edge of T. Each integer n_x depends only on the partial 2-path Γ_p which reaches x first and not on the paths γ_q with $q < p$. Similarly for each arrow, ζ_{xy}. Therefore, we obtain a global section, ζ, of φ over T. More precisely, if we lift T to a triangulation $\widetilde{T}$ of the disk D^2 such that $\widetilde{T}_{|\partial D^2}$ be a pair of arcs both projected onto the base edge (ab), then ζ is a global section of the pull-back of φ to $\widetilde{T}$.

If $\zeta_{xy} = 0$ for some edge (xy), then the transport operator along a path γ_p containing (xy) vanishes, as well as the subsequent transport operators and, at the end, we obtain $K_T = 0$. Conversely, if $K_T = 0$, then there exists an edge (at least the last one) where ζ vanishes. Consequently, we have obtained a geometric interpretation of the four-colour theorem in terms of sections of φ :

Theorem 4 : *4CT* $\iff (\zeta_{xy} \neq 0 \quad \forall\,(xy))$.

Example : Let us construct ζ on the octahedron. Let us start from $\zeta_a = \zeta_b = V$ and $\zeta_{ab}^n = \widetilde{F}$:

$$
\begin{aligned}
\zeta_e &= V^2 \\
\zeta_{ae} &= \widetilde{F} \circ (\widetilde{F} \otimes I) \\
\zeta_{eb} &= I^2 \\
\zeta_d &= V^3 \\
\zeta_{ad} &= \widetilde{F} \circ (\widetilde{F} \otimes I) \circ (\widetilde{F} \otimes I^2) \\
\zeta_{de} &= I^3 \\
\zeta_f &= V^2 \\
\zeta_{ef} &= \widetilde{F} \otimes I \\
\zeta_{fb} &= I^2 \\
\zeta_{df} &= (I \otimes \widetilde{F} \otimes I) \circ (F \otimes I^2) \\
\zeta_c &= V^3 \\
\zeta_{dc} &= (I \otimes \widetilde{F} \otimes I) \circ (F \otimes I^2) \circ (\widetilde{F} \otimes I^3) \\
\zeta_{cf} &= I^3 \\
\zeta_{ac} &= \widetilde{F} \circ (\widetilde{F} \otimes I) \circ (\widetilde{F} \otimes I^2) \circ (I^2 \otimes \widetilde{F} \otimes I) \\
&\quad \circ (I \otimes F \otimes I^2) \circ (I \otimes \widetilde{F} \otimes I^3) \circ (F \otimes I^3) \\
\zeta_{cb} &= F \otimes I \\
\zeta_{ab}^0 &= \widetilde{F} \circ (\widetilde{F} \otimes I) \circ (\widetilde{F} \otimes I^2) \circ (I^2 \otimes \widetilde{F} \otimes I) \circ (I \otimes F \otimes I^2) \\
&\quad \circ (I \otimes \widetilde{F} \otimes I^3) \circ (F \otimes I^3) \circ (I \otimes F \otimes I) \circ (F \otimes I) \\
&= \widetilde{F} \circ (\varphi_\Gamma \otimes I) \\
&= K_T \widetilde{F}
\end{aligned}
$$

7. Conclusion and perspectives

The classical approaches to the four-colour problem study the local form of a planar map to prove its global colourability. This suggests the existence of a cohomological interpretation of this property. In the present work, we have constructed a global section of a fibered category modeled on

$\mathbf{Rep}_f(\mathfrak{sl}_2)$ and proved that the validity of the four-colour theorem is equivalent to the fact that this section does not vanish. We hope that the present approach will be a first step toward an algebraic proof and the understanding of the four-colour theorem.

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