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DESCENT EQUIVALENCE

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RESUME. Soit A: $\mathbf{C}^{op} \to \mathbf{CAT}$ une catégorie indexée par **C** et soit B un objet de **C**. Une équivalence de A-descente est un morphisme de **C**/B qui induit une équivalence entre les catégories de descente relatives à A de son domaine et de son codomaine.
Dans cette note, les auteurs étudient les propriétés de ces morphismes et ils obtiennent une caractérisation complète des morphismes qui sont des équivalences de descente pour toutes les catégories A.

ABSTRACT. For a **C**-indexed category A, an A-descent equivalence is a morphism of bundles in **C** which induces an equivalence between the A-descent categories of its domain and codomain.
In this note, properties of such morphisms are studied, and those morphisms which are A-descent equivalences for all **C**-indexed categories A are fully characterized.

0. Introduction. Descent Theory was developed by Grothendieck [1], [2] in the context of fibred categories. If the category **E** is fibred over

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the category $\mathbf{C}$ with pullbacks, then each morphism $p : E \to B$ of $\mathbf{C}$ is associated with its descent category $\mathrm{Des}_{\mathbf{E}}(p)$ (see, for example, [4], for details). Having defined descent structures, it seems natural to us to compare two bundles (E, p) and (X, φ) over B in the descent sense and to ask:

When do two bundles (E, p) and (X, φ) over B have the "same descent behavior"?

More clearly, we would like to know *under which conditions a morphism of the two bundles (E, p) and (X, φ) over B would render equivalent descent categories.* To this end, we shall examine here for morphisms of bundles the notion of descent equivalence, which was introduced in the first author's Ph.D. thesis [3], and study its properties.

We formulate this notion in the (essentially equivalent) language of internal categories and of indexed categories (see [5,6,7]), rather than that of fibrations, making extensive use of some of the results of [5], which we recall here in sufficient detail.

After some preliminary observations concerning descent equivalences and their comparison with effective descent morphisms, in Theorem 1 we give a somewhat surprising necessary and sufficient condition for a morphism of bundles to be a descent equivalence (with respect to *all* indexed categories): one just needs the existence of *any* morphism of bundles in the opposite direction. In Theorem 2, we characterize those descent equivalences whose domain or codomain is given by an effective descent morphism.

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1. Internal categories. Recall that *an internal category D* (cf. [6])

in $\mathbf{C}$ is given by a diagram

$$D_2 \;\substack{\xrightarrow{\pi_2} \\ \xrightarrow{m} \\ \xrightarrow[\pi_1]{}}\; D_1 \;\substack{\xrightarrow{d} \\ \xleftarrow{e} \\ \xrightarrow[c]{}}\; D_0$$

in $\mathbf{C}$, which satisfies

I1. $de = 1_{D_0} = ce$,

I2. $dm = d\pi_2,\ cm = c\pi_1$,

I3. $m(1_{D_1} \times m) = m(m \times 1_{D_1})$,

I4. $m < 1_{D_1}, ed > = 1_{D_1} = m < ec, 1_{D_1} >$,

where $D_2,\ \pi_1,\ \pi_2$ are given by the following pullback diagram in $\mathbf{C}$:

$$\begin{array}{ccc} D_2 & \xrightarrow{\pi_2} & D_1 \\ {\scriptstyle \pi_1}\downarrow & & \downarrow{\scriptstyle c} \\ D_1 & \xrightarrow[d]{} & D_0 \end{array}$$

An internal functor $f : D \to D'$ between two internal categories D, D' in $\mathbf{C}$ is given by two morphisms $f_0 : D_0 \to D'_0, f_1 : D_1 \to D'_1$ of $\mathbf{C}$ such that

F1. $f_0 d = d' f_1,\ f_0 c = c' f_1$,

F2. $f_1 e = e' f_0,\ f_1 m = m' f_2$, where $f_2 = f_1 \times f_1 : D_1 \times_{D_0} D_1 \to D'_1 \times_{D'_0} D'_1$.

