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AXIOMS FOR CONVENIENT CALCULUS

by Alfred FRÖLICHER

Dedicated to **Dieter Pumplün** at the occasion of his 70th birthday and in memory of his active role in the foundations and early developments of convenient calculus

Résumé.

Afin de généraliser et améliorer le calcul différentiel classique, on a essayé de remplacer les normes par d'autres structures (localement convexes; bornologiques; de convergence; etc). On peut éviter un choix arbitraire. Soit Γ est une classe d'espaces vectoriels structurés quelconque et supposons qu'on a pour tout $E, F \in \Gamma$ un ensemble $\mathcal{S}(E, F)$ d'applications dites "lisses". Si les $\mathcal{S}(E, F)$ satisfont trois axiomes (qui sont valables si Γ est la classe des espaces de Banach avec les $C^\infty(E, F)$), alors tout $E \in \Gamma$ possède une unique structure d'espace vectoriel *convenable* telle que les applications "lisses" sont les applications lisses au sens du calcul convenable [7]. Donc $(\Gamma, \mathcal{S})$ est une catégorie équivalente à une souscatégorie pleine de $\underline{Con}^\infty$, la catégorie des espaces convenables avec leurs applications lisses. Réciproquement, toute souscatégorie pleine de $\underline{Con}^\infty$ qui contient l'objet $\mathbb{R}$ satisfait les 3 axiomes. Dans les sections 8 à 13 on trouve des remarques sur le calcul convenable.

Introduction.

Convenient calculus generalizes Banach space calculus and has several useful advantages. An obvious default of Banach space calculus is the fact that many of the involved function spaces belong to a larger class: $C^\infty(\mathbb{R}, \mathbb{R})$ e.g. is a Fréchet but not a Banach space. This is a serious deficiency since function spaces play an important role in modern mathematics. A good calculus should introduce, for $0 \leq k \leq \infty$, "maps of class $\mathcal{D}^k$ " from E to F which are k -times differentiable, and such that they form a function space $\mathcal{D}^k(E, F)$ belonging again to the class of spaces one started with.

For a long time the choice of the “appropriate spaces” was rather ambiguous and hence was influenced by personal preferences. Various ambient categories have been tried; we just mention locally convex spaces, convergence spaces, bornological spaces, compactly generated spaces, smooth spaces.

Our axiomatic approach avoids such an arbitrary choice. We suppose that we have any class Γ of structured real vector spaces, and for all $E, F \in \Gamma$ a set $\mathcal{S}(E, F)$ of maps $E \rightarrow F$, called “smooth” maps. For these function spaces three simple axioms are imposed. They imply that $(\Gamma, \mathcal{S})$ is a category. Examples satisfying the axioms are: (1) Γ the class of Banach spaces and $\mathcal{S}(E, F) = C^\infty(E, F)$; (2) the category $\underline{Con}^\infty$ having the convenient vector spaces as objects and the infinitely differentiable maps as morphisms, cf. [7]; (3) any full subcategory of $\underline{Con}^\infty$ which contains $\mathbb{R}$. The main result is: Every model of our axiom system is equivalent (as category) to a full subcategory of $\underline{Con}^\infty$ containing $\mathbb{R}$.

For those not familiar with convenient calculus we recall some of its *main features*.

(A) The structure of a convenient vector space includes no irrelevant information; it provides exactly what is needed for calculus.

(B) The convenient vector spaces with their linear morphisms (cf. 5.3) form a complete, cocomplete and symmetric monoidal closed category $\underline{Con}$.

(C) For $0 \leq k \leq \infty$ one has function spaces $\mathcal{D}^k(E, F)$ formed by the k -times differentiable maps $E \rightarrow F$ with some regularity condition on the last derivative. All these spaces are again convenient. More generally one gets convenient function spaces $\mathcal{D}^k(X, F)$ for rather general spaces X .

(D) Whether a map $f : E \rightarrow F$ is of class $\mathcal{D}^k$ can be tested by composing with smooth curves $\mathbb{R} \rightarrow E$ and linear smooth functions $l : F \rightarrow \mathbb{R}$; so the question of $\mathcal{D}^k$ -ness of f becomes reduced to the case of functions $\mathbb{R} \rightarrow \mathbb{R}$.

(E) The category $\underline{Con}^\infty$ is cartesian closed.

(F) Convenient calculus involves more hard analysis than classical calculus. But it is worth the effort; one obtains a very powerful tool. For elegant applications cf. [9].

The axiomatic part of the paper ends with section 7. The following sections bring various remarks and comments. Section 8 deals with the Uniform Boundedness Principle which has fundamental consequences for convenient calculus. As its name indicates, the Uniform Boundedness Principle is actu-

ally a bornological result. We refer to a bornological version which is very appropriate for being applied to convenient calculus since both work with bornologies.

The bornologies which are actually used are “*linearly generated*”. Since this notion is probably not generally known, we give in section 9 various conditions for a vector space bornology to be *linearly generated*.

In classical analysis the notions of “*continuously differentiable*” and of “ *C^k -map*” are so important that it is difficult to convince the reader that “Lipschitz-differentiable” and “*Lip^k-map*” are more important. We give in section 10 several arguments.

The category of convenient vector spaces embeds in many ambient categories, in particular into that of bornological vector spaces, of dualized vector spaces, of convergence vector spaces, of smooth vector spaces, of locally convex spaces. Each such embedding yields a different axiomatic definition of the “*same*” convenient vector spaces (just as one gets the “*same*” topological spaces if one uses “open sets” or “neighborhoods” for defining them). By far the best known of the mentioned ambient categories is the last named. It is therefore a great merit of A.Kriegl and P.Michor to have chosen it for their outstanding monograph [9]. By using locally convex spaces and by giving numerous interesting examples they convinced a much larger public that convenient calculus is useful. But unfortunately the locally convex topology behaves very badly. We show in section 11 why the best behaved ambient objects are the much less popular bornological vector spaces.

In order to calculate limits and colimits in the category of convenient vector spaces (they all exist), it is good to know the left adjoint retraction functor to the inclusion into the bornological vector spaces. Section 12 gives an explicit description.

The last section gives two simple examples: a (convex) bornological vector space which is not linearly generated; and a dualized vector space which is not bornologically generated.

It is a great pleasure for me to express my gratitude to my colleague and friend CLAUDE-ALAIN FAURE. He read carefully the manuscript and made very valuable suggestions which considerably improved the paper.

