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A NOTE ON NUCLEI OF QUANTALE MODULES

by J. PASEKA

Résumé. Le but de cet article est d'établir des théorèmes utiles de factorisation pour des Q -modules de la même manière que pour des locales. On montre qu'un noyau d'un module associé à un prénoyau d'un module est l'intersection de noyaux de modules d'une forme particulière.

Our motivating source was the paper [12] where the investigation of I-simple involutive quantales $\mathcal{Q}(M)$ and representations $Q \rightarrow \mathcal{Q}(M)$ given by a left action is developed. Such involutive quantales could play a similar role as do points in topological spaces or irreducible representations in C^* -algebras (see [9]). Since the representations $Q \rightarrow \mathcal{Q}(M)$ coincide with quantale modules we will be interested in studying factorization on quantale modules. This note is closely related to the papers [10] and [11] where the interested reader can find unexplained terms and notation concerning the subject. For facts concerning quantales and quantale modules in general we refer to [14] and [15]. For motivating examples concerning quantale modules we recommend the paper [13].

The paper is organized as follows. Section 1 introduces the notion of a left Q -module and the motivation for studying such a structure is given. Section 2 contains the basic theorems for factorization on Q -modules and the structure of the complete lattice of all module nuclei is investigated. It is shown that any module nucleus associated to a module prenucleus is a meet of module nuclei of a special form.

A *quantale* is a complete lattice Q with an associative binary multiplication satisfying

$$x \cdot \bigvee_{i \in I} x_i = \bigvee_{i \in I} x \cdot x_i \quad \text{and} \quad \left(\bigvee_{i \in I} x_i \right) \cdot x = \bigvee_{i \in I} x_i \cdot x$$

for all $x, x_i \in Q, i \in I$ (I is a set). 1 denotes the greatest element of Q , 0 is the smallest element of Q . A quantale Q is said to be *unital* if there is an element $e \in Q$ such that $e \cdot a = a = a \cdot e$ for all $a \in Q$. A *subquantale* Q' of a quantale Q is a subset of Q closed under $\vee$ and $\cdot$.

By a *morphism of quantales* will be meant a $\vee$ - and $\cdot$ -preserving mapping $f : Q \rightarrow Q'$. If a morphism preserves the top element we say that it is *strong*.

A non-trivial ($0 \neq 1$ and $1 \cdot 1 = 1$) quantale Q is said to be *simple* if any surjective homomorphism $Q \rightarrow Q'$ is either an isomorphism or a constant morphism.

By the *quantale $\mathcal{Q}(M)$ of endomorphisms of the sup-lattice M* will be meant the simple unital quantale of sup-preserving mappings from M to itself, with the supremum given by the pointwise ordering of mappings, with the multiplication corresponding to the composition of mappings, and with the unit given by the identity mapping.

1 Quantale modules

Definition 1.1 [1], [15] *Let Q be a quantale. A left module over Q (shortly a left Q -module) is a sup-lattice M , together with a module action $\bullet : Q \times M \rightarrow M$ satisfying*

$$(a \cdot b) \bullet m = a \bullet (b \bullet m) \quad (\text{M1})$$

$$(\vee S) \bullet m = \vee \{s \bullet m : s \in S\} \quad (\text{M2})$$

$$a \bullet \vee X = \vee \{a \bullet x : x \in X\} \quad (\text{M3})$$

for all $a, b \in Q, m \in M, S \subseteq Q, X \subseteq M$. So we have two maps $\dashv\dashv_l - : M \times M \rightarrow Q, \dashv\dashv_r - : Q \times M \rightarrow M$ such that, for all $a \in Q, m, n \in M, a \bullet m \leq n$ iff $a \leq m \dashv\dashv_l n$ iff $m \leq a \dashv\dashv_r n$. M is called a *unital Q -module* if Q is a unital quantale with the unit e and $e \bullet m = m$ for all $m \in M$. M is said to be a *strong Q -module* if $1 \bullet m = 1$ for all $m \in M, m \neq 0$. Let $a \not\leq b$. Then M separates a from b if there is an element $m \in M$ such that $a \bullet m \neq b \bullet m$.

Let M and N be modules over Q and let $f : M \rightarrow N$ be a sup-lattice homomorphism. f is a module homomorphism if $f(a \bullet m) = a \bullet f(m)$ for all $a \in Q, m \in M$.

Note first that if M is a sup-lattice then M is a left $\mathcal{Q}(M)$ -module such that $f \bullet m = f(m)$ for all $f \in \mathcal{Q}(M)$ and all $m \in M$. Secondly, we may dually define the notion of a (strong, separating, unital) right Q -module. Moreover, all propositions stated for left Q -modules are valid in a dualized form for right Q -modules.

We shall denote by $Q\text{-Mod}$ (resp. $\text{Mod-}Q$) the category of left Q -modules (resp. right Q -modules).

Let us make the following elementary observation. Evidently, for a quantale Q , any left Q -module M is a unital left $Q[e]$ -module with $Q[e]$ defined as in [11] by

$$Q[e] = \{a \vee \varepsilon : a \in Q, \varepsilon \in \{0, e\}\},$$

e arbitrary such that $e \notin Q$, and the multiplication $\bullet_e$ defined by

$$(a \vee \varepsilon) \bullet_e m = \begin{cases} a \bullet m & \text{if } \varepsilon = 0 \\ a \bullet m \vee m & \text{if } \varepsilon = e. \end{cases}$$

So we may always assume that any left quantale module is unital over a unital quantale.