Composition of internal functors is defined in the obvious way. Hence one obtains $\mathbf{cat}(\mathbf{C})$, the category of all internal categories and internal functors in $\mathbf{C}$. It is actually a 2-category (see [5]) since one can define the notion of *internal natural transformation* $\alpha : f \to g$ of internal functors $f,\ g : D \to D'$, given by a morphism $\alpha : D_0 \to D'_1$ in $\mathbf{C}$ such that

T1. $d'\alpha = f_0,\ c'\alpha = g_0$,

T2. $m' < \alpha c, f_1 > = m' < g_1, \alpha d >$.

The *composite* $\beta\alpha : f \to h$ of internal natural transformations $\alpha : f \to g$ and $\beta : g \to h$ is the morphism $m' < \beta, \alpha >: D_0 \to D'_1$, and the *identity internal natural transformation* $1_f : f \to f$ is the morphism $e' f_0 : D_0 \to D'_1$.

An internal functor $f : D \to D'$ of $\mathbf{C}$ is an *internal category equivalence* if there is an internal functor $g : D' \to D$ such that $gf \cong 1_D$ and $fg \cong 1_{D'}$.

For example, if $p : E \to B$ is a morphism in $\mathbf{C}$, then

$$(E \times_B E) \times_E (E \times_B E) \cong E \times_B E \times_B E \underset{\pi_{12}}{\overset{\pi_{23}}{\xrightarrow{\pi_{13}}}} E \times_B E \underset{\pi_1}{\overset{\pi_2}{\xleftarrow{e}}} E$$

is an internal category in $\mathbf{C}$, where $e =< 1_E, 1_E >$, (π_1, π_2) is the kernel pair of p, π_{12} and π_{23} are such that $\pi_1\pi_{23} = \pi_2\pi_{12}$ (pullback square) and $\pi_{13} =< \pi_1\pi_{12}, \pi_2\pi_{23} >$. This internal category is denoted by $\mathrm{Eq}(p)$. Every object B in $\mathbf{C}$ can be viewed as a discrete internal category B of $\mathbf{C}$:

$$B \underset{1_B}{\overset{1_B}{\xrightarrow{1_B}}} B \underset{1_B}{\overset{1_B}{\xleftarrow{1_B}}} B$$

Clearly, $\mathrm{Eq}(1_B)$ is isomorphic to the above discrete internal category B.

For any morphism $q : (E, p) \to (X, \varphi)$ in $\mathbf{C}/B$, as in [5] one constructs the internal functor

$$\tilde{q} : \mathrm{Eq}(p) \to \mathrm{Eq}(\varphi),$$

where $\tilde{q}_0 = q$, $\tilde{q}_1 = q \times_B q$. Then, for a fixed object B of $\mathbf{C}$, the assignments:

$$(E, p) \mapsto \mathrm{Eq}(p) \ \text{ and } \ q \mapsto \tilde{q},$$

define the functor

$$\mathrm{Eq}_B : \ \mathbf{C}/B \to \mathbf{cat}(\mathbf{C}).$$

2. Indexed categories. A $\mathbf{C}$-*indexed category* $\mathbb{A}$ or a pseudo-functor $\mathbb{A} : \mathbf{C}^{\mathrm{op}} \to \mathbf{CAT}$ (cf. [5,7,8]) consists of the following data:

· for every object E of $\mathbf{C}$ a category $\mathbb{A}^E$

· for every morphism $f : E \to D$ of $\mathbf{C}$ a functor $f^* : \mathbb{A}^D \to \mathbb{A}^E$,