1. The axiomatic approach.

We suppose given: a class Γ of structured real vector spaces; and for all $E, F \in \Gamma$ a set $\mathcal{S}(E, F)$ of maps $f : E \rightarrow F$, called *smooth maps*.

- 1.1 Notation.** 1° For a vector space E , the algebraic dual is noted E^* ;
 2° For $E \in \Gamma$, we put $E' := \mathcal{S}(E, \mathbb{R}) \cap E^*$.

We consider for the function spaces $\mathcal{S}(E, F)$, where $E, F \in \Gamma$, the following axioms.

1.2 Axioms for smooth maps.

- (S₁) $\mathbb{R} \in \Gamma$ and $\mathcal{S}(\mathbb{R}, \mathbb{R}) = C^\infty(\mathbb{R}, \mathbb{R})$;
(S₂) $f \in \mathcal{S}(E, F)$ iff $l \cdot f \cdot c \in C^\infty(\mathbb{R}, \mathbb{R}) \forall c \in \mathcal{S}(\mathbb{R}, E)$ and $\forall l \in F'$;
(S₃) $c \in \mathcal{S}(\mathbb{R}, E) \Rightarrow \exists ! v \in E$ such that $(l \cdot c)'(0) = l(v) \quad \forall l \in E'$.

- 1.3 Remark.** One has $\mathbb{R}' = \mathcal{S}(\mathbb{R}, \mathbb{R}) \cap \mathbb{R}^* = C^\infty(\mathbb{R}, \mathbb{R}) \cap \mathbb{R}^* = \mathbb{R}^*$.

2. Consequences of the axioms (S₁) and (S₂).

2.1 Lemma. (Let $E \in \Gamma$, $c : \mathbb{R} \rightarrow E$ a curve, $f : E \rightarrow \mathbb{R}$ a function. Then one has:

- 1° $c \in \mathcal{S}(\mathbb{R}, E)$ iff $l \circ c \in C^\infty(\mathbb{R}, \mathbb{R})$ for all $l \in E'$.
 2° $f \in \mathcal{S}(E, \mathbb{R})$ iff $f \circ c \in C^\infty(\mathbb{R}, \mathbb{R})$ for all $c \in \mathcal{S}(\mathbb{R}, E)$.

Proof. 1° By axiom (S₂) one has: $c \in \mathcal{S}(\mathbb{R}, E)$ iff $l \circ c \circ \varphi \in C^\infty(\mathbb{R}, \mathbb{R})$ for all $l \in E'$ and all $\varphi \in \mathcal{S}(\mathbb{R}, \mathbb{R})$. Using axiom (S₁) this simplifies to the stated condition.

2° Similarly (S₂) implies: f is smooth iff $l \circ f \circ c \in C^\infty(\mathbb{R}, \mathbb{R})$ for all $l \in \mathbb{R}'$ and all $c \in \mathcal{S}(\mathbb{R}, E)$. Since $l \in \mathbb{R}'$ is just multiplication by a constant, cf. 1.3, this condition simplifies to the one given above. $\square$

2.2 Proposition. Let $E, F \in \Gamma$ and $f : E \rightarrow F$. Then each of the following conditions is equivalent with $f \in \mathcal{S}(E, F)$:

- (1) $l \circ f \circ c \in C^\infty(\mathbb{R}, \mathbb{R})$ for all $l \in F'$ and $c \in \mathcal{S}(\mathbb{R}, E)$;
- (2) $f_*(\mathcal{S}(\mathbb{R}, E)) \subseteq \mathcal{S}(\mathbb{R}, F)$;
- (3) $f^*(\mathcal{S}(F, \mathbb{R})) \subseteq \mathcal{S}(E, \mathbb{R})$;
- (4) $f^*(F') \subseteq \mathcal{S}(E, \mathbb{R})$.

Proof. $(1 \Rightarrow 2)$. Let $c \in \mathcal{S}(\mathbb{R}, E)$. One has to show that $f_*(c) = f \circ c \in \mathcal{S}(\mathbb{R}, F)$. This is by 1° of 2.1 equivalent with hypothesis (1).

$(2 \Rightarrow 3)$. Let $g \in \mathcal{S}(F, \mathbb{R})$. Using 2° of 2.1 and (2) one has for all $c \in \mathcal{S}(\mathbb{R}, E)$:

$(f^* \circ g) \circ c = g \circ (f \circ c) \in C^\infty(\mathbb{R}, \mathbb{R})$ and hence $f^*(g) \in \mathcal{S}(E, \mathbb{R})$.

$(3 \Rightarrow 4)$. This is trivial since $F' \subseteq \mathcal{S}(F, \mathbb{R})$ by definition 1.1.

$(4 \Rightarrow 1)$. By hypothesis (4) one has $l \circ f \in \mathcal{S}(E, \mathbb{R})$ for all $l \in F'$. By 2° of 2.1 this implies $l \circ f \circ c \in C^\infty(\mathbb{R}, \mathbb{R})$, as to be shown. $\square$

2.3 Corollary. $1^\circ \Gamma$ together with the smooth maps is a category.

$2^\circ \mathcal{S}$ is completely determined if one knows $E' := \mathcal{S}(E, \mathbb{R}) \cap E^*$ for all $E \in \Gamma$.

2.4 Lemma. $\mathcal{S}(E, F)$ is a vector space (for the natural operations).

Proof. Let $f, g \in \mathcal{S}(E, F)$. Then $f + g \in \mathcal{S}(E, F)$ follows from (S_2) since $C^\infty(\mathbb{R}, \mathbb{R})$ is closed under addition. Similarly one shows that $\lambda f \in \mathcal{S}(E, F)$. $\square$

2.5 Corollary. E' is a vector subspace of E^* for every $E \in \Gamma$. This means that (E, E') is a so-called **dualized vector space**.

These and the associated bornological vector spaces play an important role for differentiation theory. We give some elementary aspects.

3. Bornological and dualized vector spaces.

3.1 Definition. 1° A **dualized vector space** is a couple (E, E') where E is a vector space (over $\mathbb{R}$) and E' is a subspace of the algebraic dual E^* of E . Since we consider E' as a structure on E , we often write $E = (E, E')$, just as one uses to write “let $G = (G, \cdot)$ be a group”.

2° DVS is the category of dualized vector spaces and has as morphisms from (E, E') to (F, F') the linear maps $f : E \rightarrow F$ satisfying $f^*(F') \subseteq E'$.

