

CAHIERS DE TOPOLOGIE ET GÉOMÉTRIE DIFFÉRENTIELLE CATÉGORIQUES

EDUARDO J. DUBUC

JORGE C. ZILBER

Infinitesimal and local structures for Banach spaces and its exponentials in a topos

Cahiers de topologie et géométrie différentielle catégoriques, tome 41, n° 2 (2000), p. 82-100

http://www.numdam.org/item?id=CTGDC_2000__41_2_82_0

© Andrée C. Ehresmann et les auteurs, 2000, tous droits réservés.

L'accès aux archives de la revue « Cahiers de topologie et géométrie différentielle catégoriques » implique l'accord avec les conditions générales d'utilisation (<http://www.numdam.org/conditions>). Toute utilisation commerciale ou impression systématique est constitutive d'une infraction pénale. Toute copie ou impression de ce fichier doit contenir la présente mention de copyright.

NUMDAM

Article numérisé dans le cadre du programme
Numérisation de documents anciens mathématiques

<http://www.numdam.org/>

INFINITESIMAL AND LOCAL STRUCTURES FOR BANACH SPACES AND ITS EXPONENTIALS IN A TOPOS

by *Eduardo J. DUBUC and Jorge C. ZILBER*

RESUME. Dans [9] nous avons construit une immersion de la catégorie des ouverts d'espaces de Banach et des fonctions holomorphes dans un topos modèle analytique de la GDS [8]. Cette immersion préserve les produits finis et elle est compatible avec le calcul différentiel

Nous étudions ici, dans un cadre général, la structure topologique interne héritée par un objet d'une topologie dans l'ensemble de ses sections globales. Et nous tenons compte des cas particuliers d'un ouvert d'un espace de Banach et de l'exponentielle de celui-ci avec un objet du site

Dans ce dernier cas, nous avons introduit une topologie qui généralise la topologie canonique (considérée dans [10]) dans l'ensemble des morphismes à valeurs complexes d'un espace analytique. Cette topologie tient compte de la convergence uniforme sur des compacts et de la topologie limite inductive dans les anneaux de germes

Introduction.

In [9] we have constructed an embedding $j: \mathcal{B} \rightarrow \mathcal{T}$ from the category \mathcal{B} of open sets of complex Banach spaces and holomorphic functions into the analytic (well adapted) model of SDG \mathcal{T} introduced in [8]. This embedding preserves finite products and is consistent with the differential calculus. Here in section 1 we study in a general context the topological structure (in the sense of [11]) inherited by an object in the topos from a topology on the set of global sections, and compare it with the Penon or intrinsic topology. We show, under a very general assumption, that the inherited topology is subintrinsic, and that its infinitesimals are exactly the intrinsic or Penon infinitesimals. Then, in sections 2 and 3 we apply these results to the objects of the form jB and the exponentials jB^X , with X any object in the site of

definition (see 0.1.1). We introduce a topology in the sets of global sections of these exponentials which is related to uniform convergence on compact subsets and the inductive limit topology on spaces of banach valued germs of holomorphic functions. When $B = \mathbb{C}$, and X is a complex space, this topology is the “canonical” Frechet topology considered in [10]. However, in our general case it is complete but not metrizable.

For the convenience of the reader and to set the notation we start recalling some facts.

0. Recall of some definitions and notation.

The topos \mathbf{T} is the topos of sheaves on the category \mathbf{H} of (affine) analytic schemes.

0.1.1 Recall briefly from [9] the construction of \mathbf{T} . We consider the category \mathbf{H} of (affine) analytic schemes. An object E in \mathbf{H} is an A -ringed space [7] $E = (E, \mathcal{O}_E)$ (by abuse we denote also by the letter E the underlying topological space of the A -ringed space) which is given by two coherent sheaves of ideals R, S in \mathcal{O}_D , where D is an open subset of \mathbb{C}^m , $R \subset S$, and where:

$$E = \{p \in D \mid h(p) = 0 \ \forall [h]_p \in S_p\},$$

$$\mathcal{O}_E = (\mathcal{O}_D/R)|_E \quad (\text{restriction of } \mathcal{O}_D/R \text{ to } E).$$

The arrows in \mathbf{H} are the morphism of A -ringed spaces. We will denote by \mathbf{T} the topos of sheaves on \mathbf{H} for the (sub canonical) Grothendieck topology given by the open coverings. There is a full (Yoneda) embedding $\mathbf{H} \rightarrow \mathbf{T}$. Notice that for an infinite dimensional banach open B , the A -ringed space (B, \mathcal{O}_B) is not in \mathbf{H} .

Let \mathbf{B} be the category of open sets of complex Banach spaces and holomorphic functions. The embedding $j: \mathbf{B} \rightarrow \mathbf{T}$ is defined in [9] as follows:

0.1.2 Let E be an object in \mathbf{H} , $E = (E, \mathcal{O}_E)$ as above, let B be an open subset of a complex Banach space, and let $t = (t, \tau)$ be a morphism of A -ringed spaces, $t: (E, \mathcal{O}_E) \rightarrow (B, \mathcal{O}_B)$ (we adopt the corresponding abuse of notation for morphisms).

We will say that t has “local extensions”, if for each $x \in E$, there is an open neighborhood U of x in \mathbb{C}^m and an extension

$$(f, f^*): (U, \mathcal{O}_U) \rightarrow (B, \mathcal{O}_B) :$$

$$\begin{array}{ccc} (E \cap U, \mathcal{O}_{E \cap U}) & \longrightarrow & (U, \mathcal{O}_U) \\ & \searrow (t, \tau) & \swarrow (f, f^*) \\ & & (B, \mathcal{O}_B) \end{array}$$

The set

$jB(E) = \{t: (E, \mathcal{O}_E) \rightarrow (B, \mathcal{O}_B) \text{ such that } t \text{ has local extensions}\}$ defines a sheaf $jB \in \mathbf{T}$.

If $g: F \rightarrow E$ is an arrow in \mathbf{H} , $jB(g): jB(E) \rightarrow jB(F)$ is given by composing with g :

$$jB(g)(t) = t \circ g: F = (F, \mathcal{O}_F) \rightarrow E = (E, \mathcal{O}_E) \rightarrow (B, \mathcal{O}_B),$$

(for $t \in jB(E)$).