· for every $f : E \to D$, $g : D \to B$ in $\mathbf{C}$, two natural isomorphisms:

$$i^D : 1_{\mathbb{A}^D} \to (1_D)^*, \quad j^{f,g} : f^*g^* \to (gf)^*$$

which make the diagrams

$$\begin{array}{ccc} f^* & \xrightarrow{f^* i^D} & f^*(1_D)^* \\ {\scriptstyle i^E f^*}\downarrow & \searrow{\scriptstyle 1_{f^*}} & \downarrow{\scriptstyle j^{f,1_D}} \\ (1_E)^* f^* & \xrightarrow[j^{1_E,f}]{} & f^* \end{array}$$

and

$$\begin{array}{ccc} f^*g^*h^* & \xrightarrow{f^* j^{g,h}} & f^*(hg)^* \\ {\scriptstyle j^{f,g}h^*}\downarrow & & \downarrow{\scriptstyle j^{f,hg}} \\ (gf)^*h^* & \xrightarrow[j^{gf,h}]{} & (hgf)^* \end{array}$$

commute.

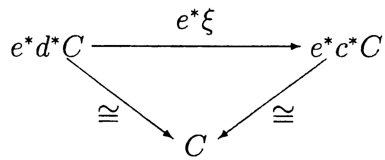
For example,

$$\mathbb{A} : \mathbf{C}^{\mathrm{op}} \to \mathbf{CAT}$$

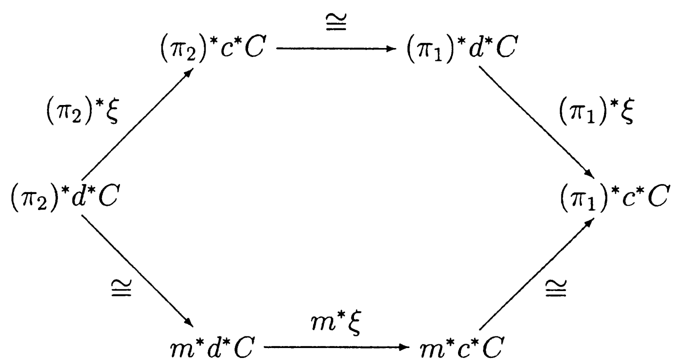
given by $B \longmapsto \mathbf{C}/B$ and $(f : E \to B) \longmapsto f^* : \mathbf{C}/B \to \mathbf{C}/E$, the pullback functor along f, is a $\mathbf{C}$-indexed category, also called the *basic* $\mathbf{C}$*-indexed category.*

Let D be an internal category in $\mathbf{C}$. One defines $\mathbb{A}^D$ (cf. [5]) to be the category with

- objects all pairs of (C, ξ), where $C \in \mathrm{ob}\mathbb{A}^{D_0}$ and $\xi : d^*C \to c^*C$ is a morphism in $\mathbb{A}^{D_1}$ such that



and



commute, in $\mathbb{A}^{D_0}$ and $\mathbb{A}^{D_2}$, respectively, with the above natural isomorphisms arising from I1 and I2,

- morphisms $h : (C, \xi) \to (C', \xi')$ of $\mathbb{A}^D$ given by morphisms $h : C \to C'$ of $\mathbb{A}^{D_0}$ such that

$$\begin{array}{ccc} d^*C & \xrightarrow{d^*h} & d^*C' \\ {\scriptstyle \xi}\downarrow & & \downarrow{\scriptstyle \xi'} \\ c^*C & \xrightarrow{c^*h} & c^*C' \end{array}$$

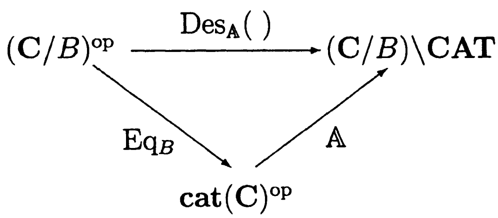
commutes in $\mathbb{A}^{D_1}$.