3° BVS is the category of bornological vector spaces with the linear bornological maps (also called “bounded linear maps”) as morphisms.

3.2 Proposition.

1° For any dualized vector space E the collection

$$\mathcal{B}_E := \{B \subseteq E \mid l(B) \text{ is bounded for all } l \in E'\}$$

is a vector bornology on the vector space E , i.e. $(E, \mathcal{B}_E)$ is a bornological vector space; we denote it by βE . If $f : E_1 \rightarrow E_2$ is a morphism of DVS, then $B \in \mathcal{B}_{E_1}$ implies $f(B) \in \mathcal{B}_{E_2}$. So we get a functor $\beta : \underline{DVS} \rightarrow \underline{BVS}$ by defining $\beta(f) := f$.

2° Similarly we get a functor $\delta : \underline{BVS} \rightarrow \underline{DVS}$; it associates to a bornological vector space $F = (F, \mathcal{B})$ the couple (F, F') where F' is the bornological dual:

$$F' := \{l \in F^* \mid B \in \mathcal{B} \Rightarrow l(B) \text{ bounded in } \mathbb{R}\}.$$

3° Both functors preserve the underlying vector spaces and the underlying maps. They form a so-called **Galois connection**, i.e. the identity maps $\delta\beta E \rightarrow E$ and $F \rightarrow \beta\delta F$ are morphisms of DVS respectively BVS. It follows that δ is left-adjoint to β and that $\delta\beta\delta = \delta$ and $\beta\delta\beta = \beta$.

4° The objects E of DVS satisfying $\delta\beta E = E$ form a full subcategory bg.DVS of DVS. Its objects are called **bornologically generated dualized vector spaces**. In [7] they were called **preconvenient vector spaces**. The functor $\delta\beta$ gives a retraction and is right-adjoint to the inclusion bg.DVS $\rightarrow$ DVS.

5° Similarly, the objects F of BVS satisfying $\beta\delta F = F$, called **linearly generated bornological vector spaces**, form a full subcategory denoted by lg.BVS. The functor $\beta\delta$ yields a retraction and is left-adjoint to the inclusion.

6° The categories bg.DVS and lg.BVS are isomorphic: The functor β restricts to an isomorphism bg.DVS $\rightarrow$ lg.BVS. The inverse is obtained by restriction of δ .

Proof. All verifications are straightforward. □

4. Difference quotients and smooth curves.

4.1 Definition. Let $c : \mathbb{R} \rightarrow E$ be a curve of a vector space E . If $t_0, \dots, t_k$ are $k + 1$ different reals, then $\delta^k c(t_0, \dots, t_k) := \sum_i \beta_i \cdot c(t_i)$ where $\beta_i := k! \cdot \prod_{j \neq i} (t_i - t_j)^{-1}$. If $A \subseteq \mathbb{R}$ we denote by $A^{<k>}$ the set $\{(a_0, \dots, a_k) \mid a_i \in A \text{ and } a_i \neq a_j \text{ for } i \neq j\}$.

We will consider this for $k \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$ and remark that $\delta^0 c = c$ and $A^{<0>} = A$.

The set $\mathbb{R}^{<k>}$ will be considered with the bornology having as bounded sets those contained in $I_n^{<k>}$ for some $n \in \mathbb{N}$ where $I_n := [-n, n]$.

4.2 Proposition. *For a function $f : \mathbb{R} \rightarrow \mathbb{R}$ one has:*

$$f \in C^\infty(\mathbb{R}, \mathbb{R}) \Leftrightarrow \delta^k f : \mathbb{R}^{<k>} \rightarrow \mathbb{R} \text{ is bornological for all } k \in \mathbb{N}.$$

Proofs can be found in some books on elementary calculus or in [7].

In order to apply this to vector spaces $E \in \Gamma$ we recall that associated to E one has the dualized vector space $D_E := (E, E')$ where $E' := \mathcal{S}(E, \mathbb{R}) \cap E^*$; cf. 1.1. And to D_E corresponds the bornological vector space $B_E := \beta D_E = (E, \mathcal{B}_E)$ where $\mathcal{B}_E := \{B \subseteq E \mid l(B) \text{ is bounded for all } l \in E'\}$; cf. 1° of 3.2.

4.3 Proposition. *Suppose $E \in \Gamma$ and let $c : \mathbb{R} \rightarrow E$ be a curve. Then*

$$c \text{ is smooth} \Leftrightarrow \delta^k c : \mathbb{R}^{<k>} \rightarrow B_E \text{ is bornological for all } k \in \mathbb{N}_0.$$

Proof. By 1° of 2.1 c is smooth iff $l \circ c \in C^\infty(\mathbb{R}, \mathbb{R})$ for all $l \in E'$, and by 4.2 this holds iff $\delta^k(l \circ c)(A^{<k>})$ is bounded (in $\mathbb{R}$) for all $k \in \mathbb{N}_0$ and all bounded $A \subseteq \mathbb{R}$. The linearity of l implies $\delta^k(l \circ c)(A^{<k>}) = l(\delta^k c(A^{<k>}))$ and so the condition is equivalent with $\delta^k c(A^{<k>})$ bounded for all bounded $A \subseteq \mathbb{R}$ and hence also with $\delta^k c : \mathbb{R}^{<k>} \rightarrow E$ being bornological. $\square$

5. Smoothness of linear maps.

5.1 Proposition. *Let $E \in \Gamma$ and $f \in E^*$. Then*

$$f \text{ is bornological} \Leftrightarrow f \text{ is smooth.}$$

Proof. $\Rightarrow$ Suppose f is not smooth. Then by 2° of 2.1 there exists $c \in \mathcal{S}(\mathbb{R}, E)$ such that $f \circ c \notin C^\infty(\mathbb{R}, \mathbb{R})$. Hence there exist $k \in \mathbb{N}_0$ and $A \subseteq \mathbb{R}$ bounded, with $(\delta^k(f \circ c))(A^{<k>})$ unbounded in $\mathbb{R}$. By linearity of f this equals $f(\delta^k c(A^{<k>}))$, and since by 4.3 the set $(\delta^k c)(A^{<k>})$ is bounded one concludes that f is not bornological.

