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# BANACH SPACES IN AN ANALYTIC MODEL OF SYNTHETIC DIFFERENTIAL GEOMETRY

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**Résumé.** Les modèles bien adaptés de Géométrie Différentielle Synthétique [5] se construisent comme des topos de faisceaux sur des sites où tous les objets sont de dimension finie. Pourtant, quand on applique à des objets représentables des constructions importantes dans le topos, comme les exponentiels et les objets des parties, on obtient des objets "de dimension infinie". Pour étudier ces objets il faut considérer le calcul différentiel dans les Espaces de Banach, comme l'a mis en evidence Douady dans sa thèse [4]. Dans cet article nous construisons une immersion de la catégorie des ensembles ouverts des Espaces de Banach complexes et des fonctions holomorphiques dans le modèle analytique bien adapté de GDS introduit dans [7], et nous prouvons ainsi quelques propriétés importantes de cette immersion. En particulier, qu'elle préserve les produits et qu'elle est consistante avec le calcul différentiel, dans le sens que le calcul différentiel intrinsèque synthétique dans le topos correspond aux constructions classiques de la théorie de I'holomorphie de dimension infinie. Ce faisant nous avons trouvé et résolu quelques problèmes intéressants dans la théorie des idéaux des fonctions analytiques de (plusieurs) variables complexes développée par Cartan dans [2] et [3].

## Introduction.

The well adapted models of Synthetic Differential Geometry [5] are constructed as Topos of sheaves on sites where all the objects are finite dimensional. However, important construction in the topos, like exponentials and objects of parts, when applied to representable objects, yield objects of "infinite dimensional" nature. To study these objects it becomes necessary to consider differential calculus on Banach Spaces, as it was demonstrated by Douady in his Thesis [4]. In this paper we construct an embedding of the category of open sets of complex Banach Spaces and holomorphic maps into the analytic (well adapted) model of SDG introduced in [7], and prove two important properties. Namely, that this embedding preserves products and that it is consistent with the differential calculus, in the sense that the intrinsic synthetic differential calculus in the topos corresponds to the classical constructions of the theory of infinite dimensional holomorphy. In doing so we found and solved some interesting problems in the theory of ideals of analytic functions of (several) complex variables developed by Cartan in [2] and [3].

## 1. Construction of the embedding.

We shall work with certain c-algebras introduced in [6] and which carry an additional structure which is taken care in the following definition: An Analytic Ring $A$ in a category $\boldsymbol{E}$ is a transversal product and final object preserving functor $A: \mathcal{C} \to \boldsymbol{E}$, from the category $\mathcal{C}$ of all open sets of some $\mathbb{C}^n$ and holomorphic functions. It has an underlying $c$-algebra that by abuse we also denote $A = A(\mathbb{C})$. The Reader can think an analytic ring just as this $c$-algebra. However, for details see [6].

Consider any open set $B$ of a banach space $C$, then the ring $\mathcal{O}(B)$ of complex valued holomorphic functions is an analytic ring, and given any point $p \in C$, the ring $\mathcal{O}_p(B) = \mathcal{O}_p(C)$ of germs at $p$ of holomorphic functions is a local analytic ring. More precisely:

**1.1 Proposition.** *The functor* $\mathrm{Holo}(B, -): \mathcal{C} \to$ ***Ens*** *(*Holo = *Holomorphic) preserves all limits, thus in particular it is an analytic ring (in sets),* $\mathcal{O}(B) = \mathrm{Holo}(B, \mathbb{C})$*. The ring of germs is the filtered colimit (as analytic rings)* $\mathcal{O}_p(C) = \mathrm{colim}_{p \in B} \mathcal{O}(B)$*, and evaluation at* $p$ *defines a local morphism* $\mathcal{O}_p(C) \to \mathbb{C}$ *into the ring of complex numbers.*

**Proof.** It follows from basic results on holomorphic maps between banach spaces and on analytic rings ([8] and [6]).

□

We have a warning here: Contrary to the finite dimensional case, a morphism of analytic rings $\mathcal{O}(B) \to \mathcal{O}(H)$ is not necessarily given by an holomorphic map $H \to B$, not even a morphism $\mathcal{O}(B) \to \mathbb{C}$ is always given by evaluation at a point $p \in B$. Also, the canonical morphism $\mathcal{O}(B) \otimes \mathcal{O}(H) \to \mathcal{O}(B \times H)$ is not an isomorphism (where $\otimes$ here indicates the coproduct of analytic rings).