In [5], it was proved that, *for every* $\mathbf{C}$*-indexed category* $\mathbb{A} : \mathbf{C}^{\mathrm{op}} \longrightarrow \mathbf{CAT}$, *the extension*

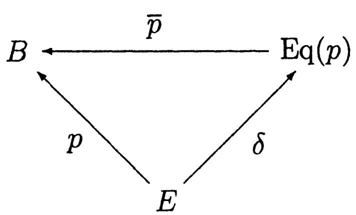
$$\mathbb{A} : \mathbf{cat}(\mathbf{C})^{\mathrm{op}} \rightarrow \mathbf{CAT}$$

given by the assignment $D \longrightarrow \mathbb{A}^D$, *is a pseudo-functor of 2-categories.* As a consequence, one obtains that *for every internal category equivalence* $f : D \longrightarrow D'$ *of* $\mathbf{C}$, *the functor* $f^* : \mathbb{A}^{D'} \longrightarrow \mathbb{A}^D$ *is an equivalence of categories.*

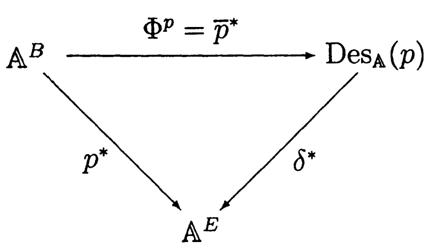
3. Effective descent, descent equivalence. Now, let $\mathrm{Des}_{\mathbb{A}}$ be the pseudo-functor $\mathbb{A}\circ\mathrm{Eq}_B$:



The discrete functor $p : E \rightarrow B$ can be factored as

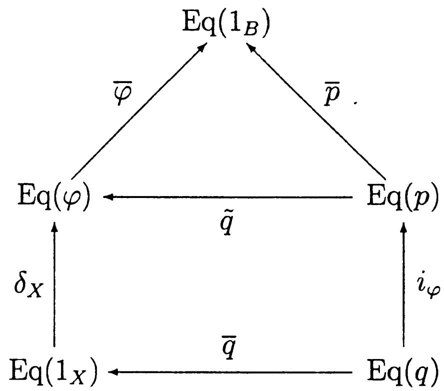


where $\overline{p}_0 = p$, $\overline{p}_1 = p\pi_1 = p\pi_2$, $\delta_0 = 1_E$, $\delta_1 = e =< 1_E, 1_E >$, with (π_1, π_2) the kernel pair of p. Applying $\mathbb{A}$ to the last diagram, one has a commutative diagram (up to natural isomorphism) in $\mathbf{CAT}$:



p is called *an $\mathbb{A}$-descent morphism (effective $\mathbb{A}$-descent morphism)* if the comparison functor Φ^p is full and faithful (an equivalence of categories). p is called *an absolute (effective) descent morphism* if it is an (effective) $\mathbb{A}$-descent morphism for every **C**-indexed category $\mathbb{A}$.

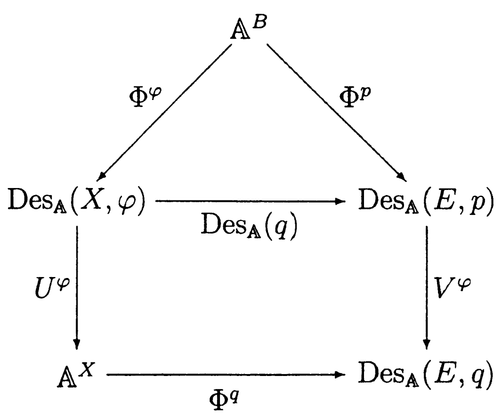
For a morphism $q : (E,p) \to (X,\varphi)$ in $\mathbf{C}/B$, the authors of [9] considered the following diagram in $\mathbf{cat}(\mathbf{C})$:



(1)

where $(i_\varphi)_0 = 1_E$, $(i_\varphi)_1 = 1_E \times_\varphi 1_E$, $\tilde{q}_0 = q$, $\tilde{q}_1 = q \times_B q$, $(\delta_X)_0 = 1_X$, $(\delta_X)_1 = \Delta_X$, $\overline{p}_0 = p$, $\overline{p}_1 = p\pi_1 = p\pi_2$, and where (π_1, π_2) is the kernel pair of p.