$\Leftarrow$ Trivially $f \in E'$ and hence f is bornological by the definition of $\mathcal{B}_E$. $\square$

5.2 Corollary. *For $E \in \Gamma$ the associated dualized vector space $D_E = (E, E')$ is bornologically generated. Hence D_E and the associated bornological vector space B_E determine each other: one has $B_E = \beta D_E$ and $D_E = \delta B_E$; cf. 3.2.*

5.3 Proposition. *Let $E, F \in \Gamma$ and $f : E \rightarrow F$ a linear map. Then the following conditions are equivalent:*

- (1) f is smooth;
 (2) $f^*(F') \subseteq E'$, i.e. f is a morphism of $\underline{DVS}$;
 (3) f is bornological, i.e. f is a morphism of $\underline{BVS}$.

Proof.

- (1 $\Leftrightarrow$ 2) f is smooth iff $f^*(F') \subseteq \mathcal{S}(E, \mathbb{R})$ iff $f^*(F') \subseteq E'$; cf. 2.2.
 (2 $\Rightarrow$ 3) One applies the functor β and uses that $\beta f = f$.
 (3 $\Rightarrow$ 2) One applies the functor δ and uses that $\delta f = f$. $\square$

5.4 Proposition. 1 $^\circ$ If one knows for $E, F \in \Gamma$ the associated dualized vector spaces D_E, D_F (or equivalently the duals E', F'), then one can recover the set $\mathcal{S}(E, F)$.

2 $^\circ$ The same holds if one knows the associated bornological vector spaces B_E, B_F (or equivalently the bornologies $\mathcal{B}_E, \mathcal{B}_F$).

Proof. 1 $^\circ$ From E' (resp. F') one recovers the smooth curves $\mathbb{R} \rightarrow E$ (resp. $\mathbb{R} \rightarrow F$) by 1 $^\circ$ of 2.1. Now one applies (2) of 2.2.

2 $^\circ$ From the bornological vector space $(E, \mathcal{B}_E)$ one recovers (E, E') ; in fact, by definition one has $(E, \mathcal{B}_E) = \beta(E, E')$. Hence $\delta(E, \mathcal{B}_E) = \delta\beta(E, E') = (E, E')$; cf. 5.2. $\square$

6. The role of axiom (S_3) .

This axiom holds iff the associated bornological vector spaces B_E satisfy a completeness and a separation condition. One can express these conditions also in terms of the associated dualized vector spaces D_E , but for the completeness property it becomes more complicated.

6.1 Theorem. Axiom (S_3) holds for E iff the associated bornological vector space $(E, \mathcal{B}_E)$ is separated and Mackey complete.

For a proof we refer to 2.5.2 and 2.6.2 in [7]. There it is shown that the different notions of a *separated* as well as those of a *complete* bornological vector space E coincide if E is *linearly generated*, and that

(a) $\forall c \in \mathcal{S}(\mathbb{R}, E) \exists v \in E$ such that $(l \circ c)(0) = l(v) \quad \forall l \in E'$ iff the bornology is Mackey complete.

(b) The vector v in (a) is unique iff the bornology is separated.

6.2 Definition. Let E be a linearly generated bornological space.

1 $^\circ$ E is *separated* if its bornological dual E' separates points of E .

2° E is called **Mackey complete** if every Mackey-Cauchy sequence is Mackey convergent.

3° A sequence $x_1, x_2, \dots$ is **Mackey convergent** to $x \in E$ if one can write it as $x_n = x + t_n \cdot b_n$ with reals t_n that converge to zero and $\{b_1, b_2, \dots\} \subseteq E$ bounded.

4° A sequence $x_1, x_2, \dots$ is **Mackey-Cauchy** if one can write:
 $x_n - x_m = t_{n,m} \cdot b_{n,m}$ with reals $t_{n,m}$ such that $t_{n,m} \rightarrow 0$ for $n, m \rightarrow \infty$ and $\{b_{n,m} / n, m \in \mathbb{N}\} \subseteq E$ bounded.

7. Models by means of convenient vector spaces.

We saw in 5.3 that if $(\Gamma, \mathcal{S})$ satisfies the axioms 1.2, then every $E \in \Gamma$ becomes a bornologically generated dualized vector space (E, E') and a linearly generated bornological vector space $(E, \mathcal{B}_E)$ which is separated and Mackey complete. E' and $\mathcal{B}_E$ determine each other (like open sets and neighborhoods of a topological space). Either one can be used in order to describe these objects. In [7], E' was used. We now give priority to the bornologies since Mackey complete is a bornological condition. For a more important argument in favour of bornologies, cf. section 11.

7.1 Definition. A convenient vector space is a bornological vector space which is linearly generated, separated and Mackey complete.

7.2 Theorem.

- 1° The axioms 1.2 hold for the category $\underline{\text{Con}}^\infty$ (cf. Introduction).
- 2° They also hold for each full subcategory of $\underline{\text{Con}}^\infty$ which contains $\mathbb{R}$.
- 3° Every model for the axioms 1.2 is equivalent to a full subcategory of $\underline{\text{Con}}^\infty$ containing $\mathbb{R}$.

Proof. 1°. We remark that with its usual bornology $\mathbb{R}$ becomes a convenient vector space. From this (S_1) follows easily. For (S_2) and (S_3) we refer to [7].

2°. This is trivial.

3°. Suppose that $(\Gamma, \mathcal{S})$ satisfies the axioms. We saw in 2.3 that $(\Gamma, \mathcal{S})$ is a category. Every object $E \in \Gamma$ has a natural structure of convenient vector space. We denote this convenient vector space by $\Phi(E)$ and the full subcategory of $\underline{\text{Con}}^\infty$ containing all the objects $\Phi(E)$ for $E \in \Gamma$ by $\mathcal{A}$. Remark that E and $\Phi(E)$ have the same underlying vector space. We first consider on $E \in \Gamma$ a linear function $l : E \rightarrow \mathbb{R}$. By 5.1 l is smooth iff l

is bornological. And according to convenient calculus (cf. 2.4.4 in [7]) one has: l is infinitely differentiable iff l is bornological. Together this gives: $l \in \mathcal{S}(E, \mathbb{R}) \Leftrightarrow l \in \underline{\text{Con}}^\infty(E, \mathbb{R}) = \mathcal{A}(E, \mathbb{R})$. Since by 2° of 2.3 all morphisms are determined by the linear ones $E \rightarrow \mathbb{R}$ we conclude that $\mathcal{S}(E, F) = \underline{\text{Con}}^\infty(E, F)$. So we can define $\Phi(f) = f$ for $f \in \mathcal{S}(E, F)$ and this gives a functor $\Phi : (\Gamma, \mathcal{S}) \rightarrow \mathcal{A}$ which is full, faithful and surjective on the objects, hence an equivalence of categories. $\square$