Moreover, given two open subsets of banach spaces, and an holomorphic function $f: B_1 \rightarrow B_2$, we consider the morphism

$$(f, f^*): (B_1, \mathcal{O}_{B_1}) \rightarrow (B_2, \mathcal{O}_{B_2}).$$

It is clear that if $E \in \mathbf{H}$ and $t \in jB_1(E)$, then $(f, f^*) \circ t \in jB_2(E)$. Thus, we have an arrow:

$$jf: jB_1 \rightarrow jB_2, (jf)_E(t) = (f, f^*) \circ t \text{ for all } E \in \mathbf{H} \text{ and } t \in jB_1(E).$$

This defines a functor $j: \mathbf{B} \rightarrow \mathbf{T}$. It is clear that $\Gamma(jB) = B$ for all $B \in \mathbf{B}$, and $\Gamma(jf) = f$ for all arrows $f: B_1 \rightarrow B_2$ in \mathbf{B} .

0.1.3 If $E \in \mathbf{H}$, $B \in \mathbf{B}$, and $q: E \rightarrow jB$ is an arrow in \mathbf{T} , q corresponds to an element $q \in jB(E)$, that is, $q: (E, \mathcal{O}_E) \rightarrow (B, \mathcal{O}_B)$ is a morphism of A -ringed spaces with local extensions. Then, $q = (q, \phi)$ where $q: E \rightarrow B$ is continuous, and it is immediate that $\Gamma(q) = q$ (notice here the abuse of language). If E is of the form (U, \mathcal{O}_U) for an open set

$U \subset \mathbb{C}^m$, then q is an holomorphic function and $\phi = q^*$, see [9, 1.2]. Thus, if $q, r: U \rightarrow jB$ are such that $\Gamma(q) = \Gamma(r)$, then $q = r$.

Notation. Given any object $F \in \mathbf{T}$, we shall write $\Omega[F] = \Omega^F$ for the internal lattice (locale) of subobjects, and $\Omega(F) = \Gamma(\Omega^F)$ for the external lattice (locale) of subobjects.

We recall now some facts due to J. Penon which hold in a context that includes the topos \mathbf{T} . See [12], also [6], [5].

0.2.0 Let \mathbf{H} now be any category with finite limits, and Λ be a class of arrows in \mathbf{H} containing the isomorphisms, closed under composition and stable under pull-backs. Consider the Grothendieck (pre) topology in \mathbf{H} which has as covers all families $U_\alpha \rightarrow X$ in Λ such that the global sections $\Gamma(U_\alpha) \rightarrow \Gamma(X)$ are a surjective family of sets, and let \mathbf{T} be the topos of sheaves.

0.2.1 The topos \mathbf{T} satisfies the Nullstellensatz. By this we mean:
For any object F in \mathbf{T} , $F = 0$ if and only if $\Gamma(F) = \emptyset$.

0.2.2 Let $F \in \mathbf{T}$ and $S \subset \Gamma(F)$. We define the subobject $E(F, S) \subset F$ in \mathbf{T} by the following universal property :

$\forall X \in \mathbf{H}, \forall q: X \rightarrow F$ in \mathbf{T} , q factors through $E(F, S)$ iff $\Gamma(q)$ factors through S .

It is easy to see that such object $E(F, S)$ exists. In fact, the reader can check that the definition above actually determines a sheaf $\mathbf{H}^{op} \rightarrow \mathbf{Ens}$ on the site \mathbf{H} . If there is no danger of confusion, we will abuse the notation and write ES in place of $E(F, S)$.

Notice that the map $E: \Omega(\Gamma F) \rightarrow \Omega(F)$ is a poset morphism right adjoint to $\Gamma: \Omega(F) \rightarrow \Omega(\Gamma F)$. That is, given any subset $S \subset \Gamma F$, and subobject $G \subset F$, then, $\Gamma G \subset S$ if and only if $G \subset ES$. We have $G \subset E\Gamma G$ and is clear that that $\Gamma ES = S$.

We shall indicate by “ \neg ” the negation in the lattices of subobjects in the topos \mathbf{T} as well that in the topos of Sets (in the latter case the usual complement). From the validity of the Nullstellensatz it follows easily (see [6]):

0.2.3 Let $F \in \mathbf{T}$ and let A be any subobject of F . Then:

$$\neg(A) = E\neg(\Gamma A) \quad \text{and} \quad \neg\neg(A) = E\Gamma(A).$$

0.2.4 Given E in \mathbf{H} and $x \in \Gamma(E)$ (that is, $x: 1 \rightarrow E$ in \mathbf{H}), an open neighborhood of x is an arrow $i: V \rightarrow E$ in Λ such that x factors $x = i \circ y$ for some $y \in \Gamma(V)$. Let $F \in \mathbf{T}$ and let A be any subobject of F . We say that A is a Λ -open subobject of F if given any $E \in \mathbf{H}$, $q: E \rightarrow F$ in \mathbf{T} , and $x \in \Gamma(E)$, if $q \circ x$ factors through A , then there is an open neighborhood $i: V \rightarrow E$ of x such that $q \circ i$ factors through A .

Λ -open subobjects form a sublocale (sublattice closed under finite intersections and all unions) $\Lambda(F) \subset \Omega(F)$. A subset of $\Gamma(F)$ is Λ -open if it is of the form $\Gamma(A)$ for a Λ -open subobject A of F . These subsets are the open subsets of a topology in the set $\Gamma(F)$. Notice that Γ preserves finite intersections (in fact all) and all unions (see [5, 2.1 and 2.2]).

Recall that a topological structure $\Lambda[F]$ in an object F [1, II, 1.3] is an internal sublocale $\Lambda[F] \subset \Omega[F] = \Omega^F$ in the topos \mathbf{T} . Λ -open subobjects generate a topological structure. This structure is the Spectral topological structure in the topos \mathbf{T} [1, appendix 1.2].

0.2.5 Given any two topological structures T, Σ on objects F, G respectively, we can consider the external sublocale

$$\pi(T, \Sigma)(F \times G) \subset \Omega(F \times G)$$

which has as a base the subobjects of the form $U \times V$, with $U \in T(F)$, $V \in \Sigma(G)$. This locale defines the product topological structure. If T is a topological structure in the topos (in the sense of [1, II, 1.12]), it follows easily that $\pi(T, T)(F \times G) \subset T(F \times G)$, but in general the equality does not hold. When considering only one object F with a topological structure T , we write $\pi(T, T) = \pi T$ for the product structure in $F \times F$.

