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ORDER CONVERGENCE AND COMPACTNESS

by D. van der ZYPEN

Résumé. Soit $(P, \leq)$ un ensemble partiellement ordonné et soit τ une topologie compacte sur P qui est plus fine que la topologie d'intervalles. Alors τ est contenu dans la topologie de convergence d'ordre. [1]

1 Topologies on a given poset

On any given partially ordered set $(P, \leq)$ there are topologies arising from the given order in a natural way (see also [2]). Perhaps the best known such topology is the *interval topology*. Set $\mathcal{S}^- = \{P \setminus (x] : x \in P\}$, and $\mathcal{S}^+ = \{P \setminus [x) : x \in P\}$ where $(x] = \{y \in P : y \leq x\}$ and $[x) = \{y \in P : y \geq x\}$. Then $\mathcal{S} = \mathcal{S}^- \cup \mathcal{S}^+$ is a subbase for the interval topology $\tau_i(P)$ on P.

There is another natural way to endow an arbitrary poset $(P, \leq)$ with a topology. We want to describe this topology in the following.

A *(set) filter* $\mathcal{F}$ on $(P, \leq)$ is a nonempty subset of the powerset of P such that

– $\emptyset \notin \mathcal{F}$

– $U, V \in \mathcal{F}$ implies $U \cap V \in \mathcal{F}$

– $U \in \mathcal{F}$ and $V \supseteq U$ imply $V \in \mathcal{F}$.

(Note that the above concept can of course be defined for arbitrary sets.) For any subset $A \subseteq P$ let *the set of lower bounds of* A be denoted by $A^l = \{x \in P : x \leq a \text{ for all } a \in A\}$ and the set of upper bounds by $A^u = \{x \in P : x \geq a \text{ for all } a \in A\}$. If $\mathcal{S}$ is a collection of subsets of P then we set $\mathcal{S}^l = \bigcup\{S^l : S \in \mathcal{S}\}$, similarly set $\mathcal{S}^u = \bigcup\{S^u : S \in \mathcal{S}\}$.

Let $A \subseteq P$ be a subset of a poset P and $y \in P$. We say that y is the infimum of A if y is the greatest element of A^l and write $\bigwedge A = y$. Dually we define

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the supremum of A, written $\bigvee A$. Note that in general, suprema and infima need not exist.

Let $\mathcal{F}$ be a filter on a poset P and let $x \in P$. We say that $\mathcal{F}$ **order-converges to** x, in symbols $\mathcal{F} \dot{\rightarrow} x$, if $\bigwedge \mathcal{F}^u = x = \bigvee \mathcal{F}^l$. Note that the principal ultrafilter consisting of the subsets of P that contain x order-converges to x.

Now we are able to define the **order convergence topology** $\tau_o(P)$ (called order topology in [1]) on any given poset P by:

$$\tau_o(P) = \{U \subseteq P : \text{for any } x \in U \text{ and any filter } \mathcal{F} \text{ with } \mathcal{F} \dot{\rightarrow} x \text{ we have } U \in \mathcal{F}\}.$$

It is straightforward to verify that this is a topology. Indeed, $\tau_o(P)$ is the finest topology on P such that order convergence implies topological convergence (which is not hard to prove either). We will make constant use of the following facts:

FACT 1.1. *Let P be a poset, let $\mathcal{F}$ be a filter on P. Then:*

1. $x \in \mathcal{F}^u \Leftrightarrow (x] \in \mathcal{F}$ *and* $x \in \mathcal{F}^l \Leftrightarrow [x) \in \mathcal{F}$.
2. *If* $\mathcal{F} \dot{\rightarrow} x$ *then* $\mathcal{F}^u \neq \emptyset \neq \mathcal{F}^l$.
3. *Suppose* $\mathcal{F} \dot{\rightarrow} x$. *If* $x \not\leq a$ *then* $P \setminus (a] \in \mathcal{F}$. *Dually if* $x \not\geq b$ *then* $P \setminus [b) \in \mathcal{F}$.
4. *If* $\mathcal{F} \dot{\rightarrow} x$ *and* $\mathcal{G}$ *is a filter on P with* $\mathcal{G} \supseteq \mathcal{F}$ *then* $\mathcal{G} \dot{\rightarrow} x$.