7.3 Remark. *The theorem shows that $\underline{\text{Con}}^\infty$ is a maximal model for our axiom system: one takes as Γ all convenient vector spaces and as “smooth” maps the infinitely differentiable ones, cf. [7].*

8. Bornological function spaces

The Uniform Boundedness Principle is fundamental for convenient calculus. As its name indicates, it actually should be formulated and proved in the frame of bornological spaces instead of locally convex spaces. We give a short survey and refer to [5] for details and the proofs of 8.2 to 8.4. Let X, Y be bornological spaces and $\mathcal{M}$ a function space consisting of (certain) bornological maps $X \rightarrow Y$. On $\mathcal{M}$ one can consider two bornologies:

8.1 Definition. 1° *The canonical bornology has as bounded sets the $B \subseteq \mathcal{M}$ which satisfy $B(A)$ bounded in Y for all A bounded in X .*

2° *The pointwise bornology has as bounded sets the $B \subseteq \mathcal{M}$ which satisfy $B(a)$ bounded in Y for all $a \in X$.*

If E, F are bornological vector spaces, the set of linear bornological maps $E \rightarrow F$ with the canonical bornology is noted $L(E, F)$. As special case we have $E' := L(E, \mathbb{R})$

8.2 Theorem. *Let E, F, G be bornological vector spaces and suppose that F is linearly generated. The following conditions are equivalent:*

- (1) *On E' the pointwise and the canonical bornology coincide;*
- (2) *On $L(E, F)$ the pointwise and the canonical bornology coincide;*
- (3) *A linear map $u : E \rightarrow L(G, F)$ is bornological iff $\text{ev}_x \circ u$ is so;*
- (4) *A bilinear map $b : E \times G \rightarrow F$ is bornological iff it is partially bornological.*

If these conditions hold for E one also says: the Uniform Boundedness Principle holds for E .

8.3 Theorem. *For every Mackey complete convex bornological vector space E the Uniform Boundedness Principle holds.*

One furthermore has to know that the properties *linearly generated* and *Mackey complete* go over from F to $L(E, F)$.

8.4 Proposition. *Let E, F be bornological vector spaces and F linearly generated.*

- 1° *The bornology of $L(E, F)$ is also linearly generated;*
- 2° *If F is Mackey complete, the same follows for $L(E, F)$.*

Proofs can be found in [5] or [7].

8.5 Corollary.

1° *A map $f : Z \rightarrow L(E, F)$ from a bornological space Z into the function space $L(E, F)$ is bornological iff $\text{ev}_x \circ f$ is bornological for all $x \in E$.*

2° *A curve $c : \mathbb{R} \rightarrow L(E, F)$ is smooth iff $\text{ev}_x \circ c$ is smooth for all $x \in E$.*

3° *A map $f : G \rightarrow L(E, F)$ is smooth iff $\text{ev}_x \circ f$ is smooth for all $x \in E$.*

Proof. 1° is a direct consequence of the Uniform Boundedness Principle.

2° follows from 1° since by 4.3 c is smooth iff $\delta^k c$ is bornological for all k .

3° follows from 2° since smoothness can be tested by smooth curves; cf. 2° of 2.1.

9. Other aspects of the linearly generated bornologies.

The bornology of a convenient vector space is *linearly generated* and this implies that it is a *convex* vector bornology (the convex hull of every bounded set is bounded). Example 13.2 will show that the converse fails. For a better insight of the notion *linearly generated*, it will be helpful to know some equivalent characterizations.

9.1 Proposition. *Let $E = (E, \mathcal{B})$ be a bornological vector space. The following conditions are equivalent:*

- (1) *The bornology is linearly generated, i.e. $\beta\delta E = E$;*

- (2) *there exists a family $\mathcal{L}$ of linear functions $l : E \rightarrow \mathbb{R}$ such that for $B \subseteq E$ one has: $B \in \mathcal{B} \Leftrightarrow l(B)$ is bounded $\forall l \in \mathcal{L}$;*
- (3) *The canonical map $j : E \rightarrow E''$ is an embedding, i.e. B bounded iff jB bounded;*
- (4) *There exists a family P of seminorms $p : E \rightarrow \mathbb{R}^+$ such that for $B \subseteq E$ one has: $B \in \mathcal{B} \Leftrightarrow p(B)$ is bounded $\forall p \in P$;*
- (5) *$\mathcal{B}$ is the von Neumann bornology of some locally convex topology $\mathcal{T}$.*

Proof. $(1 \Rightarrow 2)$ is trivial.

$(2 \Rightarrow 3)$ The canonical injection $j : E \rightarrow E''$ is defined by $(ja)(l) = l(a)$ for $a \in E$ and $l \in E'$. Since $E' = L(E, \mathbb{R})$ is Mackey complete, one knows (by the Uniform Boundedness Principle for bornological vector spaces, cf. section 8), that a subset $W \subseteq E''$ is bounded on all bounded subsets of E' iff it is pointwise bounded. Using this for $W := j(A)$ one gets

$(jA)(B)$ bounded for all bounded $B \subseteq E'$ iff $l(A)$ bounded for all $l \in E'$.

The bornology being by hypothesis (2) *linearly generated*, the right hand side says that $A \subseteq E$ bounded. The left hand side says, by the definition of the bornology of E'' , that $jA \subseteq E''$ is bounded.

$(3 \Rightarrow 4)$ The space $E' := L(E, \mathbb{R})$ is a Mackey complete convex bornological space. Hence by the Uniform Boundedness Principle the canonical and the pointwise bornology of E' coincide. So we obtain: $A \subseteq E$ bounded iff $jA \subseteq E''$ bounded iff $(jA)(B) \subseteq \mathbb{R}$ bounded for all $B \subseteq E'$ bounded iff $(jA)(l) = l(A) \subseteq \mathbb{R}$ bounded for all $l \in E'$. So we can take as seminorms $\{|l| \mid l \in E'\}$.

$(4 \Rightarrow 5)$ One considers the locally convex topology determined in the usual way by the family of seminorms. One verifies easily that the subsets bounded in the sense of von Neumann are exactly those $B \subseteq E$ which satisfy $p(B)$ bounded for all $p \in P$.

