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ON REGULAR PRESHEAVES AND REGULAR SEMI-CATEGORIES

by M.-A. MOENS, U. BERNI-CANANI and F. BORCEUX*[1]

Résumé. Nous généralisons la théorie des modules réguliers sur un anneau sans unité au cas des préfaisceaux sur une "catégorie sans unité" que nous appelons une semi-catégorie. Nous travaillons dans le contexte de la théorie des catégories enrichies. L'axiome de régularité sur un préfaisceau revient à être canoniquement une colimite de préfaisceaux représentables et la semi-catégorie elle-même est régulière quand son foncteur Hom vérifie cette condition. Nous montrons la relation avec le lemme de Yoneda et obtenons un exemple de ce que F.W. Lawvere appelle "l'unité des opposés". Nous concluons avec un théorème de Morita pour les semi-catégories régulières. Nous donnons différents exemples provenant de la théorie des matrices, des opérateurs de Hilbert–Schmidt et des Ω-ensembles.

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Introduction

Let R be a commutative ring with unit. Given a R-algebra A with unit, a right A-module M verifies in particular the axiom $m \cdot 1 = m$ for each element $m \in M$. When the algebra does not have a unit, this axiom no longer makes sense and must be replaced by an alternative version, equivalent to this classical axiom in the case where A has a unit. It is the so-called *regularity* condition on M, meaning that the scalar multiplication induces an isomorphism $M \otimes_A A \longrightarrow M$.

Regular modules play an essential role in the construction of the Brauer–Taylor group of the ring R, via the consideration of Azumaya algebras without unit (see [9]).

In [2], Azumaya categories enriched in a symmetric monoidal closed category $\mathcal{V}$ have been introduced. They allow defining the categorical Brauer group of $\mathcal{V}$ which, when $\mathcal{V}$ is the category of modules over the ring R, reduces to the classical Brauer group of R. This paper throws a bridge to an analogous theory "without identities", with the final goal of reaching a theory of enriched Azumaya graphs and the corresponding categorical Brauer–Taylor group of the base category $\mathcal{V}$. The developments of the present paper allow in particular introducing all these notions, but investigating their properties will be the topic of another work.

The present paper focuses in fact on a categorical generalization, over an arbitrary base $\mathcal{V}$, of the theory of regular modules. We introduce for this purpose enriched "categories without units", which we call more positively "semi-categories". All our notions are enriched in a base category $\mathcal{V}$ and by convention, we avoid repeating it all the time. We study the corresponding notion of "regular" presheaf, generalizing the notion of regular module. This generalization is categorically very natural, since it reduces to the fact of being a colimit of representable presheaves, a fact which is classical for categories and functors. And precisely when the semi-category turns out to be an actual category, the regular presheaves on the semi-category coincide with the actual functorial presheaves on the category.

A special attention is devoted to the regular semi-categories $\mathcal{G}$, that is, those semi-categories for which the representable presheaves are themselves regular. In that case, the category of regular presheaves on $\mathcal{G}$ is an esssential colocalization of the category of all presheaves. In particular, when $\mathcal{V}$ is abelian, the regular presheaves on a regular semi-category constitute again an abelian category. And when $\mathcal{V}$ is the category of sets, the regular presheaves on a regular semi-category constitute a Grothendieck topos.

At this stage, it is useful to throw a link with Lawvere's idea of the "unity of opposites" (see [7]). Regular presheaves on a regular semi-category verify the Yoneda lemma precisely when the semi-category is an actual category. But the regular presheaves constitute a colocalization of the category of all presheaves, thus the second right adjoint is itself full and faithful. And this second embedding identifies precisely the regular presheaves with those presheaves satisfying the Yoneda lemma.

We generalize further, to the context of regular semi-categories, various basic aspects of the theory of distributors (also called profunctors, or bimodules) and prove that regular distributors between regular semi-categories organize themselves in a bicategory. Two regular semi-categories $\mathcal{G}$ and $\mathcal{H}$ are equivalent in this bicategory precisely when the corresponding categories $\mathsf{Reg}(\mathcal{G}^{\mathrm{op}}, \mathcal{V})$ and $\mathsf{Reg}(\mathcal{H}^{\mathrm{op}}, \mathcal{V})$ of regular presheaves are equivalent. This is a Morita theorem for regular semi-categories.

It is a matter of fact that some semi-categories $\mathcal{G}$ occur quite naturally as ideals in a bigger, rather natural category $\mathcal{C}$. For example the compact operators between Hilbert spaces constitute a two-sided ideal in the category of all bounded operators, the whole situation being enriched in the base category of Banach spaces. In such cases, some people will prefer to study $\mathcal{G}$ as an ideal of $\mathcal{C}$, instead of an independant entity. It is a matter of taste, but the importance given to the study of $\mathbb{C}^*$-algebras without unit indicates that such a choice is certainly not universal.

In the spirit of the previous example, consider now the Hilbert-Schmidt operators between Hilbert spaces. This is again a situation enriched in the base category of Banach spaces, but the Hilbert-Schmidt norm can by no way be extended to more general operators. So in this case, the semi-category of Hilbert–Schmidt operators appears as a natural entity to study for itself: this is an example of a regular semi-category enriched in the category of Banach spaces.

It remains nevertheless true that every semi-category can be presented as a two-sided ideal in the category obtained by adding formally identitity arrows. This category is generally too "unnatural" to be devoted some interest, except as a tool in the proofs.

Another important example of a regular enriched semi-category is a sheaf F on a locale Ω, viewed as an Ω-set A. A locale Ω is in particular a cartesian closed category and an Ω-set A is a symmetric semi-category enriched in Ω. Requesting that A is an actual Ω-category would force the sheaf F to be generated by its global elements. In this specific case, the Ω-category of regular presheaves on A coincides with the locale of subsheaves of F.

1 A quick introduction to regular modules

This section recalls the basic idea of the theory of regular modules on an arbitrary ring and underlines crucial differences with the more classical theory involving a unit.

Let R be a commutative ring with a unit. Let A be an R-algebra, not necessarily commutative, not necessarily with a unit. A right A-module M is thus an R-module provided with a scalar multiplication by the elements of A, with axioms:

$$(m_1 + m_2)a = m_1 a + m_2 a \qquad \text{[M1]}$$

$$m(a_1 + a_2) = ma_1 + ma_2 \qquad \text{[M2]}$$

$$m(a_1 a_2) = (ma_1)a_2 \qquad \text{[M3]}$$

for all elements $a, a_1, a_2 \in A$ and $m, m_1, m_2 \in M$. The notion of left A-module is dual and the tensor product $M \otimes_A N$ is the R-module defined in the usual way.

When A has a unit 1, one requires also from a right A-module the axiom

$$m \cdot 1 = m. \qquad \text{[M4]}$$

This axiom does not make sense in the absence of a unit. Thus in the presence of a unit, we are left with two possible notions of module: with or without axiom [M4].

It is well-known that a unit can always be added formally to an R-algebra A. More precisely, given a R-algebra A, define $\overline{A} = A \oplus R$ with multiplication $(a,r)(b,s) = (ab + as + br, rs)$. Then A in an R-algebra with unit $(0,1)$.
Every right A-module verifying axioms [M1] to [M3] becomes a $\overline{A}$-module verifying axioms [M1] to [M4] when defining $m(a,r) = ma + mr$. This defines an equivalence between the corresponding categories, proving indeed that the study of modules on an algebra without unit reduces always to the study of modules on the algebra where a unit has been formally added.

The point concerning regular modules is completely different. Given a R-algebra A, can we express an axiom on right A-modules which makes sense even when A does not have a unit, and which reduces to the classical axiom [M4] when A turns out to have a unit. The answer, which is classical, is given by the following proposition (see [9]).