Note that, for any quantale Q , any left Q -module M and all $n \in M$, the antitone maps $- \multimap_l n : M \rightarrow Q$ and $- \multimap_r n : Q \rightarrow M$ form a Galois connection between Q and M since, for all $a \in Q$ and all $M \in M$, $a \leq (a \multimap_r n) \multimap_l n$ and $m \leq (m \multimap_l n) \multimap_r n$. So we have that $m \multimap_l n = ((m \multimap_l n) \multimap_r n) \multimap_l n$ and $a \multimap_r n = ((a \multimap_r n) \multimap_l n) \multimap_r n$.

A simple calculation gives us the following two lemmas.

Lemma 1.2 *Let Q be a quantale. If M is a (strong) Q -module then we have a (strong) homomorphism $f_M : Q \rightarrow \mathcal{Q}(M)$, $f_M(a)(x) = a \bullet x$. Conversely, let $f : Q \rightarrow \mathcal{Q}(M)$ be a (strong) homomorphism of quantales. Then M is a (strong) Q -module with the module action $a \bullet_f x = f(a)(x)$.*

Lemma 1.3 *Let Q be a quantale, $a \not\leq b$. If M is a Q -module separating a from b then we have a separating homomorphism $f_M : Q \rightarrow \mathcal{Q}(M)$. Conversely, let $f : Q \rightarrow \mathcal{Q}(M)$ be a homomorphism separating a from b . Then M is a separating Q -module.*

The following theorem (see [6]) is our main motivation for studying quantale modules.

Theorem 1.4 *Let Q be a quantale. Then the following conditions are equivalent:*

1. Q is spatial i.e. Q is a strong subquantale of a product of simple quantales.
2. Each element of Q is an intersection of primes.
3. Q has enough strong separating left Q -modules.

Proof. (1) $\Leftrightarrow$ (2) by [6]. (1) $\Leftrightarrow$ (3) by lemma 1.2 and by lemma 1.3. $\square$

2 Factorization on Q -modules

In the following, we shall assume that Q is an arbitrary, fixed quantale.

Definition 2.1 *Let M be a left Q -module, $j : M \rightarrow M$ an operator on M satisfying:*

- (i) $m \leq j(m)$
- (ii) $m \leq n$ implies $j(m) \leq j(n)$
- (iii) $a \bullet j(m) \leq j(a \bullet m)$

for all $a \in Q, m, n \in M$. We say that j is a module prenucleus on M . We put $M_j = \{m \in M : m = j(m)\}$. Evidently, M_j is a closure system

in M , that is, closed under arbitrary meets in M . Note that $\bigvee_j = j \circ \bigvee$. We let $\nu(j)$ be the associated closure operator, so that

$$\nu(j)(n) = \bigwedge \{m \in M_j : m \geq n\}.$$

We say that $\nu(j)$ is generated by j . If j satisfies

$$(iv) \quad j(m) = j(j(m))$$

for all $m \in M$ we shall say that j is a module nucleus on M . We shall denote by $N(M)$ the lattice of all module nuclei on M .

Note that any module prenucleus (nucleus) on a Q -module M is a module prenucleus (nucleus) on a $Q[e]$ -module M and conversely.

Lemma 2.2 *Let j be a module nucleus on a left Q -module M . Then M_j is a left Q -module with the join $\bigvee_j$ and the action $\bullet_j : Q \times M_j \rightarrow M_j$ defined as follows*

$$a \bullet_j m = j(a \bullet m)$$

for all $a \in Q$ and all $m \in M_j$. Furthermore, M_j is unital (strong) if M is.

Proof. Let $a, b \in Q$, $m \in M_j$. Then $(a \cdot b) \bullet_j m = j((a \cdot b) \bullet m) = j(a \bullet (b \bullet m)) = j(a \bullet j(b \bullet m)) = a \bullet_j (b \bullet_j m)$. Similarly, let $S \subseteq Q$, $m \in M_j$. Then $(\bigvee S) \bullet_j m = j((\bigvee S) \bullet m) = j(\bigvee \{s \bullet m : s \in S\}) = j(\bigvee \{j(s \bullet m) : s \in S\}) = \bigvee_j \{s \bullet_j m : s \in S\}$. Now, let $a \in Q$, $N \subseteq M$. Then $a \bullet_j (\bigvee_j N) = j(a \bullet j(\bigvee N)) = j(a \bullet \bigvee N) = j(\bigvee \{a \bullet n : n \in N\}) = j(\bigvee \{j(a \bullet n) : n \in N\}) = \bigvee_j \{a \bullet_j n : n \in N\}$. $\square$

Similarly as for frames (see [2]) we have

Proposition 2.3 *Let j be a module prenucleus on a left Q -module M . Then $\nu(j)$ is the smallest module nucleus greater than j . Moreover, the module homomorphism $\nu(j) : M \rightarrow M_j$ is universal among all module homomorphisms $f : M \rightarrow N$ for which $f(m) = f(j(m))$ for all $m \in M$.*