We shall consider now analytic rings in the topos $Sh(X)$ of sheaves on a topological space $X$. Recall that the sheaf $C_X$ of germs of continuos complex valued functions is a local analytic ring in $\boldsymbol{Sh}(X)$. Explicitly, $C_X$ is the functor $\mathcal{C} \to \boldsymbol{Sh}(B)$ defined by $\Gamma(H, C_X(U)) = Continous(H, U)$, for $H$ open in $X$, $U \in \mathcal{C}$. Recall also that an $A$-ringed space is (by definition) a pair $(X, \mathcal{O}_X)$, where $\mathcal{O}_X$ is a local analytic ring in $\boldsymbol{Sh}(X)$. That is, it is an analytic ring furnished with a (unique) local morphism $\mathcal{O}_X \to C_X$ of analytic rings in $\boldsymbol{Sh}(X)$ ([6], [9]). Given any point $p \in X$, the fiber is a local analytic ring $\pi: \mathcal{O}_{X,p} \to C_{X,p} \to \mathbb{C}$. If $\sigma$ is a section defined in (a neighborhood of) $p$, we shall denote by $[\sigma]_p$ the corresponding element in the ring $\mathcal{O}_{X,p}$, and by $\sigma(p)$ its value, that is, the complex number $\sigma(p) = \pi([\sigma]_p)$.

Given any open set $B$ of a banach space $C$, the pair $(B, \mathcal{O}_B)$, where $\mathcal{O}_B$ is the sheaf of germs of complex valued holomorphic functions defined on $B$, is a (reduced) $A$-ringed space. More explicitly, $\mathcal{O}_B$ is the functor $\mathcal{C} \to \boldsymbol{Sh}(B)$ defined by $\Gamma(H, \mathcal{O}_B(U)) = Holo(H, U)$, for $H$ open in $B$, $U \in \mathcal{C}$. In particular, $\Gamma(H, \mathcal{O}_B) = Holo(H, \mathbb{C})$, and if $p \in H$, the fiber of $\mathcal{O}_B$ is the local analytic ring $\mathcal{O}_p(B) = \mathcal{O}_p(C)$. We have:

**1.2 Proposition.** *The correspondence $B \mapsto (B, \mathcal{O}_B)$ defines a full embedding $\mathcal{B} \to \mathbf{A}$ from the category $\mathcal{B}$ of open sets of Banach spaces and holomorphic functions into the category $\mathbf{A}$ of $A$-ringed spaces.*

**Proof.** More precisely, to an holomorphic function $f: B \to H \subset C$ we assign the morphism $(f, f^*): (B, \mathcal{O}_B) \to (H, \mathcal{O}_H)$. This clearly defines a faithful functor. Moreover, given any morphism of $A$-ringed spaces, $(f, \phi): (B, \mathcal{O}_B) \to (H, \mathcal{O}_H)$, then $f: B \to H$ is holomorphic and $\phi = f^*$. This last assertion follows (essentially) since for all continuos linear forms $\alpha: C \to \mathbb{C}$ the composite $\alpha \circ f$ is holomorphic. In fact, let $z \in B$, let $V$ be an open subset of $H$ such that $f(z) \in V$, and let $t$ be an holomorphic function, $t: V \to \mathbb{C}$. Since $f$ is continuous, then there is an open subset $W \subset B$ such that $z \in W$ and $f(W) \subset V$. Consider the section of $\mathcal{O}_H$ (defined in $V$) given by $t$. Since $(f, \phi)$ is a morphism of $A$-ringed spaces, then $w \mapsto \phi_w([t]_{f(w)})$ is a section of $\mathcal{O}_B$ defined in $W$. Hence, there is an open subset $U \subset W$ such that $z \in U$ and an holomorphic function $g: U \to \mathbb{C}$ such that $g_w = \phi_w([t]_{f(w)})$ for all $w \in U$ (1). Since $\phi$ preserves the value of sections, then $g(w) = t(f(w))$. Thus, $g = t \circ f$ on $U$. Then, by (1), it follows that $[t \circ f]_w = \phi_w([t]_{f(w)})$ for all $w \in U$. Then, we have that $t \circ f$ is holomorphic and $[t \circ f]_z = \phi_z([t]_{f(z)})$. Thus, we have that $f$ is a continuous function such that for all $z \in B$, for all open subset $V$ of $H$ such that $f(z) \in V$, and for all holomorphic function $t: V \to \mathbb{C}$, there is an open subset $U \in B$ such that $z \in U$, $t \circ f: U \to .C$ is holomorphic and $\phi_z([t]_{f(z)}) = [t \circ f]_z$ (2). It follows [8] that $f$ is holomorphic, and by (2), $\phi = f^*$.