Proposition 1.1 *Le R be a commutative ring with unit and A, an R-algebra with unit. The following conditions are equivalent on a A-module M:*

[M4] $\forall m \in M \quad m \cdot 1 = m$;

[M5] $\mu \colon M \otimes_A A \longrightarrow M; \quad m \otimes a \mapsto ma$ *is an isomorphism of right A-modules.*

Proof Given [M4], the inverse σ of μ is given by $\sigma(m) = m \otimes 1$. Given [M5] and $m \in M$, there exist elements $m_i \in M$ and $a_i \in A$ such that $m = \sum_{i=1}^{n} m_i a_i$, which yields $m \cdot 1 = m$. □

Definition 1.2 *Let R be a commutative ring with unit and A an R-algebra, with or without unit. A regular right A-module is one verifying axioms [M1], [M2], [M3], [M5].*

For an arbitrary A, the category of regular A-modules can by no way be reduced to the category of modules with axioms [M1] to [M4], for whatever R-algebra $\overline{A}$ with unit. In general, a category of regular modules is not even abelian. But when A itself is regular, the category of regular A-modules remains abelian (see [3], or corollary 4.5).The category of regular modules on the ring $2\mathbb{Z}$ provides an counterexample.

It is well-known that a R-algebra A with unit can be seen as a one-object category $\mathcal{A}$ enriched in the symmetric monoidal closed category of R-modules. A right A-module is then just an enriched presheaf $\mathcal{A}^{\mathrm{op}} \rightarrow \mathsf{Mod}_A$. The aim of this paper is to study a corresponding categorical context generalizing the case of an A-algebra without a unit and to focus in this context on the categorical notion of regularity.

2 Semi-categories and their presheaves

This section settles the context of our theory of "enriched categories without identities", which we call "semi-categories". We avoid repeating aspects which are straightforward transcriptions of the case where identities exist. On the other hand we insist on some crucial differences.

In the rest of this paper, $\mathcal{V}$ will always denote a complete and cocomplete, symmetric monoidal closed category. All our notions will be enriched in $\mathcal{V}$ and, for the sake of brevity, we shall avoid recalling it every time.

Definition 2.1 *A semi-category consists in*

1. *a set $|\mathcal{G}|$ of objects;*
2. *for all objects $A, B \in |\mathcal{G}|$, an object $\mathcal{G}(A,B) \in \mathcal{V}$;*
3. *for all objects $A, B, C \in |\mathcal{G}|$, a composition law*

$$c_{A,B,C} \colon \mathcal{G}(A,B) \otimes \mathcal{G}(B,C) \longrightarrow \mathcal{G}(A,C);$$

those data are requested to verify the diagrammatic axiom expressing the associativity of the composition law (see [5]).

For facility, we have introduced the smallness condition in the definition of a semi-category. Of course large semi-categories are those obtained by dropping this restricition. Clearly, every category is a (possibly large) semi-category.

Definition 2.2 *A morphism $F\colon \mathcal{G} \to \mathcal{H}$ of semi-categories consists in*

1. *for each object $A \in |\mathcal{G}|$, an object $F(A) \in |\mathcal{H}|$;*
2. *for all objects $A, B \in |\mathcal{G}|$, a morphism*

$$F_{A,B}\colon \mathcal{G}(A,B) \longrightarrow \mathcal{H}\big(F(A), F(B)\big);$$

those data are requested to verify the diagrammatic axiom expressing the preservation of the composition law (see [5]).

Definition 2.3 *A natural transformation $\alpha\colon F \Rightarrow G$ between morphisms of semi-categories $F, G\colon \mathcal{G} \rightrightarrows \mathcal{H}$ consists in giving, for each object $A \in \mathcal{G}$ a morphism $\alpha_A\colon I \to \mathcal{H}\big(F(A), G(A)\big)$ in $\mathcal{V}$; those data are requested to verify the diagrammatic axiom expressing the naturality of α (see [5]).*

It is useful to recall that, while categories and functors involve an axiom about units, natural transformations do not. The reader should be aware that the absence of units in a semi-category prevents the existence of identity natural transformations, thus the possibility of defining adjunctions via the usual triangular identities.

Given semi-categories $\mathcal{G}$, $\mathcal{H}$, one defines easily new semi-categories $\mathcal{G}^{\mathsf{op}}$, $\mathcal{G} \otimes \mathcal{H}$, $\mathcal{G}^{\mathsf{op}} \otimes \mathcal{G}$ and corresponding operations between morphisms of semi-categories and natural transformations.

An essential difference with the case of categories and functors lies in the consideration of bimorphisms. A morphism of semi-categories $F\colon \mathcal{G} \otimes \mathcal{H} \longrightarrow \mathcal{K}$ does not, in general, induce morphisms $F(A,-)\colon \mathcal{H} \to \mathcal{K}$, and $F(-,B)\colon \mathcal{G} \to \mathcal{K}$, for all $A \in \mathcal{G}$, $B \in \mathcal{H}$. Indeed, already in the case $\mathcal{V} = \mathsf{Set}$, expressing the action of $F(A,-)$ on arrows requires a formula of the type $F(A,f) = F(\mathsf{id}_A, f)$ which involves clearly the identity morphism on A. This difference with the case of categories will vanish under the assumption of regularity (see section 3). As an exercice, we give the following counterexamples.

Counterexample 2.4 $\mathcal{G}$ is the multiplicative semi-group $[0, \frac{1}{10}]$, viewed as a Set-semi-category with a single object A. $\mathcal{C}$ is a category with two objects B, C, one non-trivial arrow $10\colon B \to C$ and the two identity morphisms. We get a morphism of semi-categories by defining

$$T\colon \mathcal{G} \times \mathcal{C} \longrightarrow \mathsf{Set},\quad T(A,B) = [0,1] = T(A,C),\quad T(u,v)(r) = u \cdot r \cdot v.$$

T is a two-variables morphism which does not decompose in one-variable morphisms.

Counterexample 2.5 Let R be a commutative ring with unit. We choose a commutative R-algebra A without unit, admitting two non-zero ideals I, J such that $I \cdot J = (0)$. We view I and J as one-object semi-categories and we consider the morphism of semi-categories

$$T\colon I \otimes J \longrightarrow \mathcal{V},\quad T(\star,\star) = A,\quad T(i,j)(a) = i \cdot a \cdot j.$$

T decomposes in several ways in one-variable morphisms.

Observe nevertheless that given an object A of a semi-category $\mathcal{G}$, the representable morphism $\mathcal{G}(A,-)\colon \mathcal{G} \to \mathcal{V}$, exists. Its action on arrows corresponds, by adjunction, to the composition of $\mathcal{G}$. A same argument applies to the contravariant representable morphism $\mathcal{G}(-,B)$. Finally, by associativity of the composition law, we get a morphism $\mathcal{G}\colon \mathcal{G}^{\mathsf{op}} \otimes \mathcal{G} \to \mathcal{V}$, with the property that the individual morphisms $\mathcal{G}(A,-)$ and $\mathcal{G}(-,B)$ all exist.

In particular, if $F, G\colon \mathcal{G} \rightrightarrows \mathcal{H}$ are morphisms of semi-categories, the existence of the representable morphisms allow defining the object $\mathsf{Nat}(F,G)$ of natural transformations in the usual way, via an equalizer

$$\mathsf{Nat}(F,G) \rightarrowtail \prod_{A\in\mathcal{G}} \mathcal{H}(FA,GA) \rightrightarrows \prod_{A,B\in\mathcal{G}} \big[\mathcal{G}(A,B), \mathcal{H}(FA,GB)\big],$$

where the right hand morphisms are induced respectively, via adjunction, by the action of G followed by that of $\mathcal{H}(FA,-)$ and the action of F followed by that of $\mathcal{H}(-,GB)$. This provides the morphisms of semi-categories from $\mathcal{G}$ to $\mathcal{H}$ with the structure of a semi-category $[\mathcal{G},\mathcal{H}]$. When $\mathcal{H}$ happens to be a category, $[\mathcal{G},\mathcal{H}]$ is a category as well.

The previous discussion depends only on the smallness of $\mathcal{G}$ and does not require at all the smallness of $\mathcal{H}$. Therefore:

Definition 2.6 *Given a semi-category $\mathcal{G}$, the category of presheaves on $\mathcal{G}$ is the category $[\mathcal{G}^{\mathsf{op}}, \mathcal{V}]$ of morphisms of semi-categories. The functor*

$$Y_{\mathcal{G}} \colon \mathcal{G} \longrightarrow [\mathcal{G}^{\mathsf{op}}, \mathcal{V}], \quad A \mapsto \mathcal{G}(-, A)$$

is called the Yoneda morphism of $\mathcal{G}$.