Applying $\mathbb{A}$ to diagram (1), one obtains the following commutative diagram (up to natural isomorphisms) in **CAT**:



(2)

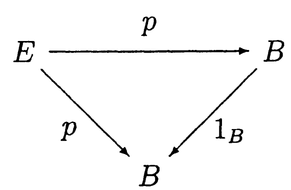
where $U^\varphi = \delta_X^*$, $V^\varphi = i_\varphi^*$, $\Phi^p = \overline{p}^*$, and $\mathrm{Des}_{\mathbb{A}}(q) = \tilde{q}^*$.

Definition. *Let $q : (E, p) \to (X, \varphi)$ be a morphism in $\mathbf{C}/B$. We call q* an $\mathbb{A}$-descent equivalence ($\mathbb{A}$-descent pre-equivalence) *if* $\mathrm{Des}_{\mathbb{A}}(q)$ *is an equivalence of categories (full and faithful). We call q* an absolute descent equivalence (absolute descent pre-equivalence) *if* $\mathrm{Des}_{\mathbb{A}}(q)$ *is an equivalence of categories (full and faithful) for every $\mathbf{C}$-indexed category $\mathbb{A}$.*

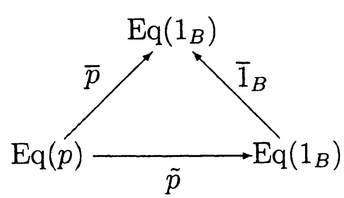
4. Properties of descent equivalences. Functoriality of $\mathrm{Des}_{\mathbb{A}}(\ )$ leads immediately to a number of consequences.

Proposition 1. *The morphism $p : (E, p) \to (B, 1_B)$ in $\mathbf{C}/B$ is an $\mathbb{A}$-descent pre-equivalence ($\mathbb{A}$-descent equivalence) if and only if p is an $\mathbb{A}$-descent (effective $\mathbb{A}$-descent) morphism.*

Proof. Applying Eq to the following commutative diagram:



we obtain the commutative diagram:



Clearly, with the notation of the previous section, $\overline{1}_B = 1_{\mathrm{Eq}(1_B)}$, $\tilde{p} = \overline{p}$. Hence, $\tilde{p}^*$ is an equivalence of categories if and only if $\overline{p}^*$ is an equivalence of categories, as desired. □

One also easily obtains:

Proposition 2. *Let $q : (E, p) \to (X, \varphi)$, $r : (X, \varphi) \to (Y, \xi)$ be morphisms in $\mathbf{C}/B$.*

(1) *If two of* q, r, *and* rq *are* $\mathbb{A}$-*descent equivalences, so is the third one.*

(2) *If* r *is an* $\mathbb{A}$-*descent equivalence, then* q *is an* $\mathbb{A}$-*descent pre-equivalence if and only if* rq *is an* $\mathbb{A}$-*descent pre-equivalence.*

□

It is also easy to show that $\mathbb{A}$-descent (pre-)equivalences have the intended invariance property:

Proposition 3. *Let* $q : (E,p) \to (X,\varphi)$ *be an* $\mathbb{A}$-*descent pre-equivalence* ($\mathbb{A}$-*descent equivalence) in* $\mathbf{C}/B$. *Then* p *is an* $\mathbb{A}$-*descent (effective* $\mathbb{A}$-*descent) morphism if and only if* φ *is an* $\mathbb{A}$-*descent (effective* $\mathbb{A}$-*descent) morphism.*

Proof. By Diagram (2), $\mathrm{Des}_{\mathbb{A}}(q)\Phi^{\varphi} = \Phi^{p}$ (up to natural isomorphism). If q is an $\mathbb{A}$-descent equivalence, then $\mathrm{Des}_{\mathbb{A}}(q)$ is an equivalence of categories. Therefore, Φ^{φ} is an equivalence of categories if and only if Φ^{p} is an equivalence of categories. Hence p is an effective $\mathbb{A}$-descent morphism if and only if φ is an effective $\mathbb{A}$-descent morphism.

