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STONE SPACES OF MORE PARTIALLY ORDERED SETS

by Elias David

RESUME. Utilisant la définition d'un idéal d'un ensemble ordonné introduite dans Doctor, l'auteur montre que plus d'ensembles ordonnés sont représentables comme des bases compact-ouvert d'espaces de Stone que lorsqu'on utilise la définition de Frink comme il l'avait fait dans un article précédent. Il obtient aussi une équivalence duale entre la catégorie des ensembles ordonnés et une catégorie faisant intervenir des espaces de Stone, étendant l'équivalence duale de cet article précédent.

Preliminaries on Notation and Terminology

In [3] we worked with quasi-ordered sets and used a rationalised notation that reduces and often obviates the use of brackets. for example we wrote $Af^{\to ul}f^{\leftarrow}$, for the usual $f^{-1}[\{f(A)\}^{ul}]$. In this paper we work with the more familiar posets and the conventional notation in order to appeal to more readers.

Let $(Z,\leq)$ be a poset and $A \subseteq Z$. We write

$\downarrow A$ for $\{x \mid \exists\, a \in A$ such that $x \leq a\}$, the *lower end closure* of A ,

A^l for $\{x \mid \forall\, a \in A, x \leq a\}$ the *lower bounds* of A ,

and similarly A^u for the *upper bounds* of A.

A function $f: M \to Z$ is said to be an *(order-)-imbedding* if it has the property $a \leq b$ iff $fa \leq fb$. It is called *residuated residual* if the inverse image of each principal ideal/filter is also a principal ideal/resp filter, *cf.* [4]. With complete lattices a residuated map is just one that preserves all joins. A *join-dense* function is one whose image is join-dense in the codomain. We abbreviate "*join-semilattice with* 0" to "*semilattice*"

Introduction

A *Stone* space is a sober space with a compact-open base for the topology Such a space is determined up to homeomorphism by its semilattice of compact-open sets and this semilattice is ideal-distributive Conversely every ideal-distributive semilattice is isomorphic to the semilattice of compact-open sets of some Stone space and there is also a dual equivalence between categories here.

We have previously extended this result in [3] by using Frink ideals. A subset W of a poset is called a *Frink ideal* if for each finite $A \subseteq W$ we have $A^{ul} \subseteq W$.

Every poset P that is Frink ideal-distributive (i.e. whose lattice of Frink ideals is distributive) is representable as a collection $\boldsymbol{P}$ of compact-open sets of a sober space such that $\boldsymbol{P}$ is a base for the topology and also such that each compact-open set is of the form $(\cap_\alpha \boldsymbol{P}_\alpha)^\circ$ for some family $(P_\alpha)_\alpha \in \boldsymbol{P}$ and conversely every such collection $\boldsymbol{P}$ forms a poset that is Frink ideal-distributive.

In this paper we find that we can extend this topological representation to more posets by using ideals as defined by Doctor [5]. Posets P that are Doctor ideal-distributive are representable as collections $\boldsymbol{P}$ of compact-open sets with the more general properties that $\boldsymbol{P}$ is a base for the topology and that all finite joins in $\boldsymbol{P}$ occur as unions.

There follows, as in [3], a dual equivalence between categories.

Doctor Ideals and Ideal-continuity

Definition 1: Let $(Z, \leq)$ be a poset. We say that $W \subseteq Z$ is a *Doctor ideal* if for every finite $A \subseteq W$ such that $\bigvee A$ exists we have $A^{ul} \subseteq W$; or alternatively if W is a lower set and for every finite $A \subseteq W$ such that $\bigvee A$ exists, $\bigvee A \in W$.

These ideals, like the Frink ideals, form an algebraic closure system for Z. As such, the closure of a subset X is the union of the closures of all finite subsets of X ; if A is finite and $\bigvee A$ exists, then A^k, the closure of A, is A^{ul}. Recall also, e.g. from Crawley and Dilworth [1], that every algebraic closure system is an algebraic lattice and that every distributive algebraic lattice is a frame.

Every Frink ideal is a Doctor ideal and in the case of a semilattice the definitions coincide. They also coincide when the lattice of Frink ideals is distributive and consequently a frame.