Proof. Evidently, $\nu(j)$ is inflationary, idempotent and order-preserving. We have to show that, for all $a \in Q$ and $m \in M$, $a \bullet \nu(j)(m) \leq \nu(j)(a \bullet m)$. We put $E_{a,m} = \{m' \in M : m \leq m' \leq \nu(j)(m), a \bullet m' \leq \nu(j)(a \bullet m)\}$. Then $m \in E_{a,m}$, $m' \in E_{a,m}$ implies $j(m') \in E_{a,m} - m \leq m' \leq \nu(j)(m)$ implies $m \leq j(m) \leq j(m') \leq j(\nu(j)(m)) = \nu(j)(m)$ and $a \bullet m' \leq \nu(j)(a \bullet m)$ implies $a \bullet j(m') \leq j(a \bullet m') \leq j(\nu(j)(a \bullet m)) = \nu(j)(a \bullet m)$. Moreover, for any non-void $M' \subseteq E_{a,m}$, $\bigvee M' \in E_{a,m}$ by the left distribution law of modules. Hence $m_0 = \bigvee E_{a,m} \in E_{a,m}$. This gives us $j(m_0) \leq m_0 \leq j(m_0)$ i.e. $m \leq m_0 \leq \nu(j)(m) \leq m_0$. So we have $a \bullet \nu(j)(m) \leq a \bullet m_0 \leq \nu(j)(a \bullet m)$.

Let us prove the second part of the lemma. Note that $\nu(j)(j(m)) = \nu(j)(m)$ for all $m \in M$. Let $f : M \rightarrow N$ be a module homomorphism such that $f(m) = f(j(m))$ for all $m \in M$. We have to show that $f(\nu(j)(m)) = f(m)$ for all $m \in M$. We put $F_m = \{m' \in M : m \leq m' \leq \nu(j)(m), f(m') = f(m)\}$. Then $m \in F_m$, $m' \in F_m$ implies $j(m') \in F_m$ by the condition on f and, for all non-void $F' \subseteq F_m$, $\bigvee F' \in F_m$ since f is a join-preserving. We put $m_1 = \bigvee F_m$. Then $j(m_1) \leq m_1 \leq j(m_1)$ i.e. $m \leq m_1 \leq \nu(j)(m) \leq \nu(j)(m_1) = m_1$ i.e. $f(m) \leq f(\nu(j)(m)) \leq f(m_1) = f(m)$. We then have a uniquely determined map $\hat{f} : M_j \rightarrow N$ defined by $\hat{f}(m) = f(m)$ for all $m \in M_j$ i.e. $\hat{f} \circ j = f$. Evidently, for all $a \in Q$, $m \in M_j$ and $S \subseteq M_j$, we have $\hat{f}(\bigvee_{M_j} S) = \hat{f}(j(\bigvee S)) = f(\bigvee S) = \bigvee_{s \in S} f(s) = \bigvee_{s \in S} \hat{f}(j(s)) = \bigvee_{s \in S} \hat{f}(s)$ and $\hat{f}(a \bullet_{M_j} m) = \hat{f}(j(a \bullet m)) = f(a \bullet m) = a \bullet f(m) = a \bullet \hat{f}(j(m)) = a \bullet \hat{f}(m)$ i.e. $\hat{f}$ is a module homomorphism. $\square$

Another way of obtaining $\nu(j)$ is to iterate the prenucleus j until it converges by the following procedure:

$$\begin{aligned} j^0(m) &= m, \\ j^{\sigma+1}(m) &= j(j^\sigma(m)) \\ j^\lambda(m) &= \bigvee \{j^\sigma(m) : \sigma < \lambda\} \end{aligned}$$

for all $m \in M$, all ordinals σ and all limit ordinals λ . Then each j^σ is a module prenucleus with the same fixpoints as j and, for sufficiently large ordinal σ , j^σ has to be idempotent.

Following Macnab (see [8]) for Heyting algebras we have

Theorem 2.4 *Let Q be a unital quantale, M a unital left Q -module, $j : M \rightarrow M$ an operator on M . The following conditions are equivalent:*

1. j is a module nucleus on M .
2. For all m, n in M , $m \rightarrow_l j(n) = j(m) \rightarrow_l j(n)$.

Proof. $1 \Rightarrow 2$. We have, for all $a \in Q$, $a \leq m \rightarrow_l j(n)$ iff $a \bullet m \leq j(n)$ iff $j(a \bullet m) \leq j(n)$ iff $a \bullet j(m) \leq j(n)$ iff $a \leq j(m) \rightarrow_l j(n)$ i.e. the condition 2 holds.

$2 \Rightarrow 1$. We have, for all $m \in M$, $j(m) \leq j(m)$ i.e. $e \bullet j(m) \leq j(m)$ i.e. $e \leq j(m) \rightarrow_l j(m)$ i.e. $e \leq m \rightarrow_l j(m)$ i.e. $e \bullet m \leq j(m)$ i.e. $m \leq j(m)$ i.e. j is inflationary.

Now, let $m, n \in M$, $m \leq n$. Then $m \leq n \leq j(n)$ i.e. $e \bullet m \leq j(n)$ i.e. $e \leq m \rightarrow_l j(n) = j(m) \rightarrow_l j(n)$ i.e. $e \bullet j(m) \leq j(n)$ i.e. $j(m) \leq j(n)$.