0.2.6 Given any $F \in \mathbf{T}$, if a subobject $A \subset F$ is Penon (or intrinsic) open, then it is Λ -open. That is, $P(F) \subset \Lambda(F)$, and thus also $P[F] \subset \Lambda[F]$ [5, 2.3]. If F is Λ -separated (or Hausdorff, see 0.2.7 below), then the converse is also true, that is $\Lambda(F) = P(F)$, $\Lambda[F] = P[F]$ [5, 2.6] (for the definition of Penon or intrinsic open subobjects see [12], or [6], [1], [5]).

0.2.7 Let $F \in \mathbf{T}$, we say that F is Λ -separated (or Hausdorff) if the negation of the diagonal $\neg\Delta \subset F \times F$ is a Λ -open subobject.

0.2.8 Let $F \in \mathbf{T}$ and let A be a Λ -open subobject of F , then $A = E\Gamma(A)$. Thus by 0.2.6 this equality also holds for Penon open subobjects [5, 2.5].

0.2.9 The map $E: \Omega(\Gamma F) \rightarrow \Omega(F)$ preserves all unions of Λ -open subsets. This follows immediately from 0.2.8 and the fact that Γ preserves all unions.

1. The inherited topological structure.

Let \mathbf{T} be any topos in the context defined in 0.2.0. Suppose we have an object F in \mathbf{T} such that the set of global sections $\Gamma(F)$ has a topology in the usual sense. In this section we shall construct a topological structure $\kappa[F]$ in the object F which is inherited from the topology of $\Gamma(F)$, and study some of its properties.

Given any object $E \in \mathbf{H}$, the set $\Gamma(E)$ is furnished with the topology defined in 0.2.4. By abuse of notation we shall write $E = \Gamma(E)$ for this topological space. It follows that a base for this topology consists of the subsets of the form $i(\Gamma(V)) \subset \Gamma(E)$ for any arrow $i: V \rightarrow E$ in Λ . Given any arrow $q: X \rightarrow E$ in \mathbf{H} , clearly the function $q = \Gamma(q): X \rightarrow E$ is continuous.

In the example which concerns this paper, 0.1.1 above, given a ringed space (E, \mathcal{O}_E) in \mathbf{H} , this topology in $E = \Gamma(E, \mathcal{O}_E)$ is just the topological space E .

1.0 Basic assumptions and notation.

- i) From now on we shall consider a (fixed) object F in \mathbf{T} such that the set $\Gamma(F)$ is furnished with a (unnamed) arbitrary topology, and when we say open of $\Gamma(F)$ we shall mean an open set in this topology.
- ii) The topology in $\Gamma(F)$ is Hausdorff.
- iii) Given any arrow $q: E \rightarrow F$ in the topos \mathbf{T} , $E \in \mathbf{H}$, the function

$$q = \Gamma(q): E \rightarrow \Gamma(F) \quad \text{is continuous.}$$

1.1 Definition. Let $E \in \mathbf{H}$, and let G be a subobject of $E \times F$ in \mathbf{T} . G is said to be an inherited test-open iff there is an open subset $S \subset E \times \Gamma(F)$

such that $G = ES$. We denote by $t\kappa(E \times F) \subset \Omega(E \times F)$ the set of inherited test-opens.

1.2 Proposition. *With the notations in definition 1.1, if G is an inherited test-open, then G is a Λ -open subobject. That is, $t\kappa(E \times F) \subset \Lambda(E \times F)$. This implies that for the topological structure defined in 1.5 below, we have $\kappa[F] \subset \Lambda(F)$.*

Proof. Let X and $E \in \mathbf{H}$, and let $q: X \rightarrow E \times F$ be an arrow in \mathbf{T} . Then, $q = \Gamma(q): X \rightarrow E \times \Gamma(F)$ is continuous. This follows since the two components of $\Gamma(q)$ (notice that Γ preserves products) are continuous. Then the statement follows directly from the definition of Λ -open, 0.2.4 above. \square

1.3 Corollary. *Notice that it follows from 0.2.8 and 0.2.9 that the maps Γ and E establish an isomorphism (preserving all the operations) $t\kappa(E \times F) \approx O(E \times \Gamma(F))$, where O indicates the local of open sets for the product topology in $E \times F$. In particular, the inherited test-opens form a sublocale of $\Omega(E \times F)$.*

1.4 Definition-Proposition. *We define the subobject $b\kappa[F] \subset \Omega[F]$ of inherited opens by the following universal property:*

$\forall X \in \mathbf{H}$, $\forall q: X \rightarrow \Omega[F]$ in \mathbf{T} , q factors through $b\kappa[F]$ iff the corresponding subobject $M_q \subset X \times F$ is inherited test-open.

Proof. We have to show that this property actually defines a subsheaf of $\Omega[F]$:

a) Given $r: E \rightarrow X$ in \mathbf{H} , and $q \circ r: E \rightarrow X \rightarrow \Omega[F]$,

if $M_q \in t\kappa(X \times F)$, then $M_{q \circ r} \in t\kappa(E \times F)$.

b) Given a covering $r_i: U_i \rightarrow X$, and $q \circ r_i: U_i \rightarrow X \rightarrow \Omega[F]$,

if $M_{q \circ r_i} \in t\kappa(U_i \times F)$ for all i , then $M_q \in t\kappa(X \times F)$.

This follows from corollary 1.3 considering the fact that $M_{q \circ r} = (r \times id)^{-1}(M_q)$. \square

1.5 Definition. *The subobject $b\kappa[F]$ is a base for the topological structure $\kappa[F]$, which is defined internally by the following statement:*

$A \in \kappa[F]$ iff $\forall x \in A \exists U \in b\kappa[F] \mid x \in U \subset A$.

1.6 Remark. Notice that for global sections, that is, the external lattices of subobjects, we have $O(\Gamma(F)) \approx t\kappa(F) = b\kappa(F) = \kappa(F)$. The isomorphism is a particular case of 1.3, the first equality holds by definition, and the second since by 1.3 the family of subobjects $b\kappa(F)$ is already a local in the category of sets.

We consider now the product topological structure $\pi\kappa(F \times F) \subset \Omega(F \times F)$, see 0.2.5 above.