$\square$

Related to our previous warning, we point out that contrary to the finite dimensional case, the embedding $\mathcal{B} \to \boldsymbol{A}$ does not preserve products. That is, the canonical morphism $(B \times H, \mathcal{O}_{B\times H}) \to (B, \mathcal{O}_B) \times (H, \mathcal{O}_H)$ is not an isomorphism.

We shall construct now an embedding from $\mathcal{B}$ into the topos $\boldsymbol{T}$ introduced in [7] which does preserve products and has other good properties necessary to allow to perform the classical differential calculus of $\boldsymbol{B}$ by the methods of Synthetic Differential Geometry in the topos $\boldsymbol{T}$.

Recall the construction of $\boldsymbol{T}$. We consider the category $\mathcal{H}$ of (affine) analytic schemes [7]. An object $E$ in $\mathcal{H}$ is an $A$-ringed space $E = (E, \mathcal{O}_E)$ (by abuse we denote also by the letter $E$ the underlying topological space of the $A$-ringed space) which is given by two coherent sheaves of ideals $R$, $S$ in $\mathcal{O}_D$, where $D$ is an open subset of $\mathbb{C}^m$, $R \subset S$, and where:

$$E = \{p \in D \mid h(p) = 0 \quad \forall\, [h]_p \in S_p\},$$
$$\mathcal{O}_E = (\mathcal{O}_D/R)|_E \quad (\text{ restriction of } \mathcal{O}_D/R \text{ to } E).$$

The arrows in $\mathcal{H}$ are the morphism of $A$-ringed spaces. We will denote by $\boldsymbol{T}$ the topos of sheaves on $\mathcal{H}$ for the (sub canonical) Grothendieck topology given by the open coverings. There is a full (Yoneda) embedding $\mathcal{H} \to \boldsymbol{T}$. Notice that for an infinite dimensional banach open $B$, the $A$-ringed space $(B, \mathcal{O}_B)$ is not in $\mathcal{H}$.

**1.3 Definition.** *Let $E$ be an object in $\mathcal{H}$, $E = (E, \mathcal{O}_E)$ as above, let $B$ be an open subset of a complex Banach space $C$, and let $t = (t, \tau)$ be a morphism of $A$-ringed spaces, $t: (E, \mathcal{O}_E) \to (B, \mathcal{O}_B)$ (we adopt the corresponding abuse of notation for morphisms). We will say that t has* "local extensions", *if for each* $x \in E$, *there is an open neighborhood* $U$ *of* $x$ *in* $\mathbb{C}^m$ *and an extension* $(f, f^*): (U, \mathcal{O}_U) \to (B, \mathcal{O}_B)$*:*

$$\begin{tikzcd} (E \cap U, \mathcal{O}_{E \cap U}) \arrow[rr] \arrow[dr, "(t,\tau)"'] & & (U, \mathcal{O}_U) \arrow[dl, "(f,f^*)"] \\ & (B, \mathcal{O}_B) & \end{tikzcd}$$

The set

$$jB(E) = \{t: (E, \mathcal{O}_E) \to (B, \mathcal{O}_B) \text{ such that } t \text{ has local extensions}\}$$

defines a sheaf $jB \in \boldsymbol{T}$.

If $g: F \to E$ is an arrow in $\mathcal{H}$, $jB(g): jB(E) \to jB(F)$ is given by composing with $g$, that is, for $t \in jB(E)$:

$$jB(g)(t) = t \circ g: F = (F, \mathcal{O}_F) \to E = (E, \mathcal{O}_E) \to (B, \mathcal{O}_B),$$

The fact that if $t$ has local extensions and $g$ is a morphism of $A$-ringed spaces, then the composite $t \circ g$ also has local extensions, follows easily by continuity. It is also straightforward to check that $jB$ is a sheaf on $\mathcal{H}$, that is $jB \in \boldsymbol{T}$.