Proof. The proofs of assertions 1 and 2 are straightforward, and assertion 3 follows directly from [1], p. 3. We prove assertion 4. Since $\mathcal{G}^u \supseteq \mathcal{F}^u$ it suffices to show that $\mathcal{G}^u \subseteq [x)$ in order to get $\bigwedge \mathcal{G}^u = x$. Assume that there is $y \in \mathcal{G}^u \setminus [x)$. By assertion 1, $(y] \in \mathcal{G}$. Since we have $x \not\leq y$, we get $P \setminus (y] \in \mathcal{F} \subseteq \mathcal{G}$ (by assertion 3). So $(y] \cap (P \setminus (y]) = \emptyset \in \mathcal{G}$, which is a contradiction. The statement $\bigvee \mathcal{G}^l = x$ is proved similarly. □

2 The result

Note that 1.1, assertion 3 implies that for any poset P, the interval topology $\tau_i(P)$ is contained in the order convergence topology $\tau_o(P)$. The following theorem connects the concepts of interval topology, order convergence and compactness.

THEOREM 2.1. *Let $(P, \leq)$ be a poset. If τ is a compact topology on P such that $\tau_i(P) \subseteq \tau$, then $\tau \subseteq \tau_o(P)$.*

Proof. Suppose that $W \in \tau \setminus \tau_o(P)$. Then there is $x \in W$ and a filter $\mathcal{F}$ on P such that $\mathcal{F} \dot{\rightarrow} x$ and $W \notin \mathcal{F}$.

The strategy now is to find an ultrafilter on the closed set $Q := P \setminus W$ of (P, τ) that does not converge to any point of Q with respect to the subspace topology of (P, τ) on Q. This will imply that Q is a non-compact closed subset of (P, τ), which in turn implies that (P, τ) is not compact.

Note that every element of $\mathcal{F}$ intersects Q (otherwise we would have $W \in \mathcal{F}$). So $\mathcal{F} \cup \{Q\}$ is a filter base which is contained in some ultrafilter $\mathcal{U}$. Moreover, by 1.1, assertion 4, the ultrafilter $\mathcal{U}$ order-converges to x.

It is easy to check that

$$\mathcal{U}|_Q = \{U \cap Q : U \in \mathcal{U}\}$$

is an ultrafilter on Q (this uses of course the fact that Q is a member of $\mathcal{U}$).

Claim: $\mathcal{U}|_Q$ does not converge to any $y \in Q$ with respect to $\tau|_Q$, the topology on Q induced by τ.

Proof of Claim: Pick any $y \in Q$. First, we know that $x \in W$ and $y \in Q$, whence $x \neq y$. Suppose that the following holds in P:

(A) For all $z \in \mathcal{U}^u$ we have $y \leq z$ and for all $z' \in \mathcal{U}^l$ we have $y \geq z'$.

Then by definition of order convergence this would imply $y \leq x$, since $x = \bigwedge \mathcal{U}^u$, and similarly we would get $y \geq x$, a contradiction to $x \neq y$. So, (A) must be false, and without loss of generality we may assume that there is a $z_0 \in \mathcal{U}^u$ with $y \not\leq z_0$. By 1.1, assertion 1, we get $(z_0] \in \mathcal{U}$ which implies

$$B := (z_0] \cap Q \in \mathcal{U}|_Q.$$

Since $y \not\leq z_0$ we also have

$$y \in P \setminus (z_0]. \quad (\star)$$

Because τ contains the interval topology $\tau_i(P)$, statement $(\star)$ above implies that the set

$$V := (P \setminus (z_0]) \cap Q = Q \setminus B$$

is an open neighborhood of y in $(Q, \tau|_Q)$. But since $B \in \mathcal{U}|_Q$ and $V = Q \backslash B$, we have $V \notin \mathcal{U}|_Q$, so $\mathcal{U}|_Q$ does not converge to y with respect to $\tau|_Q$. Since $y \in Q$ was arbitrary, the claim is proved.

The claim now shows that $Q = P \setminus W$ is a closed, non-compact subset of (P, τ). So (P, τ) cannot be compact. $\square$

This theorem has a direct consequence for Priestley spaces, i.e. compact totally order-disconnected ordered spaces as introduced in ([3], [4]).

COROLLARY 2.2. *If $(P, \tau, \leq)$ is a Priestley space, then $\tau \subseteq \tau_o(P)$.*

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