We write simply Y when no confusion can occur. It is classical to verify that $Y_{\mathcal{G}}$ is a morphism of semi-categories. It cannot be full and faithful in general, because this would imply that $\mathcal{G}$ is a category, since so is $[\mathcal{G}^{\mathsf{op}}, \mathcal{V}]$.

The existence of objects of natural transformations between presheaves on a semi-category allows generalizing at once to this context the notion of weighted colimit.

Definition 2.7 *Consider a semi-category $\mathcal{G}$, a category $\mathcal{C}$ and morphisms of semi-categories $H \colon \mathcal{G}^{\mathsf{op}} \to \mathcal{V}$, $F \colon \mathcal{G} \to \mathcal{C}$. The colimit $H \star F$ of F weighted by H, when it exists, is a pair $(L \in \mathcal{C}, \lambda) \colon H \Rightarrow \mathcal{C}\big(F(-), L\big)$, inducing for every object $C \in \mathcal{C}$ natural isomorphisms in $\mathcal{V}$*

$$\mathsf{Nat}\Big(H(-), \mathcal{C}\big(F(-), C\big)\Big) \cong \mathcal{C}(L, C).$$

In an arbitrary semi-category, the lack of identities prevents developing a good notion of isomorphism. For that reason, we limit ourselves to considering weighted colimits of functors with values in an actual category $\mathcal{C}$, recapturing so the uniqueness, up to isomorphism, of these colimits. Of course, the notion of weighted limit can be handled dually.

Let us make a strong point that given the situation $\mathcal{G}^{\mathsf{op}} \colon \mathcal{G} \otimes \mathcal{G}^{\mathsf{op}} \to \mathcal{V}$, $F \colon \mathcal{G}^{\mathsf{op}} \otimes \mathcal{G} \to \mathcal{V}$ one can indeed consider the weighted colimit $\mathcal{G}^{\mathsf{op}} \star F$, but there is a priori no reason for that colimit to be calculated by the usual coend formula $\int^{A \in \mathcal{G}} F(A, A)$. Indeed this coend formula is a coequalizer requiring the consideration of the individual morphisms $F(A, -)$ and $F(-, B)$, which do not exist in general as the previous examples show. And even when they do exist, the classical proof of the isomorphism $\int^{A} F(A, A) \cong \mathcal{G}^{\mathsf{op}} \star F$ uses explicitly the existence of identity arrows in $\mathcal{G}$. Again, this coend formula will be recaptured under the regularity assumption of section 3.

Proposition 2.8 *Let $\mathcal{G}$ be a semi-category, $\mathcal{C}$ a tensored cocomplete category and $F : \mathcal{G}^{\mathsf{op}} \longrightarrow \mathcal{V}$ and $G : \mathcal{G} \longrightarrow \mathcal{C}$ two morphisms. There is an isomorphism $F \star G \cong \int^X F(X) \otimes G(X)$.*

Proof It suffices to prove that for every $C \in \mathcal{C}$, there is an isomorphism

$$\mathsf{Nat}\Big(F, \mathcal{C}\big(G(-), C\big)\Big) \cong \mathcal{C}\left(\int^X F(X) \otimes G(X), C\right).$$

This condition is verified if and only if for each $V \in \mathcal{V}$ there is a bijection between the morphisms $V \longrightarrow \mathcal{C}\left(\int^X F(X) \otimes G(X), C\right)$ and the natural transformations $F \Rightarrow \Big[V, \mathcal{C}\big(G(-), C\big)\Big]$.

On the one hand, since $\mathcal{C}$ is tensored and the functor $V \otimes -$ preserves colimits, we get the isomorphism

$$\left[V, \mathcal{C}\left(\int^X F(X) \otimes G(X), C\right)\right] \cong \mathcal{C}\left(\int^X V \otimes F(X) \otimes G(X), C\right).$$

On the other hand, the natural transformations $F \Rightarrow \Big[V, \mathcal{C}\big(G(-), C\big)\Big]$ are in bijection with the compatible families of morphisms $V \otimes F(A) \otimes G(A) \longrightarrow C$. The result immediately follows from the definition of a coend. □

3 The regularity condition

This section is the core of the paper and shows that the regularity property of "preserving identities if they existed" reduces to being a colimit of representable morphisms.

Definition 3.1 *Le $\mathcal{G}$ be a semi-category and F: $\mathcal{G}^{\mathsf{op}} \longrightarrow \mathcal{V}$ a contravariant morphism of semi-categories. The morphism F is called a* regular presheaf on $\mathcal{G}$ *when the canonical morphism $F \star Y_{\mathcal{G}} \Longrightarrow F$ is an isomorphism.*

An analogous definition holds by duality for covariant regular morphisms. Let us establish at once a characterization of regularity in terms of a "coend" formula as a direct consequence of Proposition 2.8.

Proposition 3.2 *Let $\mathcal{G}$ be a semi-category and $F\colon \mathcal{G}^{\mathrm{op}} \to \mathcal{V}$ a contravariant morphism of semi-categories. The morphism F is regular when the canonical comparison morphism*

$$\int^{B} F(B) \otimes \mathcal{G}(A,B) \longrightarrow F(A)$$

is an isomorphism.

□

More generally, the next results allow a classical use of "coend" formulæ in the theory of regular presheaves.

Lemma 3.3 *Let $\mathcal{G}$ and $\mathcal{H}$ be semi-categories. Every regular morphism $F\colon \mathcal{G} \otimes \mathcal{H} \longrightarrow \mathcal{V}$ induces canonically regular morphisms $F(A,-)\colon \mathcal{H} \to \mathcal{V}$ and $F(-,B)\colon \mathcal{G} \to \mathcal{V}$ for all objects $A \in \mathcal{G}$, $B \in \mathcal{H}$.*

Proof Of course one defines $F(A,-)(B) = F(A,B)$ and it remains to construct the action $F(A,-)_{B,C}\colon \mathcal{H}(B,C) \longrightarrow \big[F(A,B), F(A,C)\big]$.
] Using composition of $\mathcal{H}$ and proposition 3.3, we get a morphism corresponding to $F(A,-)_{B,C}$:

$$\int^{X,Z} \mathcal{G}(X,A) \otimes F(X,Z) \otimes \mathcal{H}(Z,B) \otimes \mathcal{H}(B,C) \to F(A,C).$$

and these last morphisms are given by the action of F. It is routine to verify that this makes $F(A,-)$ a morphism of semi-categories. An analogous argument holds for each $F(-,B)$.

The regularity of $F(A,-)$ means that the canonical morphism $\int^{Z} F(A,Z) \otimes \mathcal{H}(Z,B) \to F(A,B)$ is an isomorphism. We shall construct its inverse. F is regular and thus it is enough to construct a morphism

$$\int^{X,Z} \mathcal{G}(X,A) \otimes F(A,B) \otimes \mathcal{H}(Z,B) \to \int^{Z} F(A,Z) \otimes \mathcal{H}(Z,B)$$

Since $F(-,Z)$ is a morphism of semi-categories, the expected morphism is given by the action of $F(-,Z)$ by adjunction. The rest is routine. □

Corollary 3.4 *Let $\mathcal{G}$ be a semi-category and $F\colon \mathcal{G}^{\mathrm{op}} \otimes \mathcal{G} \longrightarrow \mathcal{V}$ a regular morphism. The colimit of F weighted by $\mathcal{G}^{\mathrm{op}}\colon \mathcal{G} \otimes \mathcal{G}^{\mathrm{op}} \longrightarrow \mathcal{V}$ exists and is given by the usual "coend" formula.*

Proof The "coend" formula has been recalled at the end of section 2 and makes sense since, by lemma 3.3, the morphisms $F(A,-)$ and $F(-,B)$ are defined. The weighted colimit $\mathcal{G}^{\mathrm{op}} \star F$ is characterized by the natural isomorphisms $\mathsf{Nat}(\mathcal{G}^{\mathrm{op}}, [F,V]) \cong [\mathcal{G}^{\mathrm{op}} \star F, V]$ for all $V \in \mathcal{V}$.