Suppose now that q is an $\mathbb{A}$-descent pre-equivalence. Then $\mathrm{Des}_{\mathbb{A}}(q)$ is full and faithful. If φ is $\mathbb{A}$-descent pre-quivalence morphism, then $\Phi^{p} = \mathrm{Des}_{\mathbb{A}}(q)\Phi^{\varphi}$ (up to isomorphism) is full and faithful. Hence p is an $\mathbb{A}$-descent morphism. On the other hand, if p is $\mathbb{A}$-descent morphism, then $\mathrm{Des}_{\mathbb{A}}(q)\Phi^{\varphi} = \Phi^{p}$ (up to isomorphism) is full and faithful, and so is Φ^{φ}. □

5. A necessary and sufficient condition for absolute descent equivalences. In any category, the absolutely effective descent morphisms are precisely the split epimorphisms [5]. A characterization of the absolute descent equivalences is given by the following:

Theorem 1. *Let* $q : (E,p) \to (X,\varphi)$ *be a morphism in* $\mathbf{C}/B$. *Then* q *is an absolute descent equivalence if and only if there is any morphism* $s : (X,\varphi) \to (E,p)$ *in* $\mathbf{C}/B$.

Proof. $\Longleftarrow$: By hypothesis, we have $p = \varphi q$ and $ps = \varphi$. So there exist two internal functors

$$\tilde{s} : \mathrm{Eq}(\varphi) \to \mathrm{Eq}(p) \text{ and } \tilde{q} : \mathrm{Eq}(p) \to \mathrm{Eq}(\varphi).$$

We claim that $\tilde{s}\tilde{q} \cong 1_{\mathrm{Eq}(p)}$ and $\tilde{q}\tilde{s} \cong 1_{\mathrm{Eq}(\varphi)}$. In order to prove this it suffices to construct natural transformations between the respective pairs of functors since all natural transformations between internal functors whose codomain is a groupoid are natural isomorphisms. To this end we define $\alpha : \tilde{s}\tilde{q} \to 1_{\mathrm{Eq}(p)}$ by $\alpha =< 1_E, sq >: E \to E \times_B E$ in $\mathbf{C}$. It is easy to check that

$$\pi_2\alpha = sq,\ \pi_1\alpha = 1_E,$$

and

$$\pi_{13} < \alpha\pi_1, (s \times_B s)(q \times_B q) >= \pi_{13} < 1_{E\times_B E}, \alpha\pi_2 > .$$

Hence α is an internal natural transformation.

Similarly one shows that $\beta : 1_{\mathrm{Eq}(\varphi)} \to \tilde{q}\tilde{s}$, given by $\beta =< qs, 1_X >: X \to X \times_B X$, is an internal natural transformation. Therefore, $\mathrm{Des}_{\mathbb{A}}(q)$ is an equivalence of categories.

$\Longrightarrow$: We show more precisely:

(1) *If* $\mathrm{Des}_{\mathbb{A}}(q)$ *is essentially surjective on objects for every* $\mathbf{C}$*-indexed category* $\mathbb{A}$, *then there is a morphism* $s : X \to E$ *in* $\mathbf{C}$ *with* $psq = p$;

(2) *If, furthermore,* $\mathrm{Des}_{\mathbb{A}}(q)$ *is full and faithful for every* $\mathbb{A}$, *then* s *of* (1) *yields a morphism* $s : (X, \varphi) \to (E, p)$ *in* $\mathbf{C}/B$.