Let $a \in Q$, $m \in M$. We have $a \bullet m \leq j(a \bullet m)$ i.e. $a \leq m \rightarrow_l j(a \bullet m)$ i.e. $a \leq j(m) \rightarrow_l j(a \bullet m)$ i.e. $a \bullet j(m) \leq j(a \bullet m)$.

Let us show that j is idempotent. Assume $m \in M$. We have $j(m) \leq j(m)$ i.e. $e \bullet j(m) \leq j(m)$ i.e. $e \leq j(m) \rightarrow_l j(m)$ i.e. $e \leq j(j(m)) \rightarrow_l j(m)$ i.e. $e \bullet j(j(m)) \leq j(m)$ i.e. $j(j(m)) \leq j(m) \leq j(j(m))$ i.e. $j(j(m)) = j(m)$. So it follows that j is a module nucleus on M . $\square$

Proposition 2.5 *Let M be a left Q -module, $n \in M$. Then the map $w_n : M \rightarrow M$ defined by $w_n(m) = (m \rightarrow_l n) \rightarrow_r n$ is a module nucleus. Similarly, the map $W_n : Q \rightarrow Q$ defined by $W_n(a) = (a \rightarrow_r n) \rightarrow_l n$ is a right module nucleus on Q , seen as a right module over itself. Moreover, for all $a \in Q$ and $m \in M$, we have $a \bullet m \leq n$ iff $W_n(a) \bullet w_n(m) \leq n$.*

Proof. Clearly, the Galois connection makes $m \leq w_n(m) = w_n(w_n(m))$ and $m_1 \leq m_2$ implies $w_n(m_1) \leq w_n(m_2)$ for all $m, m_1, m_2 \in M$. Let $a \in Q$, $m \in M$. Then $y \leq a \bullet m \rightarrow_l n$ iff $(y \cdot a) \bullet m = y \bullet (a \bullet m) \leq n$ iff $(y \cdot a) \leq m \rightarrow_l n$ iff $(y \cdot a) \rightarrow_r n \geq w_n(m)$ iff $(y \cdot a) \bullet w_n(m) \leq n$ iff $y \bullet (a \bullet w_n(m)) \leq n$ iff $y \leq (a \bullet w_n(m)) \rightarrow_l n$. Hence $a \bullet m \rightarrow_l n = (a \bullet w_n(m)) \rightarrow_l n$, and thus $w_n(a \bullet m) = w_n(a \bullet w_n(m))$, i.e. $w_n(a \bullet m) \geq$

$a \bullet w_n(m)$. To prove the second part of the lemma note that Q is a right module over itself. By similar arguments as above, we have that W_n is a right module nucleus.

Finally, we show that $a \bullet m \leq n \iff W_n(a) \bullet w_n(m) \leq n$: $a \bullet m \leq n$ iff $a \leq m \multimap_l n$ iff, by the Galois connection, $a \leq w_n(m) \multimap_l n$ iff $a \bullet w_n(m) \leq n$ iff, by similar steps using $\multimap_r$, $W_n(a) \bullet w_n(m) \leq n$. $\square$

Lemma 2.6 *Let Q be a quantale, $j_\alpha, \alpha \in \Lambda$ module prenuclei (nuclei) on a left Q -module M . Then the operator $\bigwedge_{\alpha \in \Lambda} j_\alpha$ is a module prenucleus (nucleus).*

Proof. Evidently, for all $m, n \in M$, $a \in Q$, we have $m \leq \bigwedge_{\alpha \in \Lambda} j_\alpha(m) = (\bigwedge_{\alpha \in \Lambda} j_\alpha)(m)$, $m \leq n$ implies $j_\alpha(m) \leq j_\alpha(n)$ i.e. $(\bigwedge_{\alpha \in \Lambda} j_\alpha)(m) \leq (\bigwedge_{\alpha \in \Lambda} j_\alpha)(n)$ and $a \bullet (\bigwedge_{\alpha \in \Lambda} j_\alpha)(m) \leq a \bullet j_\beta(m) \leq j_\beta(a \bullet m)$ for all $\beta \in \Lambda$ i.e. $a \bullet (\bigwedge_{\alpha \in \Lambda} j_\alpha)(m) \leq (\bigwedge_{\alpha \in \Lambda} j_\alpha)(a \bullet m)$. Let $j_\alpha, \alpha \in \Lambda$ be module nuclei. Then $\bigwedge_{\alpha \in \Lambda} j_\alpha(\bigwedge_{\beta \in \Lambda} j_\beta(m)) \leq j_\gamma(j_\gamma(m)) = j_\gamma(m)$ for all $\gamma \in \Lambda$ and all $m \in M$ i.e. $\bigwedge_{\alpha \in \Lambda} j_\alpha$ is idempotent. $\square$

We have then that, for any set Λ , $\bigwedge \{j_\lambda \in N(M) : \lambda \in \Lambda\}$ (with $\bigwedge$ defined pointwise), exists and it is a module nucleus i.e. $N(M)$ is a complete lattice. Because the meet in $N(M)$ is pointwise so also is the order on it. We shall denote by $\sqcup$ the join in $N(M)$ and by $\vee$ the pointwise join, which in general is only a prenucleus. Let $j_1, j_2 \in N(M)$. Then $j_1 \circ j_2, j_2 \circ j_1 \leq j_1 \sqcup j_2$. Note that $j_1 \circ j_2$ is a module prenucleus. So it follows that if $j_1 \circ j_2$ is idempotent then $j_1 \circ j_2 = j_1 \sqcup j_2$. If $j_1 \circ j_2 \leq j_2 \circ j_1$ then $j_2 \circ j_1$ is idempotent i.e. a module nucleus.