A morphism $f\colon \mathcal{G}^{\mathrm{op}} \star F \to V$ corresponds to a natural transformation β given by $\beta_{A,B}\colon \mathcal{G}(B,A) \to [F(A,B),V]$. This induces a compatible family γ of morphisms

$$\gamma_{A,X,Y}\colon \mathcal{G}(A,X) \otimes F(X,Y) \otimes \mathcal{G}(Y,A) \longrightarrow V$$

obtained by applying first the action of $F(-,Y)$ and next the morphism corresponding by adjunction to $\beta_{A,Y}$, or equivalently by applying first the action of $F(X,-)$ and next the morphism corresponding by adjunction to $\beta_{X,A}$. By regularity of F, this corresponds to morphisms $\delta_A\colon F(A,A) \longrightarrow V$ identifying the two parallel morphisms defining the coend $\int^A F(A,A)$. This yields a morphism $g\colon \int^A F(A,A) \to V$.

Conversely, such a morphism g yields corresponding morphisms δ_A. One defines then $\alpha_{A,B}\colon \mathcal{G}(A,B) \otimes F(B,A) \longrightarrow V$ by the action of $F(-,A)$ followed by δ_A, or equivalently by the action of $F(B,-)$ followed by δ_B. The natural transformation β corresponds to α by adjunction, from which a morphism $f\colon \mathcal{G}^{\mathrm{op}} \star F \to V$.

The rest is routine. □

We want to make a strong point that the analogue of corollary 3.4 for ends and weighted limits has a priori no reason to hold: indeed, the notion of regular presheaf is not auto-dual.

Definition 3.5 *A semi-category is regular when the canonical morphism $\mathcal{G}\colon \mathcal{G}^{\mathrm{op}} \otimes \mathcal{G} \longrightarrow \mathcal{V}$ is regular.*

Particularizing proposition 3.2 yields at once:

Proposition 3.6 *A semi-category $\mathcal{G}$ is regular when the canonical comparison morphisms*

$$\int^{X,Y} \mathcal{G}(A,X) \otimes \mathcal{G}(X,Y) \otimes \mathcal{G}(Y,B) \longrightarrow \mathcal{G}(A,B)$$

are isomorphisms for all $A, B \in \mathcal{V}$. □

One should compare the previous coend formula with the isomorphism $\mu\colon A \otimes_A A \longrightarrow A$ characterizing a regular R-algebra in section 1. Applying twice this isomorphism, it follows at once that the three variables multiplication $\tau\colon A \otimes_A A \otimes_A A \longrightarrow A$ is an isomorphism as well. Conversely, if the three variables multiplication is an isomorphism, the inverse of the two variables multiplication is given by τ^{-1} followed by the multiplication of two variables. Thus the notion of a regular R-algebra in section 1 is indeed a special case of the coend formula in 3.6.

Proposition 3.7 *For a semi-category $\mathcal{G}$, the following conditions are equivalent:*

1. *the semi-category $\mathcal{G}$ is regular;*
2. *the representable morphisms on $\mathcal{G}$ are regular.*

Proof (1) $\Rightarrow$ (2) follows at once from lemma 3.3. The converse is an immediate application of the "associativity" of colimits: a coend indexed by X of coends indexed by Y is at once a coend indexed by (X,Y) (see 3.2 and 3.6). □

Lemma 3.8 *If $\mathcal{G}$ and $\mathcal{H}$ are regular semi-categories, so are $\mathcal{G}^{\mathrm{op}}$ and $\mathcal{G} \otimes \mathcal{H}$.*

Proof Routine calculation from the coend formula of corollary 3.4 and the preservation of coends by tensor product. □

We conclude this section with proving that when a semi-category turns out to have identities, regular presheaves are actual functors, thus presheaves in the ordinary sense.

Proposition 3.9 *Let $\mathcal{A}$ be a category and $F\colon \mathcal{A}^{\mathsf{op}} \longrightarrow \mathcal{V}$ a morphism of semi-categories. The following conditions are equivalent:*

1. *F is a functor;*
2. *F is a regular morphism of semi-categories.*

Proof (1) $\Rightarrow$ (2) is a classical result. Conversely, the assumption implies the isomorphism $F \cong \int^{B\in\mathcal{A}} F(B) \otimes \mathcal{A}(-,B)$. This coend is computed pointwise; therefore, considering the identity $\mathsf{id}_A\colon A \longrightarrow A$ in the category $\mathcal{A}$, $F(\mathsf{id}_A) = \int^{B\in\mathcal{A}} F(B) \otimes \mathcal{A}(\mathsf{id}_A, B)$. Since each $\mathcal{A}(\mathsf{id}_A, B)$ is the identity on $\mathcal{A}(A,B)$, it follows that $F(\mathsf{id}_A) = \mathsf{id}_{F(A)}$. □

4 The category of regular presheaves

First let us observe that given a semi-category $\mathcal{G}$, one gets at once a category $\overline{\mathcal{G}}$ by adding freely identities to $\mathcal{G}$. More precisely, $\overline{\mathcal{G}}$ has the same objects as $\mathcal{G}$ and

$$\overline{\mathcal{G}}(A,B) = \begin{cases} \mathcal{G}(A,B) & \text{when } A \neq B \\ \mathcal{G}(A,A) \amalg I & \text{when } A = B \end{cases}$$

where $I \in \mathcal{V}$ is the unit of the tensor product. The composition law is induced in the obvious way by that of $\mathcal{G}$ and the following lemma follows at once.

Lemma 4.1 *Let $\mathcal{G}$ be a semi-category and $\overline{\mathcal{G}}$, the corresponding category obtained by adding freely identities. The category $[\mathcal{G}^{\mathsf{op}}, \mathcal{V}]$ of morphisms of semi-categories is equivalent to the category of contravariant functors from $\overline{\mathcal{G}}$ to $\mathcal{V}$.* □

The basic result of this section is:

Theorem 4.2 *Let $\mathcal{G}$ be a regular semi-category.* $\mathsf{Reg}(\mathcal{G}^{\mathsf{op}}, \mathcal{V})$ *, the category of regular presheaves on $\mathcal{G}$, is an essential colocalization – thus also an essential localization – of the category $[\mathcal{G}^{\mathsf{op}}, \mathcal{V}]$ of all morphisms of semi-categories.*

Proof We must exhibit three functors $i\colon \mathsf{Reg}(\mathcal{G}^{\mathsf{op}},\mathcal{V}) \hookrightarrow [\mathcal{G}^{\mathsf{op}},\mathcal{V}]$, $j\colon [\mathcal{G}^{\mathsf{op}},\mathcal{V}] \to \mathsf{Reg}(\mathcal{G}^{\mathsf{op}},\mathcal{V})$ and $k\colon \mathsf{Reg}(\mathcal{G}^{\mathsf{op}},\mathcal{V}) \to [\mathcal{G}^{\mathsf{op}},\mathcal{V}]$, with $i \dashv j \dashv k$, where i is the canonical inclusion.

Given a morphism of semi-categories $F\colon \mathcal{G}^{\mathsf{op}} \to \mathcal{V}$, we observe first that $j(F) = F \star Y_{\mathcal{G}}\colon \mathcal{G}^{\mathsf{op}} \to \mathcal{V}$ is regular. This means that the canonical morphism $(F \star Y_{\mathcal{G}}) \star Y_{\mathcal{G}} \longrightarrow F \star Y_{\mathcal{G}}$ is an isomorphism. This follows at once from the regularity of the representable presheaves (see 3.7), which means $Y_{\mathcal{G}}(A) \star Y_{\mathcal{G}} \cong Y_{\mathcal{G}}(A)$ for each $A \in \mathcal{G}$.

It is then routine to check that the canonical morphism $F \star Y_{\mathcal{G}} \to F$ has the required universal property to be the counit of a coreflection between ordinary Set-based categories. To conclude that the coreflection holds in the context of $\mathcal{V}$-enriched categories, it suffices to verify that the $\mathcal{V}$-categories $\mathsf{Reg}(\mathcal{G}^{\mathsf{op}},\mathcal{V})$ and $[\mathcal{G}^{\mathsf{op}},\mathcal{V}]$ are tensored and the inclusion i preserves tensors: this is obvious, since all those tensors are computed pointwise.