Moreover, given two open subsets $B_1$ and $B_2$ of complex Banach spaces $C_1$ and $C_2$ respectively, and an holomorphic function $f: B_1 \to B_2$, we consider the morphism $(f, f^*)$: $(B_1, \mathcal{O}_{B_1}) \to (B_2, \mathcal{O}_{B_2})$. It is clear that if $E \in \mathcal{H}$ and $t \in jB_1(E)$, then $(f, f^*) \circ t \in jB_2(E)$. Thus, we have an arrow in $\boldsymbol{T}$:

$$jf: jB_1 \to jB_2, (jf)_E(t) = (f, f^*) \circ t \text{ for all } E \in \mathcal{H} \text{ and } t \in jB_1(E).$$

We have then a functor $j: \mathcal{B} \to \boldsymbol{T}$. It is clear that $\Gamma(jB) = B$ for all $B \in \mathcal{B}$, and that $\Gamma(jf) = f$ for all arrows $f: B_1 \to B_2$ in $\mathcal{B}$.

**1.4 Remark.** *Let* $E \in \mathcal{H}$, $B \in \mathcal{B}$ *, and let* $q: E \to jB$ *be an arrow in* $\boldsymbol{T}$. *Then,* $q$ *corresponds to an element* $q \in jB(E)$*, that is,* $q: (E, \mathcal{O}_E) \to (B, \mathcal{O}_B)$ *is a morphism of* $A$*-ringed spaces with local extensions. Then,* $q = (q, \phi)$ *where* $q: E \to B$ *is continuous, and it is immediate that* $\Gamma(q) = q$ *(notice here the abuse of language).*

## 2. On ideals of analytic functions of complex variables.

Let $\mathcal{O}_{n,p}$ be the ring of germs of holomorphic (= analytic) functions on $n$ complex variables, $p \in \mathbb{C}^n$. Given a function $f$ defined in a neighborhood of $p$, we denote its germ by $[f]_p$. Recall that all ideals in this ring are finitely generated. More than that:

**2.1 Fact.** *Given any ideal $I_p \subset \mathcal{O}_{n,p}$, there is an open set $p \in U \subset \mathbb{C}^n$ and a (finitely generated) ideal $I(U) \subset \mathcal{O}(U)$ which generate $I_p$.*

**Proof.** It follows immediately since $I_p$ is finitely generated ([2] pp. 191).

□

Consider in $\mathcal{O}_{n,p}$ the inductive limit topology for the topology of uniform convergence on compact subsets on the rings $\mathcal{O}_n(U)$, $p \in U \subset \mathbb{C}^n$. It can be proved that in this topology a sequence $[f_k]_p$ converges to a limit $[f]_p$, if there is a neighborhood where (for sufficiently large $k$) all $f_k$ and $f$ are defined and the convergence is uniform. We shall refer to this topology as "the topology of uniform convergence". In what it follows a result of Cartan is essential ([2] pp. 194, or [3] 28. Lemma 6):

**2.2 Lemma (Cartan).** *All ideals of the ring $\mathcal{O}_{n,p}$ are closed for the topology of uniform convergence.*

We shall consider now a property of ideals first utilized explicitly in [1] in the context of $C^\infty$ functions of real variables. Let $p \in U \subset \mathbb{C}^n$, $q \in V \subset \mathbb{C}^m$. Let $I_p$, $J_q$ be ideals in $\mathcal{O}_{n,p}$, $\mathcal{O}_{m,q}$ respectively. We denote $(I_p, J_q)$ the ideal in $\mathcal{O}_{n+m,(p,q)}$ generated by the germs in $I_p$ and $J_q$ (considered as functions of $n+m$ variables). Similarly for ideals $I$, $J$ in $\mathcal{O}_n(V)$, $\mathcal{O}_m(U)$ respectively, there is the ideal $(I, J)$ in the ring $\mathcal{O}_{n+m}(U \times V)$.

**2.3 Definition.** *An ideal $I_p$ in $\mathcal{O}_{n,p}$ is said to have line determined extensions if for any $m$, and any germ $[f]_{(p,q)} \in \mathcal{O}_{n+m,(p,q)}$, the implication a) $\Rightarrow$ b) below holds:*

a) *There are open sets $p \in U \subset \mathbb{C}^n$, $q \in V \subset \mathbb{C}^m$ and $f \in \mathcal{O}_{n+m}(U \times V)$ (which defines the germ) such that for all (fixed) $s \in V$, the germ $[f(-, s)]_p \in I_p \subset \mathcal{O}_{n,p}$.*

b) $[f]_{(p,q)} \in (Ip, 0) \subset \mathcal{O}_{n+m,(p,q)}$ *(where* $0$ *indicates the zero ideal).*

It is easy to see by induction that this will hold provided it holds for $m = 1$.