Lemma 2: *Let* $(Z, \leq)$ *be a poset whose Frink ideals form a frame. Then the Frink ideals and the Doctor ideals coincide.*

Proof: (This simplified proof was supplied by the referee.) Let W be a Doctor ideal and ^{k} be the Frink ideal closure operator. We prove that $W^k = W$. Let $x \in W^k$. Then

$$\{x\}^l = \{x\}^k \leq W^k = \bigvee_{w \in W}\{w\}^l .$$

So

$$\{x\}^k = \bigvee_{w \in W}\{x\}^l \cap \{w\}^l = \bigvee_{w \in W}\{x,w\}^l = [\bigcup_{w \in W}\{x,w\}^l]^k = [\{x\}^l \cap W]^k$$

as $W = \downarrow W$. So $x \in Y^k$ for some finite $Y \subseteq \{x\}^l \cap W$. But $Y^k = Y^{ul}$ so $x = \bigvee Y$ and so $x \in W$.

We call a function (*Doctor*) *ideal-continuous* if the inverse image of each Doctor ideal is a Doctor ideal.

Theorem 3: *Let* $f\colon M \to Z$ *be a function between posets. Then the following conditions are equivalent :*

1) *f is (Doctor) ideal-continuous.*
2) *For every* $x \in Z$, $f^{-1}(x^l)$ *is a (Doctor) ideal.*
3) *For every finite* $A \subseteq M$ *such that* $\bigvee A$ *exists, we have* $f(A^{ul}) \subseteq (fA)^{ul}$.
4) *For every finite* $A \subseteq M$ *such that.* $\bigvee A$ *exists,* $f(\bigvee A) = \bigvee(fA)$.

Variations of this theorem occur in Doctor [5], Jürgen Schmidt [9], and Erné [6].

Proof: $1 \Rightarrow 2$: Clear.
$2 \Rightarrow 3$: Let A be as above, $b \in A^{ul}$ and $x \in (fA)^u$. We show that $fb \leq x$. Now $f(A) \subseteq \{x\}^k$ so $A \subseteq f^{-1}\{x\}^k$ which is a (Doctor) ideal. Hence $A^{ul} \subseteq f^{-1}\{x\}^k$, but $b \in A^{ul}$ so $fb \in \{x\}^k$.
$3 \Leftrightarrow 4$: Hints: a) Both 3 and 4 imply that f is order preserving. b) If $\bigvee A$ exists then $\downarrow \bigvee A = A^{ul}$.
$3 \Rightarrow 1$: Suppose $W \subseteq Z$ is an ideal and A is finite, $A \subseteq f^{-1}(W)$, and $\bigvee A$ exists. Then $fA \subseteq W$ and $\bigvee(fA)$ exists hence $(fA)^{ul} \subseteq W$. So $f(A^{ul}) \subseteq W$.

Thus every Frink ideal-continuous function is Doctor ideal-continuous.

Ideal-continuity and a Schmidt Reflection

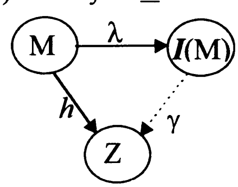
Let M be a poset and $\boldsymbol{I}$(M) be its (Doctor) ideal-lattice. We already know that $\boldsymbol{I}$(M) is an algebraic closure system that contains the principal ideals and that these are the closures of the singletons. The compact elements of $\boldsymbol{I}$(M) are the closures of the finite subsets of M. The principal ideal imbedding λ: M $\to$ $\boldsymbol{I}$(M) is a join-dense order-imbedding that takes elements of M to compact elements of $\boldsymbol{I}$(M). Let W$\in\boldsymbol{I}$(M). Then

$$\lambda^{-1}\{W\}^{l}) = W$$

and so, by Theorem 3, λ is ideal-continuous. It also has a universal mapping property as a corollary of Theorem 2 in Jürgen Schmidt [9].

Proposition 4: *For every complete lattice Z and every ideal-continuous h*: M $\to$ Z, *there exists a unique residuated map* γ: $\boldsymbol{I}$(M) $\to$ Z *to commute.*

We sketch the proof: Since h is ideal-continuous, for each $x\in$Z we have $h^{-1}(x^{l}) \in\boldsymbol{I}$(M). Define δ: Z $\to$ $\boldsymbol{I}$(M) as $\delta x = h^{-1}(x^{l})$. On the other hand we define γ: $\boldsymbol{I}$(M) $\to$ Z as γ(W) = $\bigvee(h$W) for W$\in\boldsymbol{I}$(M). Then $\gamma\circ\lambda = h$ and γ is residuated with δ as its corresponding residual map, *cf.* [4] and [9]. Further γ is unique to commute because λ is join-dense and it can also be defined as $\gamma(N^{k}) = \bigvee(hN)$ for any N $\subseteq$ M where N^{k} is the ideal-closure of N.