The right adjoint to the coreflection is given by

$$k\colon \mathsf{Reg}(\mathcal{G}^{\mathsf{op}},\mathcal{V}) \to [\mathcal{G}^{\mathsf{op}},\mathcal{V}], \quad G \mapsto k(G) = \mathsf{Nat}\big(Y_{\mathcal{G}}(-),G\big).$$

The adjointness property reduces indeed to

$$\mathsf{Nat}\Big(F, \mathsf{Nat}\big(Y_{\mathcal{G}}(-),G\big)\Big) \cong \mathsf{Nat}(F \star Y_{\mathcal{G}}, G)$$

which is exactly the definition of $F \star Y_{\mathcal{G}}$.

Since i is full and faithful, so is k and thus $\mathsf{Reg}(\mathcal{G}^{\mathsf{op}},\mathcal{V})$ is also an essential localization of $[\mathcal{G}^{\mathsf{op}},\mathcal{V}]$. □

Corollary 4.3 *The category of regular presheaves on a regular semi-category is complete and cocomplete.*

Proof By 4.1 and 4.2. □

Corollary 4.4 *Let $\mathcal{G}$ be a regular semi-category. The following conditions are equivalent for a morphism of semi-categories $F\colon \mathcal{G}^{\mathsf{op}} \to \mathcal{V}$*

1. *F is regular;*
2. *F is a colimit of representable morphisms.*

Proof (1) ⇒ (2) is by definition of regularity. Conversely, suppose given the following situation

$$G\colon \mathcal{X}^{\mathrm{op}} \to \mathcal{V}, \quad \mathcal{X} \xrightarrow{H} \mathcal{G} \xrightarrow{Y_{\mathcal{G}}} \mathsf{Reg}(\mathcal{G}^{\mathrm{op}}, \mathcal{V}) \hookrightarrow [\mathcal{G}^{\mathrm{op}}, \mathcal{V}]$$

with $\mathcal{X}$ a semi-category and $F \cong G\star(Y_{\mathcal{G}} \circ H)$ as a colimit in $[\mathcal{G}^{\mathrm{op}}, \mathcal{V}]$. By 4.3 and 4.2, the colimit $G\star(Y_{\mathcal{G}} \circ H)$ exists already in $\mathsf{Reg}(\mathcal{G}^{\mathrm{op}}, \mathcal{V})$ and is preserved by the inclusion in $[\mathcal{G}^{\mathrm{op}}, \mathcal{V}]$, proving that $F \in \mathsf{Reg}(\mathcal{G}^{\mathrm{op}}, \mathcal{V})$. □

Corollary 4.5 *When the base category $\mathcal{V}$ is abelian, the category $\mathsf{Reg}(\mathcal{G}^{\mathrm{op}}, \mathcal{V})$ of regular presheaves on a regular semi-category is abelian as well.*

Proof The category $[\mathcal{G}^{\mathrm{op}}, \mathcal{V}]$ is abelian with pointwise structure. It is well known that every localization of an abelian category is itself abelian. One concludes by 4.2. □

Corollary 4.6 *When the base category is that of sets, the category $\mathsf{Reg}(\mathcal{G}^{\mathrm{op}}, \mathcal{V})$ of regular presheaves on a regular semi-category is a topos.*

Proof The category $[\mathcal{G}^{\mathrm{op}}, \mathcal{V}]$ is a topos by 4.1 and every localization of a topos is a topos, thus one concludes by 4.2. □

5 The Yoneda presheaves

Given a regular semi-category $\mathcal{G}$, the full and faithfulness of the Yoneda morphism

$$Y_{\mathcal{G}}\colon \mathcal{G} \longrightarrow \mathsf{Reg}(\mathcal{G}^{\mathrm{op}}, \mathcal{V}) \hookrightarrow [\mathcal{G}^{\mathrm{op}}, \mathcal{V}], \quad A \mapsto \mathcal{G}(-, A)$$

is equivalent to the validity of the Yoneda lemma for all representable morphisms $\mathcal{G}(-, A)$. As already observed, this would imply that $\mathcal{G}$ is an actual category. This proves that the Yoneda lemma does not hold in general, not even for regular presheaves on regular semi-categories.

The present section intends to show that nevertheless the Yoneda lemma holds in the spirit of the "unity of opposites" described by F.W. Lawvere (see [7]).

Definition 5.1 *A morphism $P\colon \mathcal{G}^{\mathrm{op}} \to \mathcal{V}$ of semi-categories is a Yoneda presheaf when the canonical morphism $\theta_P\colon P(-) \Rightarrow \mathsf{Nat}\big(Y_{\mathcal{G}}(-), P\big)$ is an isomorphism.*

Let us recall that each $\mathsf{Nat}\big(Y_{\mathcal{G}}(A), P\big)$ is defined as an end; the A-component of the canonical morphism θ_P is the factorization through this end of the morphisms $P(A) \longrightarrow \big[\mathcal{G}(B,A), P(B)\big]$ corresponding, by adjunction, to the action of P.

Theorem 5.2 *Let $\mathcal{G}$ be a regular semi-category. The full and faithful embedding*

$$k\colon \mathsf{Reg}(\mathcal{G}^{\mathrm{op}}, \mathcal{V}) \hookrightarrow [\mathcal{G}^{\mathrm{op}}, \mathcal{V}], \quad F \mapsto \mathsf{Nat}\big(Y_{\mathcal{G}}(-), F\big)$$

of theorem 4.2 identifies, up to an equivalence, the category $\mathsf{Reg}(\mathcal{G}^{\mathrm{op}}, \mathcal{V})$ of regular presheaves with the full subcategory $\mathsf{Yon}(\mathcal{G}^{\mathrm{op}}, \mathcal{V}) \hookrightarrow [\mathcal{G}^{\mathrm{op}}, \mathcal{V}]$ of Yoneda presheaves.

Proof We must prove that a presheaf P satisfies the Yoneda lemma precisely when it is isomorphic to a presheaf of the form $\mathsf{Nat}\big(Y_{\mathcal{G}}(-), F\big)$ for some regular presheaf F.

Indeed, if P verifies the Yoneda lemma, by theorem 4.2 we get

$$P(A) \cong \mathsf{Nat}\big(i\mathcal{G}(-,A), P\big) \cong \mathsf{Nat}\big(\mathcal{G}(-,A), j(P)\big) \cong k\big(j(P)\big)(A)$$

and thus $P \cong k\big(j(P)\big)$ with $j(P)$ regular.

Conversely consider $P \cong j(F)$ with F regular.

$$\begin{aligned}
\mathsf{Nat}\big(\mathcal{G}(-,A)\big) &\cong \mathsf{Nat}\big(\mathcal{G}(-,A), j(F)\big) \\
&\cong \int_B \Big[\mathcal{G}(B,A), \mathsf{Nat}\big(\mathcal{G}(-,B), F\big)\Big] \\
&\cong \int_B \Big[\mathcal{G}(B,A), \int_C \big[\mathcal{G}(C,B), F(C)\big]\Big] \\
&\cong \int_{B,C} \Big[\mathcal{G}(C,B) \otimes \mathcal{G}(B,A), F(C)\Big] \\
&\cong \int_C \Big[\int^B \mathcal{G}(C,B) \otimes \mathcal{G}(B,A), F(C)\Big] \\
&\cong \int_C \Big[\mathcal{G}(C,A), F(C)\Big] \cong j(F)(A) \cong P(A)
\end{aligned}$$

from which P satisfies the Yoneda lemma. □

The previous result is an occurence of what F.W. Lawvere calls the "unity of opposites" (see [7]) : the functor $j\colon [\mathcal{G}^{\mathrm{op}}, \mathcal{V}] \longrightarrow \mathsf{Reg}(\mathcal{G}^{\mathrm{op}}, \mathcal{V})$ of theorem 4.2 has both a left and a right adjoint, which are full and faithful. The left adjoint i exhibits the full subcategory of those presheaves which are colimits of representables, while the right adjoint k determines exactly the full subcategory of those presheaves which satisfy the Yoneda lemma. Thus a regular presheaf F does not, in general, satisfy the Yoneda lemma; but canonically associated with it, via the "unity of opposites", we get a Yoneda presheaf $k(F)$.

At this point we should also exhibit the link with the work of G.M. Kelly and F.W. Lawvere on essential localizations (see [6]).