**2.4 Lemma.** *All ideals of the ring* $\mathcal{O}_{n,p}$ *have line determined extensions.*

For the proof we isolate a fact that has its own interest:

**Lemma A.** *Given any germ* $[h]_{(p,0)} \in \mathcal{O}_{n+1,(p,0)}$, *the implication a) $\Rightarrow$ b) below holds:*

a) *There are open sets* $p \in U \subset \mathbb{C}^n$, $0 \in V \subset \mathbb{C}$ *and* $h \in \mathcal{O}_{n+1}(U \times V)$ *(which defines the germ) such that for all (fixed)* $s \in V$, *the germ* $[s]_p[h(-, s)]_p \in I_p \subset \mathcal{O}_{n,p}$.

b) $[h(-, 0)]_p \in I_p$.

This follows from Lemma 2.2: Take a sequence of complex numbers $0 \neq s_k \mapsto 0$. It is easy to check that the sequence of functions $h_k = h(-, s_k)$ converges uniformly (in a sufficiently small neighborhood) to the function $h(-, 0)$. Since clearly each germ $[h_k]_p$ is in $I_p$, it follows that $[h(-, 0)]_p \in I_p$.

**Proof (of 2.4).** As observed above, it suffices to prove the case $m = 1$. Clearly we can also assume that $q = 0$. We work with the situation in definition 2.3, we have to prove that $[f]_{(p,0)} \in (I_p, 0)$. We indicate by $x$ a variable $x \in U \subset \mathbb{C}^n$ , and by $z$ a variable $z \in V \subset \mathbb{C}$ . Taking $U$, $V$ sufficiently small, write

$$f(x, z) = f_0(x) + zh(x, z).$$

We have $[f_0]_p = [f(x, 0)]_p \in I_p$. Then, for all fixed $s$, $[s]_p[h(-, s)]_p \in I_p$. Thus by lemma A the germ $[h(-, 0)]_p \in I_p$. Now write

$$\begin{aligned} f(x, z) &= f_0(x) + zf_1(x) + z^2g(x, z). \\ \text{Clearly} \quad zh(x, z) &= zf_1(x) + z^2g(x, z). \\ \text{Thus} \quad h(x, z) &= f_1(x) + zg(x, z). \end{aligned}$$

As before, we have $[f_1]_p = [h(x, 0)]_p \in I_p$, and for all fixed $s$, $[s]_p[g(-, s)]_p \in I_p$. Like this, by induction, it follows that in the development below all the $[f_i]p$ are in $I_p$.

$$f(x,z) = f_0(x) + zf_1(x) + z^2 f_2(x) + \cdots + z^k f_k(x) + z^{k+1} h_k(x,z),$$

$$[f](p,0) = [f_0]p + [z]_0[f1]_p + [z^2]_0[f_2]_p + \cdots$$
$$\cdots + [z^k]_0[f_k]_p + [z^{k+1}]_0[h_k]_{(p,0)}.$$

(notice the abuse of notation)

Define $[g_k]_{(p,0)} = [f]_{(p,0)} - [z^{k+1}]_0[h_k]_{(p,0)}$. Clearly $[g_k]_{(p,0)} \in (I_p, 0)$ for all $k$, and since the sequence converges in $\mathcal{O}_{n+1,(p,0)}$ to $[f]_{(p,0)}$, the result follows (this time again) by lemma 2.2.

□

**2.5 Corollary.** *Let $U$ be an open subset of $\mathbb{C}^n$ , let $p \in U$, let $J_p$ be an ideal, $J_p \subset \mathcal{O}_{n,p}$ and let $f$ be an holomorphic function, $f: U \times U \to \mathbb{C}$ such that, for all $s \in U$, $[f(-,s)]p \in J_p$*

*Then, $[f(w,w)]_p \in J_p$ (Here, $f(w,w)$ is the function $U \to \mathbb{C}$ which sends $w \in U$ to $f(w,w)$ )*

**Proof**. We have that $[f]_{(p,p)} \in (J_p, 0) \subset \mathcal{O}_{n+n,(p,p)}$ (where 0 indicates the zero ideal). It follows immediately that $[f(w,w)]_p \in J_p$.