(1) Consider the $\mathbf{C}$-indexed category $\mathbb{A}_p$ of Theorem 3.5 [5]:

$$\begin{array}{ccc} \mathbf{C}^{\mathrm{op}} & \xrightarrow{\mathbb{A}_p} & \mathbf{CAT} \\ \\ A & \longmapsto & \mathbf{C}(A,\ E) \\ \uparrow{\scriptstyle t} & & \uparrow{\scriptstyle t^*} \\ B & \longmapsto & \mathbf{C}(B,\ E) \end{array}$$

where $\mathbb{A}_p^A = \mathbf{C}(A,\ E)$ carries an equivalence relation given by

$$u \sim v \Leftrightarrow pu = pv,$$

making it a category (in fact, a groupoid), and where $t^* : \mathbf{C}(B,\ E) \to \mathbf{C}(A,\ E)$ is the composition functor with t. Since

$$p\pi_1 = p\pi_2, \pi_1^*(1_E) = \pi_1 \sim \pi_2 = \pi_2^*(1_E),$$

the object 1_E of $\mathbb{A}_p^E$ has a descent structure $\xi : \pi_2^*(1_E) \to \pi_1^*(1_E)$, where (π_1, π_2) is the kernel pair of p. Hence, by diagram (2) and the proof of Theorem 2.5 [5],

$$V^\varphi(1_E, \xi) = (i_\varphi)^*(1_E, \xi) = ((1_E)^*(1_E), \xi_{i_\varphi}) = (1_E, \xi') \in \mathrm{Des}_{\mathbb{A}_p}(E, q).$$

But $\mathrm{Des}_{\mathbb{A}_p}(q)$ is essentially surjective, so there is $(s, \mu) \in \mathrm{Des}_{\mathbb{A}_p}(X, \varphi)$ such that $\mathrm{Des}_{\mathbb{A}_p}(q)(s, \mu) \cong (1_E, \xi)$, and therefore $V^\varphi \mathrm{Des}_{\mathbb{A}_p}(q)(s, \mu) \cong V^\varphi(1_E, \xi) = (1_E, \xi')$. That is $\Phi^q U^\varphi(s, \mu) \cong (1_E, \xi')$. But Φ^q is just a lifting of q^*,

$$q^* U^\varphi(s, \mu) \cong \delta^* \Phi^q U^\varphi(s, \mu) \cong \delta^*(1_E, \xi').$$

Hence $q^* s \sim 1_E$ in $\mathbb{A}_p^E$, and therefore $psq = p$.

(2) In order to prove that $ps = \varphi$, again, we consider the C-indexed category $\mathbb{B} : \mathbf{C}^{\mathrm{op}} \to \mathbf{CAT}$ of Theorem 3.5 of [5] with $\mathbb{B}^A = \mathbf{C}(A, B)$ considered a discrete category, for every $A \in \mathbf{C}$, and with t^* the composition functor with t, for every $t : A \to B$ in $\mathbf{C}$. It is easy to check that $(ps, 1)$ and $(\varphi, 1)$ are objects of $\mathrm{Des}_{\mathbb{B}}(X, \varphi)$ and that

$$\mathrm{Des}_{\mathbb{B}}(q)(ps, 1) = \mathrm{Des}_{\mathbb{B}}(q)(\varphi, 1) = (p, 1),$$

by the fact that $psq = p$. Since $\mathrm{Des}_{\mathbb{B}}(q)$ is full and faithful, $(ps, 1)$ is isomorphic to $(\varphi, 1)$, which yields $ps = \varphi$. □

From Theorem 1 one obtains:

Corollary 1. *Let $q : E \to X$ and $\varphi : X \to B$ be two morphisms of $\mathbf{C}$. Then $q : (E, \varphi q) \to (X, \varphi)$ is an absolute descent equivalence if and only if there is a morphism $s : X \to E$ in $\mathbf{C}$ such that $\varphi q s = \varphi$*