Lemma 5.3 *A regular semi-category $\mathcal{G}$ is an idempotent two-sided ideal in the corresponding category $\overline{\mathcal{G}}$ obtained by adding freely identities.*

Proof The fact of being a left-sided ideal means that the composition $\overline{\mathcal{G}}(A,B) \otimes \mathcal{G}(B,C) \longrightarrow \overline{\mathcal{G}}(A,C)$ factors through $\mathcal{G}(A,C)$, which is obvious (see lemma 4.1). The same argument applies on the right.

In the Set-case, the idempotency of the ideal means that "every arrow in $\mathcal{G}$ is the composite of two arrows in $\mathcal{G}$", which can be translated here as the fact that composition induces a strong (or even regular) epimorphism $\coprod_{B\in\mathcal{G}} \mathcal{G}(A,B) \otimes \mathcal{G}(B,C) \twoheadrightarrow \mathcal{G}(A,C)$. This is the case by regularity of the representable functors (see 3.7 and 3.6). $\square$

Observe that regularity of $\mathcal{G}$ is stronger than idempotency: we have a coend formula instead of a strong epimorphism from a coproduct. In the Set-case, this means that every morphism in $\mathcal{G}$ can be written as the composite of two morphisms in $\mathcal{G}$, but moreover two such decompositions of a same morphism can be connected by a zig-zag.

In the Set-case, Lawvere and Kelly classify the essential localizations of a topos $[\mathcal{C}^{\mathrm{op}}, \mathrm{Set}]$ of presheaves by the idempotent two-sided ideals of $\mathcal{C}$. In the situation of lemma 5.3, their construction yields exactly the essential localization of Yoneda presheaves on $\mathcal{G}$, as attested by the following lemma, which holds over an arbitrary base $\mathcal{V}$.

Lemma 5.4 *Let $\mathcal{G}$ be a regular semi-category and $\overline{\mathcal{G}}$ the corresponding category, obtained by adding freely identities. Given a morphism $P \in$*

$[\mathcal{G}^{\mathrm{op}}, \mathcal{V}]$ *of semi-categories, write* $\overline{P}$ *for the corresponding presheaf* $\overline{P} \in [\overline{\mathcal{G}}^{\mathrm{op}}, \mathcal{V}]$ *(see 4.1). The following conditions are equivalent:*

1. *P is a Yoneda presheaf;*

2. *$\overline{P}$ is orthogonal to each $\overline{\mathcal{G}(-,A)} \longrightarrow \overline{\mathcal{G}}(-,A)$ in $[\overline{\mathcal{G}}^{\mathrm{op}}, \mathcal{V}]$, that is, the induced morphism $\mathsf{Nat}(\overline{\mathcal{G}}(-,A), \overline{P}) \longrightarrow \mathsf{Nat}(\overline{\mathcal{G}(-,A)}, \overline{P})$ is an isomorphism.*

Proof (1) $\Rightarrow$ (2) is attested by the following isomorphisms, which hold by lemma 4.1, the assumption on P and the Yoneda lemma for $\overline{\mathcal{G}}$:

$$\begin{aligned} \mathsf{Nat}(\overline{\mathcal{G}(-,A)}, \overline{P}) &\cong \mathsf{Nat}(\mathcal{G}(-,A), P) \\ &\cong P(A) = \overline{P}(A) \cong \mathsf{Nat}(\overline{\mathcal{G}}(-,A), \overline{P}). \end{aligned}$$

Conversely,

$$\begin{aligned} \mathsf{Nat}(\mathcal{G}(-,A), P) &\cong \mathsf{Nat}(\overline{\mathcal{G}(-,A)}, \overline{P}) \\ &\cong \mathsf{Nat}(\overline{\mathcal{G}}(-,A), \overline{P}) \cong \overline{P}(A) = P(A) \end{aligned}$$

which proves the result. □

6 A Morita theorem for regular semi-categories

As already noticed, the lack of identities in a semi-category, even with the regularity requirement, prevents the existence of identity natural transformations between morphisms, and therefore the development of a good theory of adjoint morphisms. This difficulty vanishes in the case of regular distributors (also called profunctors or bimodules, at least in the categorical case).

Proposition 6.1 *The following data constitute a bicategory* RDist*:*

1. *the objects are the regular semi-categories;*

2. *the arrows are the regular distributors* $\varphi\colon \mathcal{G} \multimap \mathcal{H}$*, that is, the regular presheaves* $\varphi\colon \mathcal{H}^{\mathsf{op}} \otimes \mathcal{G} \longrightarrow \mathcal{V}$*;*

3. *the 2-cells are the natural transformations;*

4. *the composition of arrows* $\mathcal{G} \overset{\varphi}{\multimap} \mathcal{H} \overset{\psi}{\multimap} \mathcal{K}$ *is given by* $(\psi \circ \varphi)(C, A) = \int^{B \in \mathcal{H}} \psi(C, B) \otimes \varphi(B, A)$ *where* $A \in \mathcal{G}$ *and* $C \in \mathcal{K}$*;*

5. *vertical composition of 2-cells is pointwise, while horizontal composition is induced as usual by that of arrows.*

Proof Proposition 3.3 indicates that the coend formula of the statement makes sense. The regularity of the composite $\psi \circ \varphi$ means

$$(\psi \circ \varphi)(C', A') \cong \int^{A,C} \mathcal{K}(C', C) \otimes (\psi \circ \phi)(C, A) \otimes \mathcal{G}(A, A')$$

which reduces to the relation

$$\int^{B} \psi(C', B) \otimes \varphi(B, A')$$

$$\cong \int^{A,B,C} \mathcal{K}(C', C) \otimes \psi(C, B) \otimes \varphi(B, A) \otimes \mathcal{G}(A, A').$$

This last relation holds by regularity of $\psi(-, B)$ and $\varphi(B, -)$.

Next the regularity of $\varphi(-, A)$ and $\psi(C, -)$ exactly means that $\mathcal{H}\colon \mathcal{H}^{\mathsf{op}} \otimes \mathcal{H} \to \mathcal{V}$ is the identity distributor $\mathcal{H} \multimap \mathcal{H}$.

Notice moreover that for a regular distributor $\varphi\colon \mathcal{G} \multimap \mathcal{H}$, the identity natural transformation on $\varphi\colon \mathcal{H}^{\mathsf{op}} \otimes \mathcal{G} \longrightarrow \mathcal{V}$ exists and is computed pointwise, since $\mathcal{V}$ is an actual category.

The rest is routine. □

Theorem 6.2 (Morita theorem) *Let* $\mathcal{G}$ *and* $\mathcal{H}$ *be regular semi-categories. The following conditions are equivalent:*

1. *the categories* $\mathsf{Reg}(\mathcal{G}^{\mathsf{op}}, \mathcal{V})$ *and* $\mathsf{Reg}(\mathcal{H}^{\mathsf{op}}, \mathcal{V})$ *of regular presheaves are equivalent;*

2. *$\mathcal{G}$ and $\mathcal{H}$ are equivalent in the bicategory* RDist *of regular distributors, that is, there exist regular distributors $\varphi\colon \mathcal{G} \to \mathcal{H}$ and $\psi\colon \mathcal{H} \to \mathcal{G}$ such that $\varphi \circ \psi = \mathcal{H}$, $\psi \circ \varphi = \mathcal{G}$.*

Proof Let us consider an equivalence $\theta\colon \mathsf{Reg}(\mathcal{G}^{\mathsf{op}}, \mathcal{V}) \xrightarrow{\cong} \mathsf{Reg}(\mathcal{H}^{\mathsf{op}}, \mathcal{V})$ with inverse equivalence τ. We recall that equivalences preserve colimits. We define

$$\varphi\colon \mathcal{H}^{\mathsf{op}} \otimes \mathcal{G} \longrightarrow \mathcal{V}, \quad \varphi(B, A) = \Big(\theta\big(\mathcal{G}(-, A)\big)\Big)(B)$$

and analogously for ψ, starting from τ. Let us also write

$$\mathsf{ev}_A\colon \mathsf{Reg}(\mathcal{G}^{\mathsf{op}}, \mathcal{V}) \longrightarrow \mathcal{V}, \quad F \mapsto F(A)$$

for the evaluation functor at $A \in \mathcal{G}$; since colimits are computed pointwise in $\mathsf{Reg}(\mathcal{G}^{\mathsf{op}}, \mathcal{V})$, they are preserved by ev_A.