1.7 Corollary. *The maps Γ and E establish an isomorphism (preserving all the operations) $\pi\kappa(F \times F) \approx O(\Gamma(F) \times \Gamma(F))$, where O indicates the local of open sets for the usual product topology in $\Gamma(F) \times \Gamma(F)$. Notice that this holds if we have two different objects with respective topologies, as in 0.2.5.*

Proof. It follows immediately from the previous remark considering that the maps Γ and E preserve products in the appropriate sense, and thus satellites an isomorphism between the bases. \square

1.8 Proposition. *The object F is Λ -separated or Hausssdorf, and also κ -separated with respect to $\pi\kappa$.*

Proof. Since $\Gamma(F)$ is a Hausssdorf topological space, it follows immediately from 1.7 that the negation of the diagonal $\neg\Delta \subset F \times F$ is a $\pi\kappa$ -open subobject. It follows then from 1.2 that it is also a $\pi\Lambda$ -open, which in turn implies it is Λ -open. \square

1.9 Proposition. *The intrinsic or Penon opens of F are exactly the Λ -opens. We have $P[F] = \Lambda[F] \subset \Omega[F]$.*

Proof. Follows from the previous proposition and 0.2.6. \square

1.10 Proposition. *The inherited opens topological structure in F is subintrinsic. That is, $\kappa[F] \subset P[F]$.*

Proof. Follows from the previous proposition and proposition 1.2. \square

1.11 Observation. *With the topology on $\Gamma(E)$ defined in 0.2.4 the objects E in the site have always a canonical inherited topological structure. This structure coincides with the spectral topological structure. That is, $\kappa[E] = \Lambda[E]$.*

All arrows in the site become automatically continuous. When the topology on $\Gamma(E)$ is Hausdorff, the basic assumption is automatically satisfied, thus, in this case, for the objects of the site we have $\kappa[E] = \Lambda[E] = P[E]$.

We record now a fact that is often useful when dealing with the inherited topological structure.

1.12 Proposition. *Let F_1 and F_2 be objects in the topos such that their sets of global sections $\Gamma(F_1)$ and $\Gamma(F_2)$ are furnished with topologies as in 1.0 above. Then, an arrow in the topos $f: F_1 \rightarrow F_2$ is continuous for the inherited topological structures if and only if the function between the global sections $\Gamma(f): \Gamma(F_1) \rightarrow \Gamma(F_2)$ is continuous.*

Proof. Assume that $\Gamma(f)$ is continuous. We have to show that if $G \in \kappa[F_2]$, then $f^{-1}(G) \in \kappa[F_1]$. We can assume that G is on the base, $G \in b\kappa[F_2]$. Then G is given by an object E in \mathbf{H} and an inherited open subobject $G \subset E \times \Gamma(F_2)$. G is of the form $G = ES$ for an open subset $S \subset E \times \Gamma(F_2)$. But $(id \times f)^{-1}(ES) = E((id \times \Gamma(f))^{-1}(S))$. This shows one implication. The proof of the other follows in the same way taking $E = 1$. \square

Recall that given any object F in the topos provided with a topological structure $T[F] \subset \Omega[F] = \Omega^F$, and a point $x \in F$, the infinitesimal T -neighborhood of x is defined as the intersection of all T -open neighborhoods:

$$T_x(F) = \bigcap \{U \in T[F] \mid x \in U\}$$

That is, $T_x(F)$ is defined by the internal validity of the formula:

$$z \in T_x(F) \iff z \in U \forall U \in T[F] \mid x \in U.$$

Given $x \in F$, we will consider now the infinitesimal objects $\kappa_x(F)$ and $P_x(F)$, and relate them with the double negation of $\{x\}$ (the object of all Penon infinitesimals), denoted $\neg\neg\{x\} = \Delta_F(x)$.

1.13 Proposition. *Given any $x \in F$, we have:*

$$\Delta_F(x) = \neg\neg\{x\} = \kappa_x(F) = P_x(F).$$

Proof. Since F is κ -separated (1.7) and therefore satisfies the separation condition T_1 , and $\kappa[F]$ is subintrinsic (1.10), the proposition follows

from [1, II, 1.9 and 1.10]. For the convenience of the reader we reproduce here the (internal) elementary proof of this result:

We shall show (1) $\kappa_x(F) \subset \neg\neg\{x\}$, (2) $\neg\neg\{x\} \subset P_x(F)$ and (3) $P_x(F) \subset \kappa_x(F)$. For (1), let $z \in \kappa_x(F)$ and suppose that $z \in \neg\{x\}$. Then $x \in \neg\{z\}$, and since by condition T_1 $\neg\{z\} \in \kappa[F]$, it follows that $z \in \neg\{z\}$. This contradiction shows that $z \in \neg\neg\{x\}$. For (2), given any Penon open U and $x \in U$, if (to contains "freshmeat-news@freshmeat.net") then save "/Mail/freshmeat" it follows immediately by definition that $\neg\neg\{x\} \subset U$. Finally, (3) holds since $\kappa[F]$ is subintrinsic. \square

2. The inherited topological structure on the objects jB

In this section we shall consider a particular case of the construction developed in section 1. Here the topos \mathbf{T} is the topos of sheaves on the category \mathbf{H} of (affine) analytic schemes (0.1.1 above). There is an embedding $j: \mathbf{B} \rightarrow \mathbf{T}$ of the category \mathbf{B} of open sets of complex Banach spaces and holomorphic functions (0.1.2 above). We consider F to be an object of the form $F = jB$ for some open set of a banach space B , and the topology in $B = \Gamma(jB)$ to be the topology determined by the banach structure. Recall that in this case the objects on the site are (affine) analytic schemes (0.1.1 above) $E = (E, \mathcal{O}_E)$, and that the topology on $E = \Gamma(E) = \Gamma(E, \mathcal{O}_E)$ is just the one of the topological space E .

Clearly the object $F = jB$ satisfy the basic assumption 1.1 in section 1.

2.1 Definition. *For each open set of a banach space B , the inherited open topological structure in jB , denoted $\kappa[B]$, is the topological structure constructed in section 1. This structure corresponds to the topology determined on B by the banach structure.*

We make now some considerations that, given an open set B of a Banach space, will permit to understand better the behavior of the negation on the object jB , and given a point p in B , the infinitesimal neighborhood of p .