$(5 \Rightarrow 1)$ It is a well-known classical result that for any locally convex space the bounded subsets B are exactly those for which $l(B)$ is bounded for every $l \in \mathcal{L}$ where $\mathcal{L}$ is the set of linear continuous functions; cf. [5] for a short simple proof. So one has $E = \beta D$ where D is the dualized vector space $D := (E, \mathcal{L})$. This implies $\beta\delta E = E$; cf. 3.2. $\square$

10. Lipschitz- versus continuous differentiability.

In classical calculus the functions $f : E \rightarrow F$ which are just differentiable (e.g. in the sense of Gâteaux or Fréchet) do not form useful function spaces. Much better are the spaces $C^k(E, F)$ based on the notion of *continuous differentiability*. Also in convenient calculus it is good to impose a regularity condition on the derivative and one might think of a continuity condition. There are several topologies available. In particular: the usual locally convex one corresponding to the bornology (cf. [7]), the weak topology, the Mackey closure topology. But one would loose the *testing property*, cf. 2° of 10.6. In fact, in the finite dimensional case all these topologies coincide with the standard one. But there exist simple examples of a function $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ such that for every C^∞ -curve $c : \mathbb{R} \rightarrow \mathbb{R}^2$ (even for every C^1 -curve) the composite $f \circ c : \mathbb{R} \rightarrow \mathbb{R}$ is a C^1 -function, but f is not C^1 ; cf. [1].

An important regularity condition for which the testing property holds is as follows.

10.1 Definition. A map $f : E \rightarrow F$ is called a *Lip⁰-map* if for every smooth curve $c : \mathbb{R} \rightarrow E$ and every function $l \in F'$ the composite $l \circ f \circ c : \mathbb{R} \rightarrow \mathbb{R}$ is locally Lipschitzian.

10.2 Definition.

A map $f : E \rightarrow F$ is called **Lipschitz-differentiable** if

1. For all $(x, v) \in E \amalg E$, $df(x, v) := w\text{-}\lim_{t \rightarrow 0} \frac{f(x+tv) - f(x)}{t}$ exists.
2. The so defined map $df : E \amalg E \rightarrow F$ is a *Lip⁰-map*.

By $w\text{-}\lim$ we understand “limit with respect to the weak topology”. So the first of the two conditions means that the curve $c_{x,v} : \mathbb{R} \rightarrow F$, defined by $t \mapsto f(x + tv)$ has at $t = 0$ a tangent vector, noted $df(x, v)$; cf. 1.2, (**S**₃).

10.3 Lemma. Suppose $f : E \rightarrow F$ is Lipschitz-differentiable. Then for any points $x, v, w \in E$

$$M\text{-}\lim_{t \rightarrow 0} \frac{f(x+t \cdot w+t \cdot v) - f(x+t \cdot w)}{t}$$

exists and is equal to $df(x, v)$. $M\text{-}\lim$ stands for Mackey limit.

For a proof, cf. 4.3.12 in [7].

10.4 Corollary. If $f : E \rightarrow F$ is Lipschitz-differentiable, then df is linear in the second variable.

Proof. $df(x, t \cdot v) = t \cdot df(x, v)$ follows trivially. For the additivity one uses

$$df(x, v + w) = \lim_{t \rightarrow 0} \frac{f(x + tv + tw) - f(x)}{t} = \\ \lim_{t \rightarrow 0} \left(\frac{f(x + tv + tw) - f(x + tw)}{t} + \frac{f(x + tw) - f(x)}{t} \right).$$

By the lemma this gives $df(x, v) + df(x, w)$. □

10.5 Proposition. *If $f : E \rightarrow F$ is Lipschitz differentiable, then $f'(x) := df(x, -) \in L(E, F)$ and the map $f' : E \rightarrow L(E, F)$ is Lip^0 . The converse is trivial.*

Proof. (a) We just saw that $df(x, -)$ is linear. Furthermore, it is Lip^0 . But for a linear map this is equivalent with bornological; cf. 2.4.4 in [7]. This proves that $f'(x) := df(x, -) \in L(E, F)$.

(b) A map $g : G \rightarrow L(E, F)$ is Lip^0 iff $ev_x \circ g$ is Lip^0 for all $x \in E$. The proof of this is like that of 3° in 8.5. For $g = f'$ one has $(ev_x \circ f')(v) = df(x, v)$, i.e. $ev_x \circ f' = df(x, -)$ and this is Lip^0 . □

10.6 Summary. Lipschitz-differentiability has the following useful properties which fail for continuous differentiability.

1° *Imposing the condition Lip^0 on df or on f' is equivalent.*

2° *The testing property holds: A map $g : E \rightarrow F$ is Lipschitz differentiable iff $f := l \circ g \circ c : \mathbb{R} \rightarrow \mathbb{R}$ is Lipschitz-differentiable for all smooth curves $c : \mathbb{R} \rightarrow E$ and all $l \in F'$. For $f : \mathbb{R} \rightarrow \mathbb{R}$ one has a simple criterion by means of difference quotients: $\delta^i f$ must be bornological for $0 \leq i \leq 2$. Equivalent is: f must be differentiable and its derivative f' locally Lipschitzian.*

More generally one can recursively define maps g of class Lip^k as follows: Lip^1 is the same as Lipschitz differentiable. And g is of class Lip^{k+1} if it is of class Lip^1 and g' (or equivalently dg) is of class Lip^k . For all these function spaces the testing property holds. Moreover, they all have a natural structure of convenient vector space. Let us remark that, more generally, one obtains convenient function spaces $Lip^k(X, F)$ if one replaces E by a set X endowed with a so-called " Lip^k -structure". In particular, X may be a subset of E or a differential Lip^k -manifold modelled on convenient vector spaces.

10.7 Remark. One might say that C^1 is more important since it is

an essential hypothesis for the inverse function theorem of classical Banach space calculus. However, the standard proof establishes in fact two results:

1° If $f : E \rightarrow F$ is continuously differentiable in a neighborhood of a point $p \in E$, then f is *strictly differentiable* at the point p ;

2° If f is *strictly differentiable* at a point $p \in E$ and if the derivative $f'(p)$ is an isomorphism $E \rightarrow F$, then the standard conclusion on local bijectivity of f holds, and f^{-1} is strictly differentiable at the point $q := f(p)$, with $(f^{-1})'(q) = (f'(p))^{-1}$.