Corollary 2. *Let $q : (E, p) \to (X, \varphi)$ be a morphism in $\mathbf{C}/B$. Then q is an absolute descent equivalence if either q is a split epimorphism in $\mathbf{C}$ or q is a split monomorphism in $\mathbf{C}/B$.*

Remark. Corollary 1 implies in particular that split epimorphisms are the absolutely effective descent morphisms (see Thm. 3.5 of [5]). In fact, if $p : E \to B$ has a splitting s with $ps = 1_B$, then we may apply Corollary 1 to $p : (E, p) \to (B, 1_B)$, so that with 1_B also p is an absolute effective descent morphisms (i.e., effective descent w.r.t. every $\mathbf{C}$-indexed category $\mathbb{A}$), by Proposition 3.

6. Descent equivalences whose domain or codomain is effective descent. With the help of Corollary 2, Proposition 3 can be refined, as follows. Given any morphism $q : (E, p) \to (X, \varphi)$ in $\mathbf{C}/B$, we form the pullback diagram

$$\begin{array}{ccc} & E \times_B X & \\ {}^{\pi_1}\swarrow & & \searrow^{\pi_2} \\ E & & X \\ {}_{p}\searrow & & \swarrow_{\varphi} \\ & B & \end{array} \qquad (3)$$

in which π_1 is a split epimorphism. Hence $\pi_1 : (E \times_B X, p\pi_1) \to (E, p)$ is an absolute descent equivalence, by Corollary 2.

Theorem 2. *The following conditions are equivalent:*

(i) *p is an effective $\mathbb{A}$-descent morphism and $\pi_2 : (E \times_B X, p\pi_1) \to (X, \varphi)$ is an $\mathbb{A}$-descent equivalence,*

(ii) *φ is an effective $\mathbb{A}$-descent morphism, and $q : (E, p) \to (X, \varphi)$ is an $\mathbb{A}$-descent equivalence.*

Proof. (i)$\Longrightarrow$(ii): By Prop.1, $p : (E, p) \to (B, 1_B)$ is an $\mathbb{A}$-descent equivalence. Since π_1 is an $\mathbb{A}$-descent equivalence, also $p\pi_1 = \varphi\pi_2 : (E \times_B X, p\pi_1) \to (B, 1_B)$ is an $\mathbb{A}$-descent equivalence, and so is φ :

$(X, \varphi) \to (B, 1_B)$, by Prop.2 and the hypothesis on π_2. Then, another application of Propositions 1 and 2 gives (ii).

(ii)$\Longrightarrow$(i): By Prop.3, p is an effective $\mathbb{A}$-descent morphism. As before then, $p\pi_1 = \varphi\pi_2$ is an $\mathbb{A}$-descent equivalence, and so are q (by hypothesis), p, φ, and then π_2, by repeated application of Propositions 1 and 2. □

Remark. We note that in (i) it is enough to assume that $\mathrm{Des}_{\mathbb{A}}(\pi_2)$ be full and faithful, rather than an equivalence of categories. Indeed, since π_1 is an $\mathbb{A}$-descent equivalence, also $p\pi_1 = \varphi\pi_2$ is an $\mathbb{A}$-descent equivalence when p is an effective $\mathbb{A}$-descent morphism, which implies $\mathrm{Des}_{\mathbb{A}}(\pi_2)$ is essentially surjective on objects.

If $\mathbb{A}$ *is the basic fibration*, Theorem 2 may be simplified, as follows:

Corollary 3. *For any morphism* $q : (E, p) \to (X, \varphi)$ *in* $\mathbf{C}/B$, p *is an effective descent morphism if and only if* φ *is an effective descent morphism and* $q : (E, p) \to (X, \varphi)$ *is a descent equivalence.*

Proof. Using pullback-stability of effective descent morphisms (see [10]) and the composition-cancellation rule of [9], for "only if" one can argue as in (i)$\Longrightarrow$(ii) of Theorem 3. Likewise for "if". □

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