2.2 Observation. *Recall that given any A -ringed space (B, \mathcal{O}_B) and any subset $S \subset B$, there is a ringed space $(S, \mathcal{O}_{B/S})$, where S has the subspace topology and $\mathcal{O}_{B/S}$ is the restriction to S of the sheaf (etal space) \mathcal{O}_B . Thus, for $p \in S$, the fiber over p is just \mathcal{O}_{B_p} , the same that the fiber over p in \mathcal{O}_B . The sheaf $\mathcal{O}_{B/S} \rightarrow S$ is just the inverse image sheaf*

$i^* (\mathcal{O}_B \rightarrow B)$ induced by the inclusion $i: S \rightarrow B$. It has the following universal property :

There is an inclusion $i: (S, \mathcal{O}_{B/S}) \rightarrow (B, \mathcal{O}_B)$ of A -ringed spaces, and given any other A -ringed space (X, \mathcal{O}_X) , a morphism

$$(f, \phi): (X, \mathcal{O}_X) \rightarrow (B, \mathcal{O}_B)$$

factors through $(S, \mathcal{O}_{B/S})$ if and only if the function $f: X \rightarrow B$ factors through S .

2.3 Definition-Proposition. Let B be an open subset of a Banach space and $S \subset B$ be any subset of B . We define the subobject $jS \subset jB$ in \mathbf{T} as follows:

Given any $X = (X, \mathcal{O}_X)$ in \mathbf{H} , a section $s: X \rightarrow jS$ is a morphism $s: (X, \mathcal{O}_X) \rightarrow (S, \mathcal{O}_{B/S})$ of A -ringed spaces such that the composite $i \circ s: (X, \mathcal{O}_X) \rightarrow (B, \mathcal{O}_B)$ has local extensions. Thus, by definition, $i \circ s$ is a section $X \rightarrow jB$ (see 0.1.2 above). This defines a sub-sheaf $jS \subset jB$. Clearly, a section $s: X \rightarrow jB$ factors through jS if and only if $\Gamma(s): X \rightarrow B$ factors through S .

2.4 Proposition. In the situation above, we have $ES = jS$. Moreover, if S is an open subset of B , jS is just the object jS defined by the embedding $j: B \rightarrow \mathbf{T}$.

Proof. Straightforward. □

To understand better the negation on the objects jB the reader should consider the statement in 0.2.3 under the light of these last two propositions. In a way, we can imagine that the negation of a subobject A of jB is given by the restriction of the ringed space (B, \mathcal{O}_B) to the complement of ΓA in B .

2.5 Proposition. Let $B \in \mathbf{B}$ be any open set of a banach space. Consider a point $p \in B$. We have $p: 1 \rightarrow jB$ in \mathbf{T} , and we denote $\{p\} \subset jB$. Then:

$$\neg\{p\} = j(B - \{p\}) \text{ and } \Delta_B(p) = \neg\neg\{p\} = j\{p\}$$

Proof. It follows immediately from 0.2.3 and proposition 2.4 □

The objects $\Delta_B(p)$ are isomorphic for all the points p in B . They define the object of infinitesimals of B . We shall write Δ_B for this object. Thus, if B is finite dimensional, $\Delta_B = \Delta(n) \subset \mathbb{C}^n$.

3. Topological structures on some exponential objects.

In this section we shall define a topological structure in objects of the form F^X , where $F \subset jB$ is any subobject in the topos, B is an open subset of a banach space, and X is any object in the site \mathbf{H} . Notice that a particular case of this is the exponential of two objects in the site.

This structure corresponds to a topology in the set $\Gamma(F^X)$ of global sections which is defined using uniform convergence on compact subsets and the inductive limit topology in the rings of germs of holomorphic functions.

Notice that $\Gamma(F^X) \subset \Gamma(jB^X)$ which is the set of sections of jB defined on X . We shall see that these are sections of a certain sheaf on X whose fibers are certain quotients $\mathcal{O}_x^n(B)/I_x$ of the space $\mathcal{O}_x^n(B)$ of B -valued holomorphic germs, where I_x is the ideal in the definition of the object X . In this way, there is a natural “pointwise” or initial topology on $\Gamma(jB^X)$ derived from the topology on the fibers.

3.1 Definition.

- a) Let x be a point $x \in \mathbb{C}^n$, and let $B \subset C$ be an open subset of a complex banach space C . We denote $\mathcal{O}_x^n(B)$ the space of germs on x of holomorphic functions with values in B , that is, the limit of the inductive system $Hol(U, B)$ with U running on the filter of open neighborhoods of x in \mathbb{C}^n . Thus, $\mathcal{O}_x^n(\mathbb{C}) = \mathcal{O}_x^n$ is the ring of germs of holomorphic functions on n variables. We consider on each $Hol(U, B)$ the topology of uniform convergence on compact subsets, and on $\mathcal{O}_x^n(B)$ the locally convex inductive limit topology (see remark below). Given an open set U , $x \in U \subset \mathbb{C}^n$, and $g: U \rightarrow B$, we shall denote by $[g]_x$ the corresponding germ.
- b) Let $I_x \subset \mathcal{O}_x^n$ be an ideal in the ring of germs of holomorphic functions. We define the quotient $\mathcal{O}_x^n(B)/I_x$ by means of the following equivalence relation:
Given two germs defined in an open set U , $x \in U \subset \mathbb{C}^n$, $f, g: U \rightarrow B$, $f = g \text{ mod}(I_x)$ if for all continuous linear forms $\alpha \in C'$,

$$[\alpha \circ (f - g)]_x \in I_x.$$

That is, $f = g \text{ mod}(I_x)$ if for all $\alpha \in C'$ $\alpha \circ f = \alpha \circ g \text{ mod}(I_x)$.
 There is a quotient map $\mathcal{O}_x^n(B) \rightarrow \mathcal{O}_x^n(B)/I_x$, and we consider on $\mathcal{O}_x^n(B)/I_x$ the quotient topology.

3.2 Remark. *With the notations in the previous definition:*

- a) *The space $Hol(U, B) \subset Hol(U, C)$ is a subspace, and $Hol(U, C)$ is a complete locally convex topological vector space, which furthermore is a Frechet space [11, II, 9.14]. Also, the space $\mathcal{O}_x^n(B) \subset \mathcal{O}_x^n(C)$ is a subspace, and $\mathcal{O}_x^n(C)$ is a complete locally convex topological vector space, and its topology is the final topology in the sense of topological spaces [2, 5.2]. However, it is known that it is not metrizable and a fortiori not a Frechet space. Notice that in particular it is a Hausdorff topological space.*
- b) *It is known that all ideals $I_x \subset \mathcal{O}_x^n$ are closed ideals [3 pp. 194, 4, lemma 6]. From this it easily follows that the equivalence relation in b) in definition 3.1 is a closed equivalence relation. It is not difficult to check that the quotient map is open. It follows that the quotient space $\mathcal{O}_x^n(B)/I_x$ is Hausdorff.*

Next we state an important fact which follows from a lemma we proved in [9].