The regularity of φ is proved as follows, where $A, A' \in \mathcal{G}$ and $B, B' \in \mathcal{H}$; the first isomorphism follows from the regularity of $\theta\big(\mathcal{G}(-, A)\big)$.

$$\begin{aligned}
\int^{A,B} & \mathcal{H}(B', B) \otimes \theta\big(\mathcal{G}(-, A)\big)(B) \otimes \mathcal{G}(A, A') \\
&\cong \int^{A} \theta\big(\mathcal{G}(-, A)\big)(B') \otimes \mathcal{G}(A, A') \\
&\cong \mathcal{G}(-, A') \star (ev_{B'} \circ \theta \circ Y_{\mathcal{G}}) \\
&\cong (ev_{B'} \circ \theta)\big(\mathcal{G}(-, A') \star Y_{\mathcal{G}}\big) \\
&\cong (ev_{B'} \circ \theta)\big(\mathcal{G}(-, A')\big) \cong \varphi(B', A').
\end{aligned}$$

An analogous argument holds for ψ.

To prove that φ and ψ are reciprocal equivalences in RDist, choose $A, A' \in \mathcal{G}$ and $B \in \mathcal{H}$.

$$\begin{aligned}
(\psi \circ \varphi)(A', A) &\cong \int^{B} \tau\big(\mathcal{H}(-, B)\big)(A') \otimes \theta\big(\mathcal{G}(-, A)\big)(B) \\
&\cong \theta\big(\mathcal{G}(-, A)\big)(?) \star \tau\big(\mathcal{H}(-, ?)\big)(A') \\
&\cong \theta\big(\mathcal{G}(-, A)\big) \star (\mathsf{ev}_{A'} \circ \tau \circ Y_{\mathcal{H}}) \\
&\cong (ev_{A'} \circ \tau)\Big(\theta\big(\mathcal{G}(-, A)\big) \star Y_{\mathcal{H}}\Big) \\
&\cong (ev_{A'} \circ \tau)\Big(\theta\big(\mathcal{G}(-, A)\big)\Big) \cong \mathcal{G}(A', A).
\end{aligned}$$

An analogous argument holds for $\varphi \circ \psi$.

Conversely, assume the existence of φ and ψ as in the statement. By regularity of φ, we get a corresponding morphism

$$\overline{\varphi}\colon \mathcal{G} \longrightarrow \mathsf{Reg}(\mathcal{H}^{\mathsf{op}}, \mathcal{V}), \quad A \mapsto \varphi(-, A).$$

We define θ by

$$\theta\colon \mathsf{Reg}(\mathcal{G}^{\mathsf{op}}, \mathcal{V}) \longrightarrow \mathsf{Reg}(\mathcal{H}^{\mathsf{op}}, \mathcal{V}), \quad F \mapsto F \star \overline{\varphi}.$$

The presheaf $\theta(F)$ is at once regular since $\overline{\varphi}$ takes values in $\mathsf{Reg}(\mathcal{H}^{\mathsf{op}}, \mathcal{V})$. Analogous arguments hold for τ, starting from ψ.

By theorem 4.2, colimits in categories of regular presheaves are computed pointwise. Using the coend formulæ describing these colimits in $\mathcal{V}$,

$$\begin{aligned}
(\tau \circ \theta)(F)(A') &\cong \big((F \star \overline{\varphi}) \star \overline{\psi}\big)(A') \\
&\cong \int^{A,B} F(A) \otimes \varphi(B, A) \otimes \psi(A', B) \\
&\cong \int^{A} F(A) \otimes (\psi \circ \varphi)(A', A) \\
&\cong \int^{A} F(A) \otimes \mathcal{G}(A', A) \cong F(A').
\end{aligned}$$

An analogous argument holds for $\theta \circ \tau$. □

The equivalence described in theorem 6.2 is referred to as the Morita equivalence.

The Morita equivalence classes of regular semi-categories are provided with a multiplication induced by the tensor product (see 3.8) of semi-categories. This provides them with the structure of a (possibly large) abelian monoid. The regular semi-categories whose Morita equivalence class is invertible in this monoid are natural candidates for being called Azumaya graphs. The Morita equivalence classes of Azumaya graphs constitute then a (possibly large) abelian group, which is a natural candidate to be chosen as the Brauer–Taylor group of the base category $\mathcal{V}$.

7 Examples

We refer to the literature on regular modules for a wide variety of examples of "regular rings" (in the sense of section 1). We want nevertheless to emphasize the following criterion which, to our knowledge, is original and allows constructing various interesting examples.

Lemma 7.1 *Let R be a ring admitting a family $(e_i)_{i\in I}$ of elements such that*

1. $\forall r \in R \quad \{i \in I \mid r \cdot e_i \neq 0\}$ *and* $\{i \in I \mid e_i \cdot r \neq 0\}$ *are finite;*
2. $\forall r \in R \quad \sum_{i\in I} e_i \cdot r = r = \sum_{i\in I} r \cdot e_i$.

In those conditions, the ring R is regular in the sense of section 1.

Proof The mapping $\sigma\colon R \longrightarrow R \otimes_R R,\ \ r \mapsto \sum_{i\in I} r \otimes e_i$ is correctly defined. Indeed, write J_r for the set of indices for which $e_j \cdot r \neq 0$. If $r \cdot e_i = 0$,

$$r \otimes e_i = \left(\sum_{j\in J_r} e_j \cdot r\right) \otimes e_i = \sum_{j\in J_r} e_j \otimes (r \cdot e_i) = 0$$

from which the sum defining $\sigma(r)$ is finite. It is now routine to check that σ is inverse to the multiplication. □

Example 7.2 *The ring of finite matrices with entries in a commutative ring with unit is regular in the sense of section 1.*

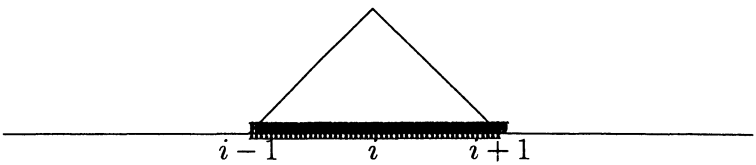
Let R be a commutative ring with unit. We consider the ring $\mathcal{M}(R)$ of matrices $(r_{i,j})_{i,j\in\mathbb{N}}$ with entries in R and such that only a finite number of these entries are non-zero. It suffices to apply lemma 7.1 to the family of matrices

$$e_n = (a_{ij})_{i,j\in\mathbb{N}}, \quad a_{ij} = \begin{cases} 1 & \text{if } i = n = j \\ 0 & \text{otherwise} \end{cases}$$

indexed by the natural numbers $n \in \mathbb{N}$. □

Example 7.3 *The ring of continuous functions $f\colon \mathbb{R} \to \mathbb{R}$ with compact support is regular in the sense of section 1.*

It suffices to apply lemma 7.1 to the family of functions



$$e_i(x) = \begin{cases} 0 & \text{if } x \leq i-1 \\ x-(i-1) & \text{if } i-1 \leq x \leq i \\ -x+(i+1) & \text{if } i \leq x \leq i+1 \\ 0 & \text{if } i+1 \leq x \end{cases}$$

for all integers $i \in \mathbb{Z}$. □

We focus now on cases of regular semi-categories with several objects.

Example 7.4 *Let K be a field. The K-vector spaces (of bounded dimension, if one insists on having a small semi-category) and K-linear mappings with finite rank constitute a regular semi-category over the category of K-vector spaces.*

To avoid size arguments, we consider the vector spaces of dimension less than some fixed cardinal α. Let us fix a vector space V with base $(e_i^V)_{i\in I}$ and a vector space W with base $(e_j^W)_{j\in J}$. We consider the linear projections

$$\varepsilon_{i_0}^V : V \to V, \quad \left(v = \sum_{i\in I} k_i \cdot e_i^V\right) \mapsto k_{i_0} \cdot e_{i_0}$$

and analogously for ε_j^W. Those projections are of course idempotent.

Let us write Fin for the semi-category of the statement. We consider the coend

$$\varphi(V,W) = \int^{A,B} \mathsf{Fin}(V,A) \otimes \mathsf{Fin}(A,B) \otimes \mathsf{Fin}(B,W)$$

and must prove that $\varphi(V,W) \cong \mathsf{Fin}(V,W)$. By definition of the coend, given the situation

$$V \xrightarrow{a} M \xrightarrow{b} A \xrightarrow{c} B \xrightarrow{d} N \xrightarrow{e} W$$

one has the equality $\left[(b\circ a)\otimes c\otimes(e\circ d)\right] = \left[a\otimes(d\circ c\circ b)\otimes e\right]$ between equivalence classes in the coend $\varphi(V,W)$.

It is well-known that for a linear mapping $f\colon V\to W$, the dimension of the image of f equals the dimension of the coimage of the kernel of f. When this dimension is finite, only finitely many $f\circ\varepsilon_i^V$ and $\varepsilon_j^W\circ f$ are non zero and the equality $f=\sum_{i\in I,j\in J}\varepsilon_j^W\circ f\circ\varepsilon_i^V$ holds trivially Observe further that in the coend $\varphi(V,W)$

$$\left[\varepsilon_i^V\otimes f\otimes\varepsilon_j^W\right] = \left[(\varepsilon_i^V\circ\varepsilon_i^V)\otimes f\otimes(\varepsilon_j^W\circ\varepsilon_j^W)\right] = \left[\varepsilon_i^V\otimes\varepsilon_j^W\circ f\circ\varepsilon_i^V\otimes\varepsilon_j^W\right],$$

proving that if $\varepsilon_j^W\circ f\circ\varepsilon_i^V=0$, then $\left[\varepsilon_i^V\otimes f\otimes\varepsilon_j^W\right]=0$.

This allows defining a linear mapping

$$\sigma_{V,W}\colon \mathsf{Fin}(V,W)\longrightarrow\varphi(V,W),\quad f\mapsto\sum_{i\in I,j\in J}\left[\varepsilon_i^V\otimes f\otimes\varepsilon_j^W\right]$$

which is easily seen to be the inverse of the canonical morphism $\varphi(V,W)\to\mathsf{Fin}(V,W)$ induced by composition. □

Example 7.5 *The Hilbert spaces (of bounded Hilbert dimension, if one insists on having a small semi-category) and Hilbert–Schmidt operators between them constitute a regular semi-category over the category of Banach spaces and linear contractions.*

We fix two Hilbert spaces V and W with respective Hilbert basis $(e_i^V)_{i\in I}$ and $(e_j^W)_{j\in J}$, with I and J of cardinality less than some fixed cardinal α. An operator $f\colon V\to W$ is Hilbert–Schmidt when

$$\sum_{i,j\in\mathbb{N}}\left|\langle e_j^W, f(e_i^V)\rangle\right|^2<\infty.$$

This definition is independent of the choice of the Hilbert basis and, taking the square root of this sum as norm of f provides the set of

Hilbert–Schmidt operators with the structure of a Banach space (it is even a Hilbert space, but we shall not need this). Let us write $\mathsf{HS}(V, W)$ for this Banach space. Up to routine verifications, this defines the semi-category of the statement. We refer to [10] or [8] for more details concerning this example.

Let us write ε_i^V and ε_j^W for the orthogonal projections on the axis determined by e_i^V and e_j^W. Notice that $\|\varepsilon_j^W \circ f \circ \varepsilon_i^V\| = \left|\langle e_j^W, f(e_i^V)\rangle\right|$ thus f is an Hilbert–Schmidt operator when

$$\|f\|^2 = \sum_{i\in I, j\in J} \|\varepsilon_j^W \circ f \circ \varepsilon_i^V\|^2 < \infty.$$

Next observe that the equality $f = \sum_{i\in I,j\in J} \varepsilon_j^W \circ f \circ \varepsilon_i^V$ holds. Indeed, every $v \in V$ can be written $v = \sum_{i\in I} \varepsilon_i^V(v)$, from which $f(v) = \sum_{j\in J} \varepsilon_j^W(f(v)) = \sum_{i\in I,j\in J} (\varepsilon_j^W \circ f \circ \varepsilon_i^V)(v)$. This proves that the given serie converges pointwise to the operator f, and it remains thus to prove that the serie converges in $\mathsf{HS}(V, W)$. But this follows at once from the definition of the norm of $\mathsf{HS}(V, W)$, the Hilbert–Schmidt condition and the Cauchy criterion.

Now we can repeat here the argument of example 7.4 about the coend $\varphi(V, W)$ and the relations existing between its elements. They allow concluding that the serie $\sum_{i\in I,j\in J} [\varepsilon_i^V \otimes f \otimes \varepsilon_j^W]$ converges in $\varphi(V, W)$. Indeed, as in example 7.4 and because the various orthogonal projectors have norm 1,

$$\begin{aligned}
\left\|[\varepsilon_i^V \otimes f \otimes \varepsilon_j^W]\right\| &= \left\|[\varepsilon_i^V \circ \varepsilon_i^V \otimes f \otimes \varepsilon_j^W \circ \varepsilon_j^W]\right\| \\
&= \left\|[\varepsilon_i^V \otimes \varepsilon_j^W \circ f \circ \varepsilon_i^V \otimes \varepsilon_j^W]\right\| \\
&\leq \left\|\varepsilon_i^V \otimes \varepsilon_j^W \circ f \circ \varepsilon_i^V \otimes \varepsilon_j^W\right\| \\
&\leq \left\|\varepsilon_i^V\right\| \cdot \left\|\varepsilon_j^W \circ f \circ \varepsilon_i^V\right\| \cdot \left\|\varepsilon_j^W\right\| \\
&= \left\|\varepsilon_j^W \circ f \circ \varepsilon_i^V\right\|
\end{aligned}$$

The convergence of the serie follows thus again from the Cauchy criterion and the Hilbert–Schmidt condition.

As in example 7.4, we conclude that the mapping

$$\sigma_{V,W} \colon \mathsf{HS}(V, W) \longrightarrow \varphi(V, W), \quad f \mapsto \sum_{i\in I, j\in J} [\varepsilon_i^V \otimes f \otimes \varepsilon_j^W]$$

is inverse to the canonical morphism $\varphi(V, W) \longrightarrow \mathsf{HS}(V, W)$ induced by composition. □

Example 7.6 *Given a locale Ω, viewed as a cartesian closed category, every Ω-set A is a regular Ω-semi-category. The category of regular presheaves on A is the locale of subobjects of A.*

The category of Ω-sets is equivalent to the category of sheaves on Ω. An Ω-set is a set A provided with an Ω-valued equality

$$[\bullet = \bullet] \colon A \times A \longrightarrow \Omega$$

which verifies the following axioms:

$$\begin{aligned} [a = b] \wedge [b = c] &\leq [a = c] \\ [a = b] &= [b = a]. \end{aligned}$$

The Ω-valued equality provides A with the structure of a Ω-semi-category, as attested by the first axiom. The regularity of the Ω-set A reduces to the property $\bigvee_{x,y \in A} [a = x] \wedge [x = y] \wedge [y = b] = [a = b]$ which is well-known to hold (see [4] or [1], volume 3).

The same references show that the locale of subsheaves of a given sheaf is isomorphic to the locale of Ω-subsets of the corresponding Ω-set A, where an Ω-subset S of A is a mapping $[\bullet \in S] \colon A \longrightarrow \Omega$ verifying the axioms

$$\begin{aligned} [a = b] \wedge [b \in S] &\leq [a \in S] \\ [a \in S] &\leq [a = a]. \end{aligned}$$

These axioms can equivalently be restated as (see the references)

$$\begin{aligned} [a = b] &\leq [b \in S] \Rightarrow [a \in S] \\ [a \in S] &= \bigvee_{x \in A} [a = x] \wedge [x \in S] \end{aligned}$$

where $\Rightarrow$ indicates the "implication" of the locale Ω, that is, the internal Hom-functor of the cartesian closed category Ω. The first condition expresses that $[\bullet \in S]$ is a morphism of Ω-semi-categories and the second condition expresses its regularity. □

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