□

**Remark:** *Notice that the same implication holds with the symmetric assumption $[f(s,-)]_p \in J_p$ (Use the function $g: U \times U \to \mathbb{C}$ , $g(w,z) = f(z,w)$ ).*

We pass now to our next result:

**2.6 Lemma.** *Let $B$ be an open subset of a complex Banach space $C$. Let $U$ be an open subset of $\mathbb{C}^n$, let $q \in U$, and let $J_q \subset \mathcal{O}_{n,q}$ be an ideal. Let $f$ and $g$ be holomorphic functions, $U \to B$, such that $f(q) = g(q) = p \in B$. Suppose that for all linear continuos forms $\alpha \in C'$, it holds that $[\alpha \circ f - \alpha \circ g]_q \in J_q$. Then, for all germs $[r]_p \in \mathcal{O}_{B,p}$, it also holds $[r \circ f - r \circ g]_q \in J_q$.*

**Proof.** Let $u = f - p$ and $v = g - p$, ($u$ and $v$ are functions $U \to C$). Given $\alpha \in C'$, by linearity it follows that $\alpha \circ u - \alpha \circ v = \alpha \circ f - \alpha \circ g$, thus also $[\alpha \circ u - \alpha \circ v]_q \in J_q$

Let $b$ be any bilinear symmetric continuos mapping $b: C \times C \to \mathbb{C}$. For each $s \in U$, consider the linear map $\alpha = b(-, u(s))$. Since

$$(\alpha \circ u - \alpha \circ v)(x) = \alpha(u(x)) - \alpha(v(x)) = b(u(x), u(s)) - b(v(x), u(s)),$$

we have $[h(-, s)]_q \in J_q$, where $h$ indicates the holomorphic function $h: U \times U \to \mathbb{C}$, $h(x, z) = b(u(x), u(z)) - b(v(x), u(z))$. Then, by corollary 2.5, $[h(x, x)]_q \in J_q$. That is, $[b(u(x), u(x)) - b(v(x), u(x))]_q \in J_q$. Similarly $[b(v(x), u(x)) - b(v(x), v(x))]_q \in J_q$. It follows that

$$[b(u(x), u(x)) - b(v(x), v(x))]_q \in J_q.$$

Proceeding inductively, for all multilinear symmetric continuous mappings $b: C \times \cdots \times C \to \mathbb{C}$ , we have:

$$[b(u(x), \ldots, u(x)) - b(v(x), \ldots, v(x))]_q \in J_q \tag{1}$$

Let $[r]_p \in \mathcal{O}_{B,p}$. Then, by definition, there exists a ball $B(p, \delta)$ and a sequence of continuous homogeneous polynomials $P_k$ such that $r(c) = \sum_{k \geq 0} P_k(c - p)$ uniformly on $B(p, \delta)$. Let $W$ be an open subset of $U$ such that $q \in W$ and $f(W) \subset B(p, \delta)$, $g(W) \subset B(p, \delta)$. It follows that

$$r(f(x)) = \sum_{k \geq 0} P_k(f(x) - p) = \sum_{k \geq 0} P_k(u(x))$$

and $$r(g(x)) = \sum_{k \geq 0} P_k(g(x) - p) = \sum_{k \geq 0} P_k(v(x))$$

uniformly on $W$. Since $P_0$ is the constant polynomial $r(p)$, we have

$$r(f(x)) - r(g(x)) = \sum_{k \geq 1} (P_k(u(x)) - P_k(v(x)))$$

uniformly on $W$. But $P_k(c) = b_k(c, c, \ldots, c)$ for a multilinear symmetric continuous mapping $b_k$, thus by (1) above $[P_k(u(x)) - P_k(v(x))]_q \in J_q$. Then, by lemma 2.2, it follows that $[r(f(x)) - r(g(x))]_q \in J_q$, that is, $[rof - rog]_q \in J_q$.

$\square$

When $B$ is of finite dimension, $B \subset \mathbb{C}^m$, functions $f$ and $g$ as above are just $m$-tuples of elements in $\mathcal{O}(U)$, and the result means that for the ring $\mathcal{O}_{n,q}$, the $c$-algebra congruence defined by an ideal $J_q$ is actually a congruence for the theory of analytic rings. This is established in (theorem 1.18 [6]), where the result is proved for any local analytic ring. Here, the coordinates in $C = \mathbb{C}^m$ have to be replaced by all continuos forms.

## 3. Preservation of products.

We want to show now that the functor $j: \mathcal{B} \to \boldsymbol{T}$ does preserve the product of two objects in $\mathcal{B}$. The following proposition essentially do this. Namely, it shows that the natural candidate for the morphism into $j(B_1 \times B_2)$ which proves that this object is a product is actually well defined.