3.3 Proposition. *With the notations in definition 3.1, b), if $f = g \text{ mod}(I_x)$, then $f(x) = g(x) = p$, and for all germs $[t]p \in \mathcal{O}_{B,p}$, $[t \circ f - t \circ g]_x \in I_x$ (notice that the reverse implication obviously holds).*

Proof. Clearly, for all continuous linear forms $\alpha \in C'$, $\alpha \circ (f - g)(x) = 0$, that is $\alpha(f(x)) = \alpha(g(x))$. By the Hahn-Banach theorem it follows that $f(x) = g(x)$. The second part of the statement is lemma 2.6 in [9]. □

In [3] and [4] Cartan considers a notion of convergence of sequences in the ring \mathcal{O}_x^n which requires that the whole sequence “lift” to some $\mathcal{O}^n(W) = Hol(W, \mathbb{C})$, and converges uniformly over all compact subsets there. It is known that this characterizes the convergence of sequences in the inductive limit topology. This not only holds for \mathcal{O}_x^n , but also more generally for $\mathcal{O}_x^n(B)$, any B . For the interested reader we give now a proof of this fact.

3.4 Fact. *With the notations in definition 3.1, a sequence $f_k \in \mathcal{O}_x^n(B)$ converges to $f \in \mathcal{O}_x^n(B)$ in the inductive limit topology if and only if there is an open neighborhood W of x in \mathbb{C}^n such that f and f_k for all k have an holomorphic extension g , respectively g_k , defined in W , and g_k converges to g in $Hol(W, B)$ uniformly over all compact subsets of W .*

Proof. Let $\lambda_U: Hol(U, B) \rightarrow \mathcal{O}_x^n(B)$ be the inductive system diagram. Given a descending chain $\dots U_i \supset K_i \supset U_{i+1} \supset K_{i+1} \supset \dots \{x\}$, with the U_i a basis of open neighborhoods and the K_i compact, and an arbitrary sequence of real numbers $\varepsilon_i > 0$, it is easy to check that the set $H = \{[g]_x \mid \exists i g \in Hol(U_i, B) \text{ and } |g|_{K_i} < \varepsilon_i\}$ is open for the final topology defined by the λ_U , and thus open in $\mathcal{O}_x^n(B)$ (see remark 3.2). Now let $[g_k]_x$ be a sequence converging to 0, and suppose that for no index i , there is a tail of the sequence defined on U_i . It follows that for all $i \geq 1$, there exists $k_i > k_{i-1}$ and j_i such that $g_{k_i} \notin Hol(U_{j_{i-1}}, B)$ and j_i is the first index such that $g_{k_i} \in Hol(U_{j_i}, B)$ (where we have set $j_0 = 1, k_0 = 1$). Each integer s lies in an interval $j_i \leq s < j_{i+1}$. Let $0 < \varepsilon_s < \inf\{|g_{kt}|_{K_s}, t \leq i\}$. Then, for all t and all s , if $t \leq i$, $|g_{kt}|_{K_s} > \varepsilon_s$, and if $t > i$, $g_{kt} \notin Hol(U_s, B)$. It follows then that for the subsequence g_{kt} , $[g_{kt}]_x \notin H$ for all t , contradicting the fact that the sequence $[g_k]_x$ converges to 0. With the same ideas it can actually be proved that the lifted sequence converges uniformly on compact subsets (on a perhaps smaller open set). \square

For a general converging net it is not the case that it will lift into some W . Consider the double indexed family $f_{k,n} = (kx)_n + 1/k$ in \mathcal{O}_0^1 . Then $f_{k,n}$ converges (on n) to the constant function $1/k$ (for each k), and $1/k$ converges (on k) to the constant 0. Thus 0 is in the closure of the set $\{f_{k,n}\}$. Clearly, no net in this set converging to 0 can lift to a same $\mathcal{O}^1(W)$ for an open W , $0 \in W$. By the way, this example, (together with 3.4) shows that \mathcal{O}_0^1 can not be a Frechet space.

Actually, since $Hol(W, B)$ is a subspace of the Frechet space $Hol(W, C)$, a converging net in $\mathcal{O}_x^n(B)$ has a tail that lifts to the same $Hol(W, B)$, for some open W , if and only if it has a subsequence (that is a subnet which is a sequence).

We consider now a subobject in the topos, $F \subset jB$, where B is an open subset of a complex banach space, and an object $X = (X, \mathcal{O}_X)$ in the site \mathbf{H} , $X \subset \mathbb{C}^n$. Let $[X, F] = \Gamma(F^X)$ be the set of arrows $X \rightarrow F$ in

T . Remark that $[X, F] \subset \Gamma(jB^X) = [X, jB]$, which is the set of sections of jB defined on X . Thus, an element $s \in [X, F]$ is a pair $s = (s, \sigma)$, $s = \Gamma(s): X \rightarrow \Gamma(F) \subset B$ a continuous function, and for each $x \in X$, $\sigma_x: \mathcal{O}_{B,p} \rightarrow X_x = \mathcal{O}_x^n/I_x$ a morphism of A -local rings, where $p = s(x)$, and I is the sheaf of ideals defining X (see 0.1.1, 0.1.2 above). Since s has local extensions, for each x there is an open $U \subset \mathbb{C}^n$, $x \in U$, and an holomorphic extension $g: U \rightarrow B$ of s , such that $\sigma_x = \rho_x \circ g^*$, where $\rho_x: \mathcal{O}_x^n \rightarrow X_x$ is the quotient map (notice that the extension g does not take values in $\Gamma(F)$, but on the ambient space B).

3.5 Proposition. *For each $x \in X$, there is a map $\gamma_x: [X, F] \rightarrow \mathcal{O}_x^n(B)/I_x$, defined by $\gamma_x(s) = [g]_x \text{ mod}(I_x) = \rho_x([g]_x)$, where g is any local extension of s around x and ρ_x is the quotient map $\mathcal{O}_x^n(B) \rightarrow \mathcal{O}_x^n(B)/I_x$ in definition 3.1, b).*

Proof. We just have to check that this map is well defined, that is, it does not depend on the local extension chosen. Let f, g be any two local extensions. We have $\rho_x \circ f^* = \rho_x \circ g^*$ since both composites are equal to σ_x . This clearly gives $[f]_x = [g]_x \text{ mod}(I_x)$. □

3.6 Definition. *With the notations in 3.5. The CU topology on the set $[X, F]$ is the initial topology with respect to the family of maps*

$$\gamma_x: [X, F] \rightarrow \mathcal{O}_x^n(B)/I_x, \quad x \in X,$$

and $\Gamma: [X, F] \rightarrow C(X, \Gamma(F)) \subset C(X, B)$. Here $\mathcal{O}_x^n(B)/I_x$ is given the quotient topology of the inductive limit topology on $\mathcal{O}_x^n(B)$ (see 3.1, b), and $C(X, B)$ the topology of uniform convergence on compact subsets (see 3.1, a).

3.7 Remark. *The CU topology in the set $[X, F]$ is Hausssdorf.*

Proof. It follows immediately from remark 3.2, b) and the obvious fact that by 3.3 the family $\gamma_x: [X, F] \rightarrow \mathcal{O}_x^n(B)/I_x$, $x \in X$, is jointly injective. □

Clearly, the CU topology on $[X, F]$ is the subspace topology of the CU topology on $[X, jB]$, $[X, F] \subset [X, jB]$.

Example 1. Consider $\Delta_n \subset \mathbb{C}^n$, $\Delta_n = \Delta_{\mathbb{C}^n}(0) = \neg\neg\{0\}$. By 1.15. we have $\Delta_n = j\{0\}$. That is, Δ_n is representable by the analytic scheme

$(\{0\}, \mathcal{O}_n)$. We have then $\Gamma(\mathbb{C}^{\Delta n}) = [\Delta_n, \mathbb{C}] = \mathcal{O}_{n,0}$. the ring of germs of holomorphic functions on n variables. In this case the CU topology is the usual inductive limit topology considered in this ring, see [9 pp. 122]

Example 2. Let $U \subset \mathbb{C}^n$ be any open set, and consider $\Gamma(jB^U) = [U, jB] = Hol(U, B)$ the set of holomorphic functions $f: U \rightarrow B$. Then, the CU topology is the topology of uniform convergence on compact subsets.

Example 3. Consider $\Gamma(jB^{\Delta n}) = [\Delta_n, jB] = [(0, \mathcal{O}_n), (B, \mathcal{O}_B)] = \{\text{morphisms of } A\text{-ringed spaces with local extensions}\}$. By definition such a morphism is of the form $f^*: \mathcal{O}_{B,p} \rightarrow \mathcal{O}_{n,0}$, $p = f(0)$, where f is an holomorphic function $f: U \rightarrow B$ defined in an open neighborhood of 0 , $U \subset \mathbb{C}^n$. Given any other such function g , if $g^* = f^*$, it follows immediately from the Hahn-Banach theorem that $g = f$ on U . Thus, $[\Delta_n, jB] = \mathcal{O}_n(B)_0$, the space of germs of B valued holomorphic functions in n variables (see 3.1 a). As in the example 1, the CU topology here is the inductive limit topology of the inductive system $Hol(U, B)$ with U running on the filter of open neighborhoods of 0 in \mathbb{C}^n .

3.8 Proposition. *Let B be an open subset of a complex Banach space and $F \subset jB$ be any subobject. Let $E, X \in \mathbf{H}$ be any two objects in the site, $E \subset \mathbb{C}^m$, $X \subset \mathbb{C}^n$, and let $f: E \rightarrow F^X$ be an arrow in \mathbf{T} . Then, $\Gamma(f): E \rightarrow [X, F]$ is continuous for the CU topology.*

Proof. Let s_k be a net in E such that s_k converges to s in E . We have to prove that the arrows $f(s_k): X \rightarrow F$ converge to $f(s): X \rightarrow F$ in $[X, F]$ with the CU topology. To the arrow f corresponds an arrow $E \times X \rightarrow F \subset jB$ in the topos that we denote also f . We have $f(s_k) = f(s_k, -)$ and $f(s) = f(s, -)$. Let now $x \in X$ be a point in X . We have $(s, x) \in E \times X$, and there are open neighborhoods U of s in \mathbb{C}^m , W of x in \mathbb{C}^n and an holomorphic extension of f , $g: U \times W \rightarrow B$. Let k_0 be an index such that, for $k > k_0$, $s_k \in U$. It follows that $g(s, -), g(s_k, -): W \rightarrow B$ are extensions of $f(s, -), f(s_k, -)$ respectively. Since the topology of uniform convergence on compact subsets is an exponential topology on $Hol(W, B)$, $g(s_k, -)$ converge to $g(s, -)$ uniformly on compact subsets. This clearly implies that the corresponding germs converge in the inductive limit $\mathcal{O}_x^n(B)$. Thus, $\gamma_x(f(s_k, -)) = [g(s_k, -)]_x \text{ mod } (I_x) = \rho_x([g(s_k, -)]_x)$ (see 3.5), converge to $\gamma_x(f(s, -))$ in the quotient $\mathcal{O}_x^n(B)/I_x$. It remains to see that the functions $u_k = \Gamma(f(s_k))$ converge to the function $u = \Gamma(f(s))$ uniformly on

compact subsets in $C(X, B)$. Let $K \subset X$ be any compact subset. By the previous considerations, for each $x \in K$, there is a k_0 such that for $k > k_0$, u_k and u all have an extension to a comun open set W , $x \in W$. It follows that there is a finite open covering of K , A_1, A_2, \dots, A_m , $A_i \subset X$, such that the restrictions $u_k|_{A_i}$ converge uniformly on compact subsets to $u|_{A_i}$, for each i . Since X is a Hausssdorf topological space, the compact set K is of the form $K = K_1 \cup K_2 \cup \dots \cup K_m$, $K_i \subset A_i$, compact subsets of A_i . It follows that u_k converge uniformly to u on K . \square

3.9 Corollary. *With the notation in 3.5 and 3.6. the object F^X and the CU topology in the set $[X, F] = \Gamma(F^X)$ satisfy the basic assumption 1.0 in section 1.*

Proof. Immediate by propositions 3.7 and 3.8. \square

3.10 Definition. *With the notation in 3.5 and 3.6. The CU topological structure in F^X , denoted $CU[F^X]$, is the topological structure constructed in section 1 associated to the CU topology in $[X, F] = \Gamma(F^X)$. Thus, $CU(F^X) \approx O([X, F])$, the lattice of open sets for the CU topology on $[X, B]$.*

3.11 Proposition. *Let B be an open subset of a complex Banach space and $F \subset jB$ be any subobject. Let $E, X \in H$ be any two objects in the site, $E \subset \mathbb{C}^m$, $X \subset \mathbb{C}^n$, with respective sheaves of ideals J and I . Let $h = (h, \eta): E \rightarrow X$ be an arrow in H . Then, the induced arrow $F^h: F^X \rightarrow F^E$ is continuous for the CU topological structures.*

Proof. By proposition 1.12 (see also 3.10) it suffices to show that the function $[h, F]: [X, F] \rightarrow [E, F]$ is continuous for the CU topology. Furthermore, this follows from the continuity of

$$[h, B] = h^*: [X, B] \rightarrow [E, B].$$

We have to check that the composites $\Gamma \circ h^*: [X, B] \rightarrow [E, B] \rightarrow C(E, B)$, and for each $y \in E$, $\gamma_y \circ h^*: [X, B] \rightarrow [E, F] \rightarrow \mathcal{O}_y^n(B)/J_y$, are all continuous. Consider the following commutative diagrams:

$$\begin{array}{ccccccc}
 [X, B] & \xrightarrow{\Gamma} & C(X, B) & [X, B] & \xrightarrow{\gamma_x} & \mathcal{O}_x^n(B)/I_x & \longleftarrow \mathcal{O}_x^n(B) & \longleftarrow \text{Hol}(W, B) \\
 h^* \downarrow & & \Gamma(h)^* \downarrow & h^* \downarrow & & \eta \downarrow & g^* \downarrow & g^* \downarrow \\
 [E, B] & \xrightarrow{\Gamma} & C(E, B) & [E, B] & \xrightarrow{\gamma_y} & \mathcal{O}_y^m(B)/J_y & \longleftarrow \mathcal{O}_y^m(B) & \longleftarrow \text{Hol}(V, B)
 \end{array}$$

The diagram on the left is clear. For the diagram on the right, $x = h(y)$, g is an holomorphic extension of h defined in an open set of \mathbb{C}^m , W runs on the filter of open neighborhoods of x , and V is such that g is defined and $g(V) \subset W$. The vertical arrow η can be seen to be well defined since it can be proved that all extensions g induce the same arrow. This follows since all extensions g do define the same arrow $\eta: \mathcal{O}_x^n/I_x \rightarrow \mathcal{O}_y^m/J_y$ as they are extensions of the morphism $h = (h, \eta)$.

It is easy to check that $\Gamma(h)^*$ in the left diagram is continuous, so $\Gamma \circ h^*$ is continuous. In the same way the rightmost arrow g^* is continuous. From this it follows that the other g^* and subsequently also η are continuous. So $\gamma_y \circ h^*$ is continuous. \square

Due to the fact stated in 3.4, convergence of sequences in the CU topology is some times easily handled. This is the case when the ideal I is the zero ideal $I = \{0\}$, and thus $\mathcal{O}_x^n(B)/I_x = \mathcal{O}_x^n(B)$.

3.12 Proposition. *With the notations in 3.5 and 3.6. assume $X = (X, \mathcal{O}_X)$ be defined by a pair of coherent sheaves of ideals $I \subset J$, with $I = \{0\}$ (see 0.1.1 above). Then, a sequence s_k converges to s in $[X, F]$ in the CU topology if and only if for each $x \in X$ there is an open neighborhood W of x in \mathbb{C}^n such that s and s_k for all k have an holomorphic extension g , respectively g_k , defined in W with values in B , and g_k converges to g uniformly over all compact subsets of W .*

Proof. This is clear by 3.4. It only remains to prove that if s_k and s satisfy the condition, then $\Gamma(s_k)$ converge to $\Gamma(s)$ uniformly on compact subsets in $C(X, B)$. This is done exactly in the same way that in proposition 3.8. \square

References.

- [1] Bunge M., Dubuc E. J. *Local Concepts in Synthetic Differential Geometry and Germ Representability*, Lectures Notes in Pure and Applied Mathematics, Marcel Dekker, New York, (1989).
- [2] Carboni G., Larotonda A. *Some Results on Inductive Limits*, Preprint, Dept. of Math, F.C.E.Y.N., U.B.A. (1999). To appear in Revista de la Unión Matemática Argentina.
- [3] Cartan H. *Idéaux de Fonctions Analytiques de n variables complexes*, Annales de L' Ecole Normale, 3e serie, **61**, (1944).
- [4] Cartan H. *Idéaux et Modules de Fonctions Analytiques de Variables Complexes*, Bulletin de la Soc. Math. de France, t **78**, (1950).
- [5] Dubuc E. J., *Logical Opens and Real Numbers in Topoi*, Journal of Pure and Applied Algebra, North Holland, **43** (1986).
- [6] Dubuc E. J., Penon J., *Objects Compacts Dans les Topos*, J. Austral. Math Soc. (Series A) **40** (1986).
- [7] Dubuc E. J., Taubin G., *Analytic Rings*, Cahiers de Topologie et Geometrie Differentielle Categoriques, Vol **XXIV-3** (1983).
- [8] Dubuc E. J., Zilber J. C., *On Analytic Models of Synthetic Differential Geometry*, Cahiers de Topologie et Geometrie Differentielle Categoriques, Vol **XXXV-1** (1994).
- [9] Dubuc E. J., Zilber J. C., *Banach Spaces in an Analytic Model of Synthetic Differential Geometry*, Cahiers de Topologie et Geometrie Differentielle Categoriques, Vol **XXXIX-2** (1998).
- [10] Kaup L., Kaup B., *Holomorphic Functions of Several Variables*, Walter de Gruyter, Berlin, New York (1983).
- [11] Mujica J., *Holomorphic Functions and Domains of Holomorphy in Finite and Infinte Dimensions*, North Holland Mathematics Studies **120** (1986).
- [12] Penon J., *De L'infinitésimal au local*, Diagrammes, Paris VII (1985).

Eduardo J. Dubuc, Jorge C. Zilber

Departamento de Matemática.
 Facultad de Ciencias Exactas y Naturales
 Ciudad Universitaria
 1428 Buenos Aires, Argentina