We recall that f is called *strictly differentiable* at a point p if there exists $l \in L(E, F)$ such that the remainder function $r(x) := f(p+x) - f(p) - l(x)$ satisfies

$$\lim_{x \neq y \rightarrow 0} \frac{r(x) - r(y)}{\|x - y\|} = 0.$$

Obviously 1° and 2° together prove the inverse function theorem. The proof of 1° is based on the mean value theorem which estimates for a curve $c : [a, b] \rightarrow E$ of a Banach space E the increment $c(b) - c(a)$ in terms of the values $c'(t)$ for $t \in [a, b]$; see e.g. [2], (8.5.1), p. 153. The result 2° follows from the fixed point theorem. This shows that for the inverse function theorem *strict differentiability* is the adequate hypothesis. Continuous differentiability only plays a role since 1° is a useful lemma. The essential result is 2°. Furthermore, Lipschitz-differentiability implies strict differentiability.

Finally we remark that using a Lipschitz-type condition is a special case of using a general Hölder condition; cf. [3] and [4].

11. The most convenient description of Con.

Besides the two possible descriptions of convenient vector spaces considered in section 7 there exist many more.

For the category Con embeddings into various categories of structured vector spaces are described in [7]. Those which allow an efficient access to differentiation should be preferred. This is the case if one uses bornological or dualized vector spaces: the smooth curves (analogous results hold for Lip^k -curves) can be described easily; cf. 4.3, resp. 2.1, 1°.

In section 6 we saw that in order to be suitable for differentiation theory the spaces must have a certain completeness property, namely Mackey com-

pleteness. This property is of bornological nature and thus bornologies seem to be the most natural structures for general calculus.

There is still another reason in favour of bornologies. The category $\underline{Con}$ is complete and cocomplete, i.e. all categorical limits and colimits exist. But a simple, efficient way for calculating them is desirable. There is one if one uses an ambient category $\underline{\mathcal{Z}}$ in which all limits and colimits are known, and which contains $\underline{Con}$ as *reflective* (*coreflective*) subcategory, i.e. such that the inclusion functor $\underline{Con} \rightarrow \underline{\mathcal{Z}}$ admits a left (right) adjoint retraction. It is well known that then any limit (colimit) in $\underline{Con}$ coincides with the limit (colimit) taken in $\underline{\mathcal{Z}}$, and colimits (limits) in $\underline{Con}$ are obtained by applying the retraction functor to the colimit (limit) formed in $\underline{\mathcal{Z}}$.

We first consider the category $\underline{BVS}$ of bornological vector spaces as ambient category. In $\underline{BVS}$ limits (colimits) are easy to describe: one forms them in the category of vector spaces and endows these with the initial (final) vector bornology; cf. [8], 2:6.

From $\underline{BVS}$ one can get to $\underline{Con}$ in several steps by restricting the objects successively by the conditions: the bornology has to be linearly generated; it has to be separated; finally it has to be Mackey complete. So we get the following inclusion functors:

$$\underline{Con} \rightarrow \underline{sep.lg.BVS} \rightarrow \underline{lg.BVS} \rightarrow \underline{BVS}.$$

The nice thing is that all these inclusion functors and hence also their composite admit left adjoint retraction functors. These are: the completion functor and the separation functor (for these one can refer to [7]), and the functor $\beta \circ \delta$ (cf. 3.2). For the inclusion functor $\underline{Con} \rightarrow \underline{BVS}$ we give an explicit description of its left adjoint retracting functor $\omega : \underline{BVS} \rightarrow \underline{Con}$ in section 12.

By the results on reflective subcategories one now obtains:

11.1 Theorem. *The category $\underline{Con}$ is complete and cocomplete. Limits in $\underline{Con}$ can be formed in $\underline{BVS}$. Colimits are obtained by applying the functor ω to the colimit taken in $\underline{BVS}$. In particular, if E_i are convenient vector spaces, the product of the underlying vector spaces endowed with the product bornology is a convenient vector space E which (together with the projections $\pi_i : E \rightarrow E_i$) is the product in $\underline{Con}$ of the given objects E_i .*

11.2 Proposition. *If $F \subseteq E$ is a vector subspace of a convenient vector space E , then F endowed with the subspace bornology ($B \subseteq F$ is bounded in F iff B is bounded in E) is also convenient.*

If one would however choose $\underline{DVS}$ as ambient category the inclusion $\underline{Con} \rightarrow \underline{DVS}$ would decompose into inclusions of reflective and coreflective subcategories. This would make the construction of limits and colimits more complicated than with $\underline{BVS}$ as ambient category.

12. The left adjoint retraction to $\underline{Con} \rightarrow \underline{BVS}$.

12.1 Remark. We already saw in section 11 that the inclusion functor

$$\underline{Con} \rightarrow \underline{BVS}$$

admits a left adjoint retraction functor. We give an explicit description of such a functor $\underline{BVS} \rightarrow \underline{Con}$.

12.2 Proposition. *A left adjoint retraction $\omega : \underline{BVS} \rightarrow \underline{Con}$ of the inclusion functor $\underline{Con} \rightarrow \underline{BVS}$ is obtained as follows.*

We first consider the (contravariant) duality functor of $\underline{BVS}$. It associates to an object E of $\underline{BVS}$ its bornological dual E' , endowed with the canonical bornology, cf. 8.1; and to a morphism $f : E \rightarrow F$ it associates the map $f^* : E' \rightarrow F'$. The square of this functor is the covariant biduality functor with $E \mapsto E''$ and $f \mapsto f^{**}$. The canonical maps $j_E : E \rightarrow E''$ are morphisms. They form a natural transformation j of the identity functor into the biduality functor. In fact, one verifies that for any morphism $f : E \rightarrow F$ the following diagram commutes:

$$\begin{array}{ccc} E & \xrightarrow{j_E} & E'' \\ f \downarrow & & \downarrow f^{**} \\ F & \xrightarrow{j_F} & F'' \end{array}$$

By $\mathcal{T} = \mathcal{T}_E$ we denote the Mackey closure topology of E'' , i.e. the topology which is final with respect to the Mackey convergent sequences (equivalently: with respect to the smooth curves) of E'' . Though $\mathcal{T}$ is in general

not a vector space topology (addition is partially continuous), the closure of a vector subspace is a subspace and we define $\omega(E) := \overline{j(E)}$. Since E'' is linearly generated and separated, the same holds for $\omega(E)$. In order to show that $\omega(E)$ is Mackey complete, let $a_1, a_2, \dots$ be a Mackey-Cauchy sequence of $\overline{j(E)}$. Considered in E'' it is a Mackey-Cauchy sequence of E'' . Since E'' is Mackey complete, the sequence is Mackey convergent in E'' to some $a \in E''$. But Mackey convergence implies $\mathcal{T}$ -convergence. So $a \in \overline{j(E)}$ and Mackey completeness of ωE is proved.

We consider now the functor ω on a morphism $f : E \rightarrow F$ of BVS.

According to the above diagram, the morphism $f^{**} : E'' \rightarrow F''$ restricts to a morphism $\tilde{f} : j_E(E) \rightarrow j_F(F)$. Furthermore, morphisms are continuous with respect to Mackey closure topology, and this implies that $f^{**}(\overline{j_E(E)}) \subseteq \overline{j_F(F)}$. So we can define $\omega f : \omega E \rightarrow \omega F$ to be the restriction of f^{**} . Functoriality of ω is obvious.

We now show that the functor ω is a retraction to the inclusion.

Let E be convenient. Then $j_E(E)$ is closed and hence $\omega E = j_E(E)$. Furthermore, $j_E : E \rightarrow j_E(E)$ is injective and E is linearly generated. By 9.1 $j_E : E \rightarrow E''$ is an embedding and so one deduces that $E \rightarrow \omega E$ is an isomorphism.

We finally show that the functor ω is left adjoint to the inclusion Con $\rightarrow$ BVS. This follows since we have a natural transformation from the identity functor of BVS to the functor ω , formed by the maps $E \rightarrow j_E(E) \rightarrow \overline{j_E(E)} = \omega E$.

13. Two examples.

As we have seen in section 11, convenient vector spaces can be described:

- (i) as certain *bornologically generated* dualized vector spaces E ;
- (ii) as certain *linearly generated* (convex) bornological vector spaces.

We give an elementary example showing that the condition “bornologically generated” in (i) is not automatically satisfied, and an other example showing that the condition “linearly generated” in (2) is not trivial.

Let F be a Banach space. We associate to F a bornological vector space as follows: it has the same underlying vector space, and its bounded subsets are those which are bounded in norm. To the dual Banach space F' then corresponds the bornological dual and its bornology is the canonical function

space bornology. For a Banach space the standard map $j : F \rightarrow F''$ is always an injective homomorphism. If j is surjective, F is called *reflexive*. One has $jF \subseteq F'' \subseteq (F')^*$ and so one has the following dualized vector spaces: $E_1 := (F', jF)$ and $E_2 := (F', F'')$. Let $\mathcal{B}_i$ be the bornology of βE_i . For $B \subseteq F'$ one has: $B \in \mathcal{B}_1$ iff $(ja)(B) = B(a)$ is bounded (in $\mathbb{R}$) for all $a \in F$ which shows that $\mathcal{B}_1$ is the pointwise bornology. Furthermore, $B \in \mathcal{B}_2$ iff $l(B)$ is bounded (in $\mathbb{R}$) for all $l \in (F')'$. It is well known that for a Banach space the bounded subsets are those which are bounded under every element of the dual Banach space. Applying this to F' we get $B \in \mathcal{B}_2$ iff $B \subseteq E'$ is bounded in norm. Hence $\mathcal{B}_2$ is the canonical bornology of E' . Since E is Mackey complete one has $\mathcal{B}_1 = \mathcal{B}_2$ by the Uniform Boundedness Principle. So $\beta E_1 = \beta E_2$. Now one concludes that $\delta\beta E_1 = \delta\beta E_2 = E_2$. If the Banach space F is non-reflexive we have $E_1 \neq E_2$. So we have:

13.1 Example. *If F is a non-reflexive Banach space, then the dualized vector space (F', jF) is not bornologically generated. The associated bornologically generated dualized vector space is (F', F'') .*

The following spaces yield an explicite version for 13.1:

$$c_o := \{s : \mathbb{N} \rightarrow \mathbb{R} \mid \lim_n s_n = 0\} \text{ with } \|s\| := \max_n |s_n|;$$

$$\ell_1 := \{s : \mathbb{N} \rightarrow \mathbb{R} \mid \sum_n |s_n| < \infty\} \text{ with } \|s\| := \sum_n |s_n|;$$

$$\ell_\infty := \{s : \mathbb{N} \rightarrow \mathbb{R} \mid \sup_n |s_n| < \infty\} \text{ with } \|s\| := \sup_n |s_n|.$$

They are Banach spaces and satisfy $(c_o)' = \ell_1$ and $(c_o)'' = (\ell_1)' = \ell_\infty$. One can consider c_o and similarly ℓ_∞ as subspaces of $(\ell_1)^*$. So (ℓ_1, c_o) and (ℓ_1, ℓ_∞) are dualized vector spaces. Since c_o is strictly included in ℓ_∞ one has by 13.1:

$$\delta\beta(\ell_1, c_o) = \delta\beta(\ell_1, \ell_\infty) = (\ell_1, \ell_\infty) \neq (\ell_1, c_o).$$

13.2 Example. *Let F be a Banach space of infinite dimension. Then the compact bornology is a convex vector bornology on the underlying vector space of F . But this bornology is not linearly generated.*

We recall that the compact bornology associated to a topology has as bounded sets all subsets of compact sets. We leave the verification of the properties stated in 13.2 as exercise. But one can also combine the following results. The bornology is a convex vector bornology, cf. [8] 1.3, Example (3). Let E be the so obtained convex bornological space. It remains to show: $\beta\delta E \neq E$. Let $\nu : \underline{LCS} \rightarrow \underline{CBS}$ be the functor which associates to a

locally convex space the underlying vector space endowed with the so-called *von Neumann bornology* and $\gamma : \underline{CBS} \rightarrow \underline{LCS}$ its left-adjoint; and, finally $\mu : \underline{DVS} \rightarrow \underline{LCS}$ the functor which associates to a dualized vector space (E, E') the space E structured by the *Mackey topology*. By [8], exercise 4.E.3, E is non-bornological in the sense of Hogbe, i.e. $\nu\gamma E \neq E$. One has $\gamma E = \mu\delta E$ and $\nu\mu E = \beta E$, cf. (iii) and (iv) of 2.1.21 in [7]. So we obtain $\beta\delta E = \nu\mu\delta E = \nu\gamma E \neq E$.

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