**3.1 Proposition.** *Let $B_1$ and $B_2$ be open subsets of complex Banach spaces, let $E$ be an object in $\mathcal{H}$, and let $U$ be an open subset of $\mathbb{C}^n$ such that $E \subset U$.*

*Let $g_1: U \to B_1$, $h_1: U \to B_1$, $g_2: U \to B_2$, $h_2: U \to B_2$ be holomorphic functions and consider the holomorphic functions $g: U \to B_1 \times B_2$ and $h: U \to B_1 \times B_2$ given by $g(z) = (g_1(z), g_2(z))$ and $h(z) = (h_1(z), h_2(z))$.*

*If $(g_1, g_1^*)|_E = (h_1, h_1^*)|_E$ and $(g_2, g_2^*)|_E = (h_2, h_2^*)|_E$,*
*then $(g, g^*)|_E = (h, h^*)|_E$*

**Proof.** $E$ is given by two coherent sheaves of ideals $J$, $S$ in $\mathcal{O}_H$, where $H$ is an open subset of $\mathbb{C}^n$, $J \subset S$. $E$ is the Zero set of $S$, and $\mathcal{O}_{E,x} = \mathcal{O}_{n,x}/J_x$ for $x \in E$.

Let $x \in E$. Since $(g_1, g_1^*)|_E = (h_1, h_1^*)|_E$ and $(g_2, g_2^*)|_E = (h_2, h_2^*)|_E$, we have that $g_1(x) = h_1(x) = p$, $g_2(x) = h_2(x) = q$. Thus $g(x) = h(x) = (p, q)$ (1). Moreover, for all $[t]_p \in \mathcal{O}_{B_1,p}$, $[(t \circ g_1) - (t \circ h_1)]_x \in J_x$ and for all $[t]_q \in \mathcal{O}_{B_2,q}$, $[(t \circ g_2) - (t \circ h_2)]_x \in J_x$.

Let $[r]_{(p,q)} \in \mathcal{O}_{B_1 \times B_2,(p,q)}$ be given by an holomorphic function $r: A_1 \times A_2 \to \mathbb{C}$, with $A_1$ an open subset of $B_1$, $p \in A_1$, and $A_2$ an open subset of $B_2$, $q \in A_2$. Let $V$ be an open subset of $\mathbb{C}^n$ such that $x \in V$, $g_1(V) \subset A_1$, $h_1(V) \subset A_1$, $g_2(V) \subset A_2$ and $h_2(V) \subset A_2$, and consider the holomorphic function

$$f(w,z) = r(g_1(w), g_2(z)) - r(h_1(w), g_2(z)), \qquad f: V \times V \to \mathbb{C}.$$

Let $z \in V$ be any point (fixed) and let $t: A_1 \to \mathbb{C}$ be given by $t = r(-, g_2(z))$. Since for all $w \in W$,

$$\begin{aligned}((t \circ g_1) - (t \circ h_1))(w) =& t(g_1(w)) - t(h_1(w))\\ =& r(g_1(w), g_2(z)) - r(h_1(w), g_2(z)) = f(w,z),\end{aligned}$$

we have that $t \circ g_1 - t \circ h_1 = f(-, z)$. Thus, $[f(-, z)]_x \in J_x$, and this holds for all (fixed) $z \in V$. By corollary 2.5 we have then that $[f(w, w)]_x \in J_x$, that is $[r(g_1(w), g_2(w)) - r(h_1(w), g_2(w)]_x \in J_x$. In the same way $[r(h_1(w), g_2(w)) - r(h_1(w), h_2(w)]_x \in J_x$. Combining these two equations it follows that $[r(g_1(w), g_2(w)) - r(h_1(w), h_2(w)]_x \in J_x$, that is, $[r \circ g - r \circ h]_x \in J_x$ (2).

Equations (1) and (2), which hold for all $x \in E$ and $[r]_{(p,q)} \in \mathcal{O}_{B_1 \times B_2, (p,q)}$, mean exactly that $(g, g^*)|_E = (h, h^*)|_E$ holds.

$\square$

**3.2 Theorem.** *The functor $j$ preserves finite products.*

**Proof.** Let $B_1$ and $B_2$ be open subsets of complex Banach spaces. We shall prove that $j(B_1 \times B_2) = j(B_1) \times j(B_2)$.

Let $E$ be an object in $\mathcal{H}$, and consider arrows in $\boldsymbol{T}$, $\xi_1: E \to jB_1$ and $\xi_2: E \to jB_2$. We have to prove that there is a unique arrow

$$\xi: E \to j(B_1 \times B_2)$$

such that $j(\pi_1) \circ \xi = \xi_1$ and $j(\pi_2) \circ \xi = \xi_2$, where $\pi_1: B_1 \times B_2 \to B_1$ and $\pi_2: B_1 \times B_2 \to B_2$ are the projections.

We have that $\xi_1: (E, \mathcal{O}_E) \to (B_1, \mathcal{O}_{B_1})$ and $\xi_2: (E, \mathcal{O}_E) \to (B_2, \mathcal{O}_{B_2})$ are morphisms of $A$-ringed spaces which have local extensions. That is, for each $x \in E$, there exits an open subset $V$ of $\mathbb{C}^n$ such that $x \in V$ and holomorphic functions $g_1: V \to B_1$ and $g_2: V \to B_2$ such that $(g_1, g_1^*)$ is an extension of $\xi_1$ and $(g_2, g_2^*)$ is an extension of $\xi_2$. We consider the holomorphic function $g: V \to B_1 \times B_2$ given by $g(z) = (g_1(z), g_2(z))$.

Let $E' = V \cap E$. In this way we have an open covering of $E$, $E = \bigcup E'$, and for each set $E'$, a morphism of $A$-Ringed spaces

$$(g, g^*)|_{E'}: E' \to (B_1 \times B_2, \mathcal{O}_{B_1 \times B_2}) \tag{1}$$

Let $E''$ be any other set in the covering,

$$(h, h^*)|_{E''}: E'' \to (B_1 \times B_2, \mathcal{O}_{B_1 \times B_2}),$$

with $h: W \to B_1 \times B_2$ given by $h(z) = (h_1(z), h_2(z))$, $E'' = W \cap E$, and $h_1: W \to B_1$, $h_2: W \to B_2$ holomorphic functions such that $(h_1, h_1^*)$ and $(h_2, h_2^*)$ are extensions of $\xi_1$ and $\xi_2$ respectively. Let $U = V \cap W$. On $U \cap E = E' \cap E''$, we have $\xi_1|_{E' \cap E''} = (g_1, g_1^*)|_{E' \cap E''} = (h_1, h_1^*)|_{E' \cap E''}$.

Similarly, $\xi_2|_{E'\cap E''} = (g_2, g_2^*)|_{E'\cap E''} = (h_2, h_2^*)|_{E'\cap E''}$. It follows at once by proposition 3.1 (applied to the functions $g_1|_U$, $h_1|_U$, $g_2|_U$ and $h_2|_U$), that $(g, g^*)|_{E'\cap E''} = (h, h^*)|_{E'\cap E''}$.

This shows that there exists a unique morphism $\xi$ of $A$-ringed spaces, $\xi: E \to (B_1 \times B_2, \mathcal{O}_{B_1\times B_2})$, such that , for all $E'$ in the covering, $\xi|_{E'} = (g, g^*)|_{E'}$, with $(g, g^*)|_{E'}: E' \to (B_1 \times B_2, \mathcal{O}_{B_1\times B_2})$. Clearly this morphism $\xi$ has local extensions. Moreover, for each $E'$ in the covering, we have that

$$\begin{aligned}(\pi_1, \pi_1^*) \circ \xi|_{E'} =&(\pi_1, \pi_1^*) \circ (g, g^*)|_{E'}\\ =&(\pi_1 \circ g, (\pi_1 \circ g)^*)|_{E'} = (g_1, g_1^*)|_{E'} = \xi_1|_{E'}.\end{aligned}$$

That is $(\pi_1, \pi_1^*)\circ\xi = \xi_1$. In the same way $(\pi_2, \pi_2^*)\circ\xi = \xi_2$. Thus, we have an arrow $\xi: E \to j(B_1 \times B_2)$ in $\boldsymbol{T}$ such that $j(\pi_1)\circ\xi = \xi_1$ and $j(\pi_2)\circ\xi = \xi_2$.

It remains to prove the uniqueness of such an arrow. This follows by showing uniqueness on a covering such as the covering above. In turn, this is straightforward utilizing again proposition 3.1 in a similar way as in the preceding argument.

$\square$

## 4. Compatibility with the construction of the tangent bundle.

Here we show that the functor $j$ is compatible with the construction of the tangent bundle in the sense that $j(T(B)) = j(B)^D$, where $D$ is the object of infinitesimals in the topