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METRIZABILITY OF σ-FRAMES

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RESUME. En imposant les changements nécessaires à la définition de diamètre métrique d'un cadre donnée par Banaschewski et Pultr, les auteurs donnent une définition de σ-cadre, et donc de la catégorie **MσFrm** des σ-cadres métriques et des applications uniformes de σ-cadres. Ils prouvent alors en particulier l'analogue du théorème de métrisabilité sans point de Banaschewski et Pultr. Finalement, ils caractérisent la catégorie **MσFrm** comme étant l'intersection des catégories **MLFrm** des cadres de Lindelöf métriques et **RσFrm** des σ-cadres réguliers.

1 Preliminaries

Here, we recall some definitions from [1], [2], [4], [6], and [12].

1.1 A *frame* (σ*-frame*) is a lattice L which has arbitrary (countable) joins and satisfies the arbitrary (countable) distributive law $x \wedge \bigvee S = \bigvee\{x \wedge s : s \in S\}$, for all $x \in L$, and arbitrary (countable) subset $S \subseteq L$. A *frame* (σ*-frame*)*map* $h : L \to M$ is a lattice morphism preserving arbitrary (countable) joins. The resulting category is denoted by **Frm** (σ**Frm**).

1.2 A *cover* of a frame (σ-frame) is any (countable) subset $A \subseteq L$ such that $\bigvee A = 1$. The set of all covers of L is denoted by $CovL$. If A, B are covers we say that A *refines* B, and write $A \leq B$, if, for each $a \in A$, there exists $b \in B$ such that $a \leq b$. For a cover A of L and $x \in L$ we put $Ax = \bigvee\{a : a \in A, a \wedge x \neq 0\}$. Let L be a frame (σ-frame) and $\mathcal{U} \subseteq CovL$. We write $x \lhd_{\mathcal{U}} y$ or simply $x \lhd y$ if there exists $A \in \mathcal{U}$ such that $\{a : a \in A, a \wedge x \neq 0\} \subseteq \downarrow y$. We say that $\mathcal{U}$ is an *admissible system* if $x = \bigvee T$, for some (countable) subset T of $\{y : y \lhd x\}$, for each $x \in L$. Note that, if L is a frame, then we have $Ax \leq y$ if and only if $\{a : a \in A, a \wedge x \neq 0\} \subseteq \downarrow y$, for each $A \in CovL$.

1.3 In a bounded lattice, we say that a is rather below b, and write $a \prec b$, if there exists a *separating element* s of L with $a \wedge s = 0$ and $s \vee b = 1$. A frame (σ-frame) L is called *regular* if each of its elements is a (countable) join of elements rather below it. Notice that, in a frame, $a \prec b$ if and only if $a^* \vee b = 1$, where $a^* = \bigvee\{y : y \wedge a = 0\}$. An element x of a frame L is said to be a *Lindelöf element* if whenever $x = \bigvee S$ for some $S \subseteq L$ then, there exists a countable subset T of S such that $x = \bigvee T$. A frame L is said to be a *Lindelöf frame* if 1 is a Lindelöf element. A frame (σ-frame) L is said to be *paracompact* if each cover has a locally finite refinement. In [2] it is shown that any regular σ-frame is paracompact.

1.4 A *basis* of a frame (σ-frame) L is a subset $B \subseteq L$ such that each element of L is a (countable) join of elements of B. For the elements a, b of a frame (σ-frame) L, we say that a *meets* b if $a \wedge b \neq 0$. A subset X of a frame (σ-frame) L is said to be *locally finite* if, there is a cover W of L such that each $w \in W$ meets only finitely many $x \in X$, and it is said to be *discrete* if, each $w \in W$ meets at most one $x \in X$. The above cover W is said to *witness* the local finiteness respectively discreteness of X. A basis is called *σ-locally finite* (*σ-discrete*) if, it is a countable union of locally finite (discrete) sets, and it is called σ-admissible if, it can be written as union of an admissible system of covers.

1.5 Note Let L be a frame. If $\mathcal{U}$ is an admissible system of covers of L, then $\bigcup \mathcal{U}$ is a basis of the frame L [1]. We can show, in the same way, that this is also true for σ-frames. Given $a \in L$, let $a = \bigvee\{y_n : y_n \lhd a\}$. Take $B_n \in \mathcal{U}$ such that $B_n y_n \leq a$. Then, $a = \bigvee\{b : b \in B_n, b \wedge y_n \neq 0, n \in \mathbb{N}\}$.

1.6 Lemma *Any σ-frame with a countable basis is a frame.*

Proof: Let B be a countable basis of a σ-frame L and X be an arbitrary subset of L. It is easy to show that $\bigvee X$ exists and it is equal to $\bigvee\{b \in B, b \leq x,$ for some $x \in X\}$.

Also, for each $X \subseteq L$ and $a \in L$, $a \wedge \bigvee X = \bigvee\{a \wedge b : b \in B, b \leq x,$ for some $x \in X\} \leq \bigvee\{a \wedge x : x \in X\} \leq a \wedge \bigvee X$. Hence, L is a frame. □

1.7 Note In [1] it is shown that, if $X \subseteq L$ is locally finite and $x \prec a$, for each $x \in X$, then $\bigvee X \prec a$. By the above lemma, if L is a σ-frame with a countable basis and $X \subseteq L$ is locally finite, then $x \prec a$, for each $x \in X$, implies $\bigvee X \prec a$.

2 Some properties of bases of σ-frames

Here, we prove the following theorem, which is the counter part of the properties of bases of frames, proved in [1].

2.1 Theorem *The following are equivalent for a σ-frame L:*

(1) *L has a countable basis.*

(2) *L has a σ-discrete basis.*

(3) *L has a σ-locally finite basis.*

Moreover, these are equivalent to the following, if L is regular

(4) *L has a σ-admissible basis.*

Proof: ($1 \Rightarrow 2$) Let $B = \{b_n : n \in \mathbb{N}\}$ be a countable basis. It is enough to take $B_n = \{b_n\}$, for each $n \in \mathbb{N}$.

($2 \Rightarrow 3$) This follows trivially from the definitions.

($3 \Rightarrow 1$) Let $B = \bigcup_{n\in\mathbb{N}} B_n$, where each B_n is a locally finite set. We show that each B_n is a countable set. Let W be the witnessing cover of B_n. Thus each $w \in W$ meets only finitely many $x \in B_n$. Let

$$\{b : b \in B_n, b \wedge w \neq 0\} = \{b(w,1), ..., b(w,l_w)\},$$

for each $w \in W$. Let $b \in B_n$ and $b \neq 0$. There exists $w_i \in W$ such that $b \wedge w_i \neq 0$ since, if $b \wedge w = 0$, for each $w \in W$, then $b = b \wedge \bigvee W = 0$. This shows that $b \in \{b(w_i,1), ..., b(w_i,l_{w_i})\}$ and so there exists $j \leq l_{w_i}$ such that $b = b(w_i,j)$. Therefore, $B_n = \bigcup\{b(w,j) : w \in W, j \in \{1,...,l_w\}\}$. This gives that B_n is a countable set, since W is a countable set. Hence $B = \bigcup B_n$ is a countable basis of L.

($4 \Rightarrow 3$) Let $\{B_n : n \in \mathbb{N}\}$ be an admissible system of covers of L. Regularity of L implies that L is paracompact, and so each B_n has a locally finite refinement, say A_n. Now, $\{A_n : n \in \mathbb{N}\}$ is an admissible system, and so $\bigcup A_n$ is a basis, by Note 1.5. Thus $A = \bigcup A_n$ is a σ-locally finite basis.

($2 \Rightarrow 4$) Let $B = \bigcup B_n$ be a σ-discrete basis, and B_n be witnessed by W_n. Take $S_w = \{b : b \in B_n, w \wedge b \neq 0\}$. We have $S_w = \emptyset$ or $S_w = \{b^w\}$. For each $x \in L$, put $T = \{b : b \in B_k, b \prec x\}$ and $x_k = \bigvee T$. Since $T \subseteq B_k$, T is a discrete subset of L, and so $\bigvee T \prec x$, by Note 1.7. If $S_w = \{b^w\}$, then ${b^w}_k \prec b^w$ implies that there exists $t(w,k) \in L$ such that ${b^w}_k \wedge t(w,k) = 0$ and $b^w \vee t(w,k) = 1$. Take $\mathcal{U} = \{A_{nk} : n, k \in \mathbb{N}\}$, where

$$A_{nk} = \{w \wedge s : w \in W_n, s \in \{b^w, t(w,k)\} \text{ if } S_w = \{b^w\}, \text{ and } s = 1 \text{ if } S_w = \emptyset\}.$$

It is easy to show that $\mathcal{U} \subseteq CovL$. We claim that $A_{nk}b_k \leq b$, for any $b \in B_n$. Let $x \in A_{nk}$, and $x \wedge b_k \neq 0$. We show that $x = w \wedge b$, for some $w \in W_n$. We have, $x \wedge b_k \neq 0$ implies $w \wedge b_k \neq 0$ and so $w \wedge b \neq 0$. Thus $S_w \neq \emptyset$ and also $s \neq t(w,k)$, since if $x = w \wedge t(w,k)$ then $x \wedge b_k \neq 0$ implies $w \wedge t(w,k) \wedge b_k \neq 0$ and so $t(w,k) \wedge b_k \neq 0$. This contradiction shows that $x = w \wedge b^w$. Now, we have $x \wedge b_k \neq 0$ implies $w \wedge b \neq 0$ which gives $b = b^w$. Therefore $x = w \wedge b \leq b$. Hence $\{x : x \in A_{nk}, x \wedge b_k \neq 0\} \subseteq \downarrow b$ and so $b_k \lhd_{\mathcal{U}} b$. Also, by regularity of L

we have that

$$b = \bigvee\{c \in B : c \prec b\} = \bigvee\{b_k : k \in N\}.$$

Thus, for each $x \in L$, $x = \bigvee\{b \in B : b \leq x\} = \bigvee\{\bigvee\{b_k : k \in N\} : b \leq x\}$, where $b_k \lhd b \leq x$, and hence $x = \bigvee\{b_k : b_k \lhd x\}$. Therefore $\mathcal{U}$ is an admissible system, and thus $\bigcup \mathcal{U}$ is a σ-admissible basis. □

Note that any countable basis B of a regular σ-frame L is in fact σ-admissible. Since, by the above theorem, L has a σ-admissible basis $A = \bigcup A_n$. Now, putting $B_1 = B$, $B_{n+1} = \{b : b \in B, b \leq a$ for some $a \in A_n\}$, one can show that $B = \bigcup B_n$.

3 Metrization Theorems for σ-frames

In this section, interpreting the definition of a metric diameter on a frame given in [1], we prove the counterparts of the metrization theorems for σ-frames.

3.1 Definition A *metric diameter* on a σ-frame L is a monotone zero-preserving map $d : L \to \overline{R}_+$ such that

(1) for all $a, b, d(a \vee b) \leq d(a) + d(b)$, whenever $a \wedge b \neq 0$,

(2) for each $\varepsilon > 0$, there is a countable subset S of ${D_\varepsilon}^L = \{a : d(a) < \varepsilon\}$ such that $\bigvee S = 1$,

(3) for all $a \in L$, there is a countable subset T of $\{y : y \lhd a\}$ such that $\bigvee T = a$, where $y \lhd a$ means that, there exists $\varepsilon > 0$ such that $\{b : b \in D_\varepsilon, b \wedge y \neq 0\} \subseteq \downarrow a$.

(4) for all $a \in L$, and $\varepsilon > 0, d(a) = \sup\{d(x \vee y) : x, y \in D_\varepsilon \cap \downarrow a\}$, whenever $d(a) \geq \varepsilon$.

A σ-frame that admits a metric diameter is said to be *metrizable.* Also, a σ-frame map $f : L \to M$ between metric σ-frames is said to be *uniform* if, for each $\varepsilon > 0$, there exists $\delta > 0$ such that ${D_\delta}^M \leq f[{D_\varepsilon}^L]$.

Metric frames (σ-frames) together with uniform frame (σ-frame) maps form a category, denoted by **MFrm** (**MσFrm**). Metric Lindelöf frames together with uniform frame maps form a category, denoted by **MLFrm**, which is a full subcategory of **MFrm**. See [7], [9], [10], [11] for details.

Having Theorem 2.1, we prove the remaining parts of the following.

3.2 Theorem *For a regular σ-frame, the following statements are equivalent.*

(1) *L is metrizable.*

(2) *L has a σ-locally finite basis. (Nagata-Smirnov)*

(3) *L has a σ-discrete basis. (Bing)*

(4) *L has a σ-admissible basis.*

(5) *L has a countable basis.*

(6) *L has a countable admissible system of covers. (Moore)*

Proof: ($5 \Rightarrow 1$) Let L be a regular σ-frame with a countable basis B. By Lemma 1.6, L is a frame and the regularity of L as a σ-frame implies the regularity of L as a frame. Thus, by Urysohn metrization theorem for frames [1], L is a metrizable frame. Let $d : L \to \overline{R}_+$ be a metric diameter on the frame L.

To show that d is a metric diameter on the σ-frame L, it is enough to check the conditions (2) and (3) of the definition. For each $\varepsilon > 0$, the set $\{b \in B : b \leq x, \text{ for some } x \in D_\varepsilon\}$ is a countable subcover of D_ε. Also, for $x \in L$, $x = \bigvee\{b \in B : b \lhd x\}$. Hence, L is a metric σ-frame.

($1 \Rightarrow 6$) Let L be a metric σ-frame with metric diameter d. For each $n \in \mathbb{N}$, let A_n be a countable subcover of $D_{1/n}$. Take $\mathcal{U} = \{A_n : n \in \mathbb{N}\}$. We show that $\mathcal{U}$ is an admissible system. It is enough to show that $x \lhd_d y$ implies $x \lhd_{\mathcal{U}} y$. Let $x \lhd_d y$. Take $\varepsilon > 0$ such that $\{a : a \in D_\varepsilon, a \wedge x \neq 0\} \subseteq\downarrow y$. We choose $n > 1/\varepsilon$. Then, $\{a : a \in A_n, a \wedge x \neq 0\} \subseteq\downarrow y$, and so $x \lhd_{\mathcal{U}} y$, since $A_n \in \mathcal{U}$. Hence, $\mathcal{U}$ is an admissible system of covers.

($6 \Rightarrow 5$) Let L be a σ-frame and $\mathcal{U} = \{A_n : n \in \mathbb{N}\}$ be a countable admissible system of covers of L. Then, by Note 1.5, $B = \bigcup A_n$ ia a countable basis. □

Compare the following with [1, 4.3].

3.3 Proposition *A regular Lindelöf frame is metrizable if and only if it has a countable basis.*

Proof: By Urysohn metrization theorem [1], a regular frame with countable basis is metrizable. Conversely, let L be a metric Lindelöf frame with a metric diameter d.By Lindelöfness of L, there exists a countable cover A_n such that $A_n \subseteq D_{1/n}$, for each $n \in \mathbb{N}$, since $\bigvee D_{1/n} = 1$. Take $B = \bigcup A_n$. We claim that B is a countable basis of L. Given $x \in L$, we have $x = \bigvee\{y : y \lhd x\}$. Consider $C = \{y : y \lhd x\}$. For each $y \in C$, there exists $\varepsilon_y > 0$ such that $\{a : a \in D_{\varepsilon_y}, a \wedge y \neq 0\} \subseteq \downarrow x$. We choose $n_y > 1/\varepsilon_y$. It is easy to show that

$$x = \bigvee\{a \in A_{n_y} : a \wedge y \neq 0, y \in C\}.$$

Hence, B is a countable basis of L. □

4 Characterization of metric σ-frames

In this section we characterize the category **MσFrm**, of metric σ-frames, as the intersection of the categories **MLFrm**, of metric Lindelöf frames, and **RσFrm**, of regular σ-frames.

4.1 Lemma *Any metric σ-frame is a metric Lindelöf frame.*

Proof: Let L be a metric σ-frame with metric diameter d. By Theorem 3.2, L has a countable basis, say B, and so it is a frame, by Lemma 1.6. Clearly L is a metric frame with metric diameter d. It is enough to show that L is a Lindelöf frame.

Let $\bigvee S = 1$, for some $S \subseteq L$. We have $1 = \bigvee S = \bigvee\{b \in B : b \leq s_b$, for some $s_b \in S\}$. Thus, $1 = \bigvee\{s_b : b \in B\}$. Hence L is a Lindelöf frame. □

4.2 Note Let L be a σ-frame with a countable basis B. Then, any σ-frame map $f : L \to M$ preserves arbitrary joins. Given $S \subseteq L$, then

$$f(\bigvee S) \leq f(\bigvee\{s : s \leq b, \text{ for some } b \in B\}) \leq \bigvee f(S) \leq f(\bigvee S).$$

4.3 Proposition *The category* **MσFrm** *is a full subcategory of* **MLFrm**.

Proof: By Lemma 4.1, it is enough to show that any uniform σ-frame map between metric σ-frames is a uniform frame map. Let $f : L \to M$ be a uniform σ-frame map. By Theorem 3.2, L has a countable basis and so, by the above note, f is a frame map. Uniformity of f as a σ-frame map implies the uniformity of f as a frame map. Thus, **MσFrm** $\subseteq$ **MLFrm**. Also, any uniform frame map is a uniform σ-frame map. Thus, **MσFrm** is a full subcategory of **MLFrm**. □

4.4 Lemma *Let L be a Lindelöf frame with countable regularity property (each element of L is a countable join of elements rather below it). Then, each element of L is Lindelöf.*

Proof: Let $x = \bigvee S$, for some subset S of L. By countable regularity of L, we have $x = \bigvee\{y_n : y_n \prec x\}$. For each $n \in \mathbb{N}$, we have $1 = y_n^* \vee x = y_n^* \vee \bigvee S = \bigvee\{y_n^* \vee s : s \in S\}$. Lindelöfness of L implies that there exists a countable subset $T_n \subseteq S$ such that $\bigvee\{y_n^* \vee s : s \in T_n\} = 1$ and so $y_n \prec \bigvee T_n$, for each $n \in \mathbb{N}$. Take $T = \bigcup T_n$. Then, $x = \bigvee\{y_n : y_n \prec x\} \leq \bigvee T \leq x$. Therefore, $x = \bigvee T$ and T is a countable subset of S. □

4.5 Lemma **MLFrm** $\cap$ **RσFrm** $\subseteq$ **MσFrm**.

Proof: Let L be a metric Lindelöf frame, with a metric diameter d, as well as a regular σ-frame. We show that d is a metric diameter on the σ-frame L. Lindelöfness of L as a frame gives a countable cover A, for each D_ε. Given $x \in L$, we have $x = \bigvee\{y \in L : y \lhd x\}$.

By Lemma 4.4, x is a Lindelöf element and so there exists a countable subset T of $\{y : y \lhd x\}$ such that $x = \bigvee T$. Thus L is a metric σ-frame. Therefore $Ob(\mathbf{MLFrm}) \cap Ob(\mathbf{R\sigma Frm}) \subseteq Ob(\mathbf{M\sigma Frm}.)$ Also, clearly any uniform frame map is a uniform σ-frame map. Thus $\mathbf{MLFrm} \cap \mathbf{R\sigma Frm} \subseteq \mathbf{M\sigma Frm}$. □

4.6 Note Any metric frame (σ-frame) is a regular frame (σ-frame). To see this, it is enough to show that $x \lhd y$ implies $x \prec y$. Let $x \lhd y$. Take $\varepsilon > 0$ such that $\{a : a \in D_\varepsilon, a \wedge x \neq 0\} \subseteq \downarrow y$. Then, there exists a (countable) subset $S \subseteq D_\varepsilon$ such that $\bigvee S = 1$. Take $t = \bigvee\{s : s \in S, s \wedge x = 0\}$. It is easy to show that $x \wedge t = 0$ and $y \vee t = 1$, and so $x \prec y$.

4.7 Theorem *The category* $\mathbf{M\sigma Frm}$ *is exactly the intersection of the categories* $\mathbf{MLFrm}$ *and* $\mathbf{R\sigma Frm}$.

Proof: By Lemma 4.5, we have $\mathbf{MLFrm} \cap \mathbf{R\sigma Frm} \subseteq \mathbf{M\sigma Frm}$, and by Proposition 4.3, $\mathbf{M\sigma Frm}$ is a full subcategory of $\mathbf{MLFrm}$. Also, any metric σ-frame is a regular σ-frame, by the above note. Therefore the category $\mathbf{M\sigma Frm}$ is a subcategory of the category $\mathbf{R\sigma Frm}$. Hence $\mathbf{MLFrm} \cap \mathbf{R\sigma Frm} = \mathbf{M\sigma Frm}$. □

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