Recall the following notions from [3]. Let Q be a semigroup, $a \in Q$. We then put $Q\langle a \rangle = \{a^k : k \in \mathbb{N}\}$. We say that $Q\langle a \rangle$ is a *cyclic subsemigroup* of Q generated by the element a . For any $a \in Q$, we have either that $Q\langle a \rangle$ is infinite or that $Q\langle a \rangle$ is finite i.e. there exist $k_1, k_2 \in \mathbb{N}$ minimal such that $a^{k_1+k_2} = a^{k_2}$; here $k_1 = p(a)$ is the *period* of a and $k_2 = i(a)$ the *index* of a . We say that an element $a \in Q$ is *periodic* if $Q\langle a \rangle$ is finite. We shall denote by $Q_{\mathbf{P}} = \{a \in Q : a \text{ is periodic}\}$ and $Q_{\mathbf{E}} = \{a \in Q : a \text{ is idempotent}\}$. If $Q = Q_{\mathbf{P}}$ ($Q = Q_{\mathbf{E}}$) we say that Q is *periodic* (*idempotent*).

Now we generalise the nuclei w_n of Proposition 2.5.

Lemma 2.7 *Let Q be a unital quantale, j a module prenucleus on a unital left Q -module M , $k \in \mathbb{N}$, $n \in M$, $z \leq j(n) \multimap_l n$. Then the operator $\nu_{z,n}^k : M \rightarrow M$ defined by the prescription*

$$\nu_{z,n}^k(m) = (z^k(m \multimap_l n)) \multimap_r n$$

is a module prenucleus on M greater than j . Especially, the operator $\nu_n^k(j) = \nu_{j(n) \multimap_l n, n}^k$ is a module prenucleus on M greater than j . Moreover, if $z \in Q_{\mathbf{P}}$ and $k \geq \mathbf{i}(z)$ we have that $\nu_{z,n}^k$ is a module nucleus greater than j .

Proof. We first show that, for all $m \in M$, $j(m) \leq \nu_{z,n}^k(m)$ – we have then $m \leq \nu_{z,n}^k(m)$. It suffices to show that

$$(z^k(m \multimap_l n)) \bullet j(m) \leq n.$$

We have

$$\begin{aligned} (z^k(m \multimap_l n)) \bullet j(m) &= z^k \bullet ((m \multimap_l n) \bullet j(m)) \\ &\leq z^k \bullet j((m \multimap_l n) \bullet m) \\ &\leq z^k \bullet j(n) \\ &\leq n, \end{aligned}$$

where the last inequality follows by induction on k :

- $z \bullet j(n) \leq (j(n) \multimap_l n) \bullet j(n) \leq n$,
- $z^{k+1} \bullet j(n) = z \bullet (z^k \bullet j(n)) \leq z \bullet n \leq z \bullet j(n) \leq n$.

Note that the function $(z^k(- \multimap_l n)) \multimap_r n$ is order-preserving. Now, let us show that $a \bullet \nu_{z,n}^k(m) \leq \nu_{z,n}^k(a \bullet m)$ for all $a \in Q$ and $m \in M$, that is, we check that $a \bullet [(z^k(m \multimap_l n)) \multimap_r n] \leq (z^k((a \bullet m) \multimap_l n)) \multimap_r n$. Evidently, $(a \bullet m \multimap_l n) \cdot a \leq m \multimap_l n$. We have

$$\begin{aligned} [(z^k((a \bullet m) \multimap_l n)) \cdot a] \bullet [(z^k(m \multimap_l n)) \multimap_r n] &\leq \\ (z^k(m \multimap_l n)) \bullet [(z^k(m \multimap_l n)) \multimap_r n] &\leq n. \end{aligned}$$

So we have shown that $\nu_{z,n}^k$ is a module prenucleus.

Now assume that $z \in Q_{\mathbf{P}}$ and $k \geq \mathbf{i}(z)$. We show that $\nu_{z,n}^k(\nu_{z,n}^k(m)) \leq \nu_{z,n}^k(m)$. This is equivalent to the following inequality $(\star)$

$$(z^k(m \multimap_l n)) \bullet (z^k((z^k(m \multimap_l n)) \multimap_r n \multimap_l n)) \multimap_r n \leq n.$$

Since $\multimap_l n$ and $\multimap_r n$ form a Galois connection and z is periodic the inequality $(\star)$ follows from the inequality $(\star\star_i)$, $i \in \mathbf{N}$

$$\begin{aligned} & [z^{i\mathbf{p}(z)} \cdot ((z^k(m \multimap_l n)) \multimap_r n \multimap_l n)] \bullet \\ & \bullet (z^k((z^k(m \multimap_l n)) \multimap_r n \multimap_l n)) \multimap_r n \leq n. \end{aligned}$$

Let us prove $(\star\star_i)$ implies $(\star)$. We have

$$\begin{aligned} & (z^k(m \multimap_l n)) \bullet (z^k((z^k(m \multimap_l n)) \multimap_r n \multimap_l n)) \multimap_r n = \\ & [z^{i\mathbf{p}(z)} \cdot (z^k(m \multimap_l n))] \bullet (z^k((z^k(m \multimap_l n)) \multimap_r n \multimap_l n)) \multimap_r n = \\ & z^{i\mathbf{p}(z)} \bullet ((z^k(m \multimap_l n)) \bullet (z^k((z^k(m \multimap_l n)) \multimap_r n \multimap_l n)) \multimap_r n) \leq \\ & z^{i\mathbf{p}(z)} \bullet (\multimap_l n) \bullet (z^k((z^k(m \multimap_l n)) \multimap_r n \multimap_l n)) \multimap_r n \leq \\ & z^{i\mathbf{p}(z)} \bullet n \leq n. \end{aligned}$$

(in fact, we also have $(\star)$ implies $(\star\star_i)$ since we obtain by $(\star)$ and the Galois connection that

$$((z^k(m \multimap_l n)) \multimap_r n \multimap_l n) \bullet (z^k((z^k(m \multimap_l n)) \multimap_r n \multimap_l n)) \multimap_r n \leq n.$$

Then

$$\begin{aligned} & [z^{i\mathbf{p}(z)} \cdot ((z^k(m \multimap_l n)) \multimap_r n \multimap_l n)] \bullet \\ & \bullet (z^k((z^k(m \multimap_l n)) \multimap_r n \multimap_l n)) \multimap_r n = \\ & = z^{i\mathbf{p}(z)} \bullet (((z^k(m \multimap_l n)) \multimap_r n \multimap_l n) \bullet \\ & \bullet (z^k((z^k(m \multimap_l n)) \multimap_r n \multimap_l n)) \multimap_r n) \leq z^{i\mathbf{p}(z)} \bullet n \leq n. \end{aligned}$$

Now, for a large enough i , such that $i\mathbf{p}(z) = i_0 + k$ for some $i_0 \in \mathbb{N}$, the condition $(\star\star_i)$ becomes equivalent to

$$\begin{aligned} z^{i_0} \bullet [z^k \left((z^k(m \twoheadrightarrow_l n)) \twoheadrightarrow_r n \twoheadrightarrow_l n \right) \bullet \\ \bullet \left(z^k \left((z^k(m \twoheadrightarrow_l n)) \twoheadrightarrow_r n \twoheadrightarrow_l n \right) \twoheadrightarrow_r n \right] \leq n. \end{aligned}$$

The last inequality evidently holds since $z^{i_0} \bullet n \leq n$ and

$$z^k \left((z^k(m \twoheadrightarrow_l n)) \twoheadrightarrow_r n \twoheadrightarrow_l n \right) \bullet \left(z^k \left((z^k(m \twoheadrightarrow_l n)) \twoheadrightarrow_r n \twoheadrightarrow_l n \right) \twoheadrightarrow_r n \right) \leq n.$$

We have then that $\nu_{z,n}^k$ is a module nucleus. $\square$

Johnstone gives in ([4]) an explicit formula for $\nu(j)$ for frames. The following proposition is a slight modification of it for Q -modules.

Proposition 2.8 *Let Q be a unital periodic quantale, j a module pre-nucleus on a unital left Q -module M . Then the operator $\nu_{\mathbf{p}}(j) : M \rightarrow M$ defined by the prescription*

$$\nu_{\mathbf{p}}(j)(m) = \bigwedge \{ ((j(n) \twoheadrightarrow_l n)^{i(j(n) \twoheadrightarrow_l n)}(m \twoheadrightarrow_l n)) \twoheadrightarrow_r n : n \in M \}$$

is the associated module nucleus $\nu(j)$.

Proof. Note that $\nu_{\mathbf{p}}(j) = \bigwedge_{n \in M} \{ \nu_n^{i(j(n) \twoheadrightarrow_l n)}(j) \}$ i.e. $\nu_{\mathbf{p}}(j)$ is a module nucleus greater than j . Let $j(m) = m$. We put

$$m' = ((m \twoheadrightarrow_l m)^{i(m \twoheadrightarrow_l m)}(m \twoheadrightarrow_l m)) \twoheadrightarrow_r m, \quad k = i(m \twoheadrightarrow_l m).$$

We then have $\nu_{\mathbf{p}}(j)(m) \leq m'$ and

$$\begin{aligned} m' = (e \cdot e) \bullet m' &\leq (e \cdot (m \twoheadrightarrow_l m)) \bullet m' \\ &\leq ((m \twoheadrightarrow_l m)^k(m \twoheadrightarrow_l m)) \bullet m' \\ &\leq m \end{aligned}$$

i.e. $\nu_{\mathbf{p}}(j)(m) = j(m)$. Hence, $\nu_{\mathbf{p}}(j)$ has the same fixed points as j i.e. $\nu_{\mathbf{p}}(j)$ is the least module nucleus greater than j . $\square$

Corollary 2.9 *Let Q be a unital idempotent quantale, j a module prenucleus on a unital left Q -module M . Then the operator $\nu_e(j) : M \rightarrow M$ defined by the prescription*

$$\nu_e(j)(m) = \bigwedge \{ ((j(n) \rightarrow_l n)(m \rightarrow_l n)) \rightarrow_r n : n \in M \}$$

is the associated module nucleus $\nu(j)$.

Theorem 2.10 *Let Q be a unital periodic quantale, M a unital left Q -module. Then, for any set Λ , $\bigsqcup \{j_\lambda \in N(M) : \lambda \in \Lambda\} = \bigsqcup^\dagger \{\nu_p(j_{\lambda_1} \circ \dots \circ j_{\lambda_s}) : \{\lambda_1, \dots, \lambda_s\} \subseteq \Lambda\} = \nu_p(\bigvee \{j_\lambda \in N(M) : \lambda \in \Lambda\})$.*

Proof. The first equality follows from the proposition 2.8 and from the fact that a directed join of module nuclei is computed pointwise. The second one follows from an easy observation that the pointwise join of module prenuclei is a module prenucleus. $\square$

Corollary 2.11 *Let Q be a unital idempotent quantale, M a unital left Q -module. Then, for any set Λ , $\bigsqcup \{j_\lambda \in N(M) : \lambda \in \Lambda\} = \bigsqcup^\dagger \{\nu_e(j_{\lambda_1} \circ \dots \circ j_{\lambda_s}) : \{\lambda_1, \dots, \lambda_s\} \subseteq \Lambda\} = \nu_e(\bigvee \{j_\lambda \in N(M) : \lambda \in \Lambda\})$.*

Proposition 2.12 *Let Q be a unital quantale, j a module prenucleus on a unital left Q -module M . Then the operators $\nu_P(j), \nu_E(j) : M \rightarrow M$ defined by the prescriptions*

$$\begin{aligned} \nu_P(j) &= \bigwedge \{ \nu_{i(z)}^{z,n}(j) : n \in M, z \in Q_P, z \leq j(n) \rightarrow_l n \} \\ \nu_E(j) &= \bigwedge \{ \nu_1^{z,n}(j) : n \in M, z \in Q_E, z \leq j(n) \rightarrow_l n \} \end{aligned}$$

coincide with the associated nucleus $\nu(j)$.

Proof. By 2.6 and 2.7 we know that both operators are module nuclei on M greater than j .

Now, let $j(m) = m$. Then $e \in Q_E \subseteq Q_P$ and $e \leq m \rightarrow_l m$. We then have

$$\begin{aligned}
 j(m) &\leq \nu_{\mathbf{P}}(j)(m) \leq \nu_{\mathbf{E}}(j)(m) \leq (e \cdot (m \twoheadrightarrow_l m)) \twoheadrightarrow_r m \\
 &= e \bullet [(m \twoheadrightarrow_l m) \twoheadrightarrow_r m] \\
 &\leq (m \twoheadrightarrow_l m) \bullet [(m \twoheadrightarrow_l m) \twoheadrightarrow_r m] \\
 &\leq m = j(m)
 \end{aligned}$$

i.e. $\nu_{\mathbf{P}}(j)(m) = \nu_{\mathbf{E}}(j)(m) = j(m)$.

Since both $\nu_{\mathbf{E}}(j)$ and $\nu_{\mathbf{E}}(j)$ have the same fixed points as j we have

$$\nu_{\mathbf{E}}(j) = \nu(j) = \nu_{\mathbf{P}}(j). \quad \square$$

Theorem 2.13 *Let Q be a unital quantale, M a unital left Q -module. Then, for any set Λ , $\sqcup\{j_\lambda \in N(M) : \lambda \in \Lambda\} = \sqcup^\uparrow\{\nu_{\mathbf{P}}(j_{\lambda_1} \circ \dots \circ j_{\lambda_s}) : \{\lambda_1, \dots, \lambda_s\} \subseteq \Lambda\} = \sqcup^\uparrow\{\nu_{\mathbf{E}}(j_{\lambda_1} \circ \dots \circ j_{\lambda_s}) : \{\lambda_1, \dots, \lambda_s\} \subseteq \Lambda\} = \nu_{\mathbf{P}}(\bigvee\{j_\lambda \in N(M) : \lambda \in \Lambda\}) = \nu_{\mathbf{E}}(\bigvee\{j_\lambda \in N(M) : \lambda \in \Lambda\})$.*

Proof. Similar to the proof of theorem 2.10, with $\nu_{\mathbf{P}}$ replaced by $\nu_{\mathbf{P}}$ and $\nu_{\mathbf{E}}$ and using proposition 2.12 instead of proposition 2.8. $\square$

Another way how to construct new module nuclei is the method of R -saturated elements introduced for frames by Kríž (see [7]).

Definition 2.14 *Let M be a left Q -module, $R \subseteq M \times M$ a relation on M . We shall say that an element $m \in M$ is R -saturated if*

$$a \bullet n_1 \leq m \text{ iff } a \bullet n_2 \leq m \quad \text{and} \quad n_1 \leq m \text{ iff } n_2 \leq m$$

holds for all $a \in Q$, $n_1, n_2 \in M$ such that $(n_1, n_2) \in R$. We shall denote by M_R the set of all R -saturated elements.

Note that M_R coincides with the sup-lattice quotient of M by the relation $R' = R \cup \{(a \bullet m_1, a \bullet m_2) : a \in Q, (m_1, m_2) \in R\}$ [5]. Equivalently the map $j_R : M \rightarrow M$ defined by $j_R(m) = \bigwedge\{n \in M_R : m \leq n\}$ is a closure operator on M , $M_{j_R} = M_R$, and the sup-lattice homomorphism $j_R : M \rightarrow M_R$ is universal amongst all sup-lattice homomorphisms $f : M \rightarrow N$ for which $f(m_1) = f(m_2)$ for all $(m_1, m_2) \in R'$.

Lemma 2.15 *Let $a \in Q$, $m \in M$ and m is R -saturated. Then the element $a \twoheadrightarrow_r m = \bigvee \{n \in M : a \bullet n \leq m\}$ is R -saturated.*

Proof. Let $n_1, n_2 \in M$, $(n_1, n_2) \in R$, $c \in Q$. Then $c \bullet n_1 \leq a \twoheadrightarrow_r m$ iff $a \bullet (c \bullet n_1) \leq m$ iff $(a \cdot c) \bullet n_1 \leq m$ iff $(a \cdot c) \bullet n_2 \leq m$ iff $a \bullet (c \bullet n_2) \leq m$ iff $c \bullet n_2 \leq a \twoheadrightarrow_r m$. Similarly, $n_1 \leq a \twoheadrightarrow_r m$ iff $n_2 \leq a \twoheadrightarrow_r m$. $\square$

Theorem 2.16 *Let M be a left Q -module, $R \subseteq M \times M$. Then the closure operator j_R is a module nucleus, and the module homomorphism $j_R : M \rightarrow M_R$ is universal amongst all module homomorphisms $f : M \rightarrow N$ for which $f(m_1) = f(m_2)$ for all $(m_1, m_2) \in R$.*

Proof. Assume $a \in Q$, $m \in M$. Then $a \bullet m \leq j_R(a \bullet m)$ i.e. $m \leq a \twoheadrightarrow_r j_R(a \bullet m)$ i.e. $j_R(m) \leq a \twoheadrightarrow_r j_R(a \bullet m)$ i.e. $a \bullet j_R(m) \leq j_R(a \bullet m)$. Hence, j_R is a module nucleus.

Now, let $f : M \rightarrow N$ be a module homomorphism such that $f(m_1) = f(m_2)$ for all $(m_1, m_2) \in R$, $m \in M$. Then f factors uniquely through j_R and a sup-lattice homomorphism $f' : M_R \rightarrow N$, because $(m_1, m_2) \in R'$ implies $f(m_1) = f(m_2)$. Let us prove that f' preserves the action. Denoting by $\bullet_{j_R}$ the action on M_R and by $\bullet_N$ the action on N we have $f'(a \bullet_{j_R} j_R(m)) = f'(j_R(a \bullet j_R(m))) = f(a \bullet j_R(m)) = a \bullet_N f(j_R(m)) = a \bullet_N f'(m)$. $\square$

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References

- [1] S. Abramsky, S. Vickers, Quantales, observational logic and process semantics, *Math. Struct. in Comp. Science*, **3** (1993) 161-227.

- [2] B. Banaschewski, Another look at the localic Tychonoff theorem, *Comm. Math. Univ. Carolinae*, **29** No. 4 (1988) 647-656.
- [3] A.H. Clifford and G.B. Preston, *The algebraic theory of semigroups*, (American Mathematical Society, Providence, 1964).
- [4] P. T. Johnstone, Two Notes on Nuclei, *Order*, **7** (1990) 205-210.
- [5] A. Joyal, M. Tierney, An extension of the Galois theory of Grothendieck, *Amer. Math. Soc. Memoirs No. 309*, (1984).
- [6] D. Kruml, Spatial quantales, *Applied Categorical Structures*, (in print).
- [7] I. Kříž, (in Czech); PhD thesis, Dept. of Mathematics, Charles University, (1988).
- [8] D.S. Macnab, Modal operators on Heyting algebras, *Algebra Universalis*, **12** (1981) 5-29.
- [9] C. J. Mulvey, J. W. Pelletier, On the quantisation of points, *Journal of Pure and Applied Algebra*, (in print).
- [10] J. Paseka, Simple quantales, Proceedings of the Eight Prague Topological Symposium 1996, (Topology Atlas 1997) 314-328.
- [11] J. Paseka, D. Kruml, Embeddings of quantales into simple quantales, *Journal of Pure and Applied Algebra*, *Journal of Pure and Applied Algebra*, **148** (2000) pp. 209-216.
- [12] J. W. Pelletier, J. Rosický, Simple involutive quantales, *Journal of Algebra*, **195** (1997) 367-386.
- [13] P. Resende, Quantales, finite observations and strong bisimulation, *Theoretical Computer Science*, (in print).
- [14] K. I. Rosenthal, *Quantales and their applications*, (Pitman Research Notes in Mathematics Series 234, Longman Scientific & Technical, Essex, 1990).

- [15] K. I. Rosenthal, *The theory of quantaloids*, (Pitman Research Notes in Mathematics Series 348, Longman Scientific & Technical, Essex, 1996).

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