From the universal mapping property of λ, it follows that there is a reflector $\boldsymbol{I}$ from the category of posets and ideal-continuous maps to the subcategory of complete lattices and residuated maps, *cf.* allied results in Fleischer [7], and Erné [6].

Let us now call a poset *Doctor ideal-distributive* if its lattice of (Doctor) ideals is distributive.

Let M be (Doctor) ideal-distributive, Z a complete lattice, and let h: M $\to$ Z be ideal-continuous. If the commuting γ described above is an isomorphism then Z is an algebraic frame and h is an ideal-continuous imbedding that is join-dense and whose image is contained in the compact elements of Z. We also have a converse:

Proposition 5: *If* M *is a poset,* Z *is an algebraic frame and* h: M $\to$ Z *is a join-dense ideal-continuous imbedding whose image is contained in the compact elements of* Z*, then* γ *of Proposition* 4 *is an isomorphism and so* M *is ideal-distributive.*

Proof: Since h is join-dense, γ is onto. We now prove it an order-imbedding. Suppose J, K$\in\mathcal{I}$(M) and $\gamma J \leq \gamma K$. We prove J $\subseteq$ K. For each $a\in$J we have $ha \leq \bigvee h$K so $ha = \bigvee_{m\in K}(ha\wedge hm)$. For each $m\in$K there exists some $B_m \subseteq$ M such that $\bigvee(hB_m) = ha\wedge hm$ and so $ha = \bigvee h(\bigcup_{m\in K}B_m)$. Further, each $B_m \subseteq \{m\}^l \subseteq$ K as h is an imbedding. Now since ha is compact $ha = \bigvee h$B for some finite B $\subseteq \bigcup_{m\in K}B_m \subseteq$ K and again as h is an order-imbedding, $a = \bigvee$B. Hence $a\in$K.

The Axiom of Choice implies that every algebraic frame is isomorphic to the topology of a space with a compact-open base and consequently :

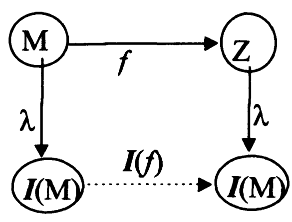
Theorem 6: a) *Every ideal-distributive poset* P *is isomorphic to a collection* ***P*** *of compact-open sets of an essentially unique sober space with the following properties:* 1) ***P*** *is a basis for the topology, and* 2) *finite joins in* ***P*** *occur as unions.*

b) *Conversely, every collection* ***P*** *of compact-open sets of a space with these two properties is an ideal-distributive poset and the inclusion* i: ***P*** $\to$ ***G****, where* ***G*** *is the topology, has the universal mapping property described in Proposition* 4.

We show a *Doctor* ideal-distributive poset that is not *Frink* ideal-distributive.

Example: Let P be the 3-element antichain $\{a,b,c\}$. Its Doctor ideal lattice is its power set which is certainly distributive. Hence P is representable as the singletons of a 3-element discrete space. But P is not even distributive as defined in [3] because $a \in \{b,c\}^{ul}$ and $c \in \{a,b\}^{lu}$ but a is not less than c. Its Frink-ideal lattice is the "*diamond*" D_5.

We now define the previously mentioned reflector ***I*** on morphisms. Let *f*: M → Z be ideal-continuous. Then ***I***(*f*): ***I***(M) → ***I***(Z) is the unique residuated map to commute in the diagram below:



Let N ⊆ M and ^{k} be the ideal-closure operator. Then $[I(f)](N^k) = (fN)^k$ and the corresponding residual map from ***I***(Z) to ***I***(M) is the restriction of the "inverse image" map f^{-1}.

Extension of the Stone Duality

Let M be an ideal-distributive poset. Then ***I***(M) is a frame. Let α be the *spec* functor from the category of frames to that of topological spaces. Then we can define the *(Doctor-)Stone space* of M to be α***I***(M). This space has as underlying set the (meet-)primes of ***I***(M), i.e. the *prime ideals* of M. If M is Frink ideal-distributive then we can also define the *(Frink-) Stone space* of M ; by Lemma 2, this is the same space.

We now determine the properties of functions *f*: M → Z between ideal-distributive posets that give rise to continuous functions between their Stone spaces. Let P_M, P_Z be their respective prime ideals. The continuous function θ: $P_Z \to P_M$ is to be defined as $\theta P = f^{-1}(P)$, *cf.* Johnstone [8], and so it is necessary (and also sufficient) that *f* has the property that the inverse image of every prime ideal of Z is a prime ideal of M. Since the ideal-lattices are distributive, every ideal is the intersection of prime ideals, *cf.* [1], and so this property implies that *f* is ideal-continuous. Now the map ***I***(*f*): ***I***(M) → ***I***(Z) is a frame map iff the inverse image of each prime ideal is a prime ideal because the (meet-)primes of the ideal-lattices are meet-dense (*cf.* [2] where it is done in terms of *coframe* maps between complete lattices). The function θ: $\boldsymbol{P}_Z \to \boldsymbol{P}_M$ is just $\alpha\boldsymbol{I}(f)$.

Definition 7: We call a function between ideal-distributive posets a *Doctor-Stone map* if it has the property that the inverse image of each prime ideal is a prime ideal.

Proposition 8: *Let f: M $\to$ Z be an ideal-continuous function between ideal-distributive posets. Then the following conditions are equivalent :*
1) *f is a (Doctor-)Stone map.*
2) *$\boldsymbol{I}(f)$: $\boldsymbol{I}$(M) $\to$ $\boldsymbol{I}$(Z) is a frame map.*
3) *For each finite* $A \subseteq M$ *and each* $B \subseteq M$, *if* $A^l \subseteq B^k$ *then* $(fA)^l \subseteq (fB)^k$, *where* ^{k} *is the ideal-closure operator.*
4) *For each finite* $A \subseteq M$, $(fA)^l \subseteq [f(A^l)]^k$ *(the converse inclusion always holds).*

Proof: We already have $1 \Leftrightarrow 2$.
$3 \Rightarrow 4$: Take $B = A^l$.
$4 \Rightarrow 3$: $A^l \subseteq B^k$ so

$$(fA)^l \subseteq [f(A^l)]^k \subseteq [f(B^k)]^k = (fB)^k$$

as f is ideal-continuous.

$2 \Rightarrow 4$: Since $\boldsymbol{I}(f)$ preserves finite meets, $[\boldsymbol{I}(f)](\wedge\lambda A) = \wedge(\boldsymbol{I}(f)\circ\lambda)A$. Now

$$[\boldsymbol{I}(f)](\wedge\lambda A) = [f(\wedge\lambda A)]^k = [f(A^l)]^k$$

whereas $\boldsymbol{I}(f)\circ\lambda = \lambda\circ f$ so

$$\wedge[\boldsymbol{I}(f)\circ\lambda]A = \wedge(\lambda\circ f)A = (fA)^l.$$

$4 \Rightarrow 1$: Let $P \in \boldsymbol{P}_Z$. We show that $f^{-1}(P)$ is a meet-prime of $\boldsymbol{I}$(M), i.e. that its complement is downwards directed. Let A be finite and $A \subseteq [f^{-1}(P)]^c$. We show that $A^l \not\subseteq f^{-1}(P)$. Now $(fA)^l \subseteq [f(A^l)]^k$ so if $A^l \subseteq f^{-1}(P)$ then we have $f(A^l) \subseteq P$, but fA is finite and $fA \subseteq P^c$ and P is a meet-prime of $\boldsymbol{I}$(Z).

The obvious categories for a Stone duality would be on the one hand Stone spaces with functions with the property that the inverse images of compact-open sets are compact-open and on the other hand ideal-distributive semilattices with Stone maps. (With semilattices there is no dispute as to what an ideal really is!)

We now present an extension of this Stone duality:

Theorem 9: *The following categories are dually equivalent:*
Category 1: *(Doctor) ideal-distributive posets with (Doctor-)Stone maps.*
Category 2: ***Objects**: Sober spaces* (Z,***P***) *where* Z *is the underlying set and **P** is a collection of compact-open sets with the properties :* 1) ***P** is a base for the topology, and* 2) *all finite joins in **P** occur as unions.* ***Morphisms**: Functions* $f: Z_1 \to Z_2$ *such that for each* $P \in \boldsymbol{P}_2$ *we have* $f^{-1}(P) \in \boldsymbol{P}_1$.

The corresponding pair of categories that we obtain in [3] are full subcategories of the above.

NOTE: This paper was typed on a PC only thanks to the instruction and assistance of Bernard Kestelman, son of the late Prof. H. Kestelman of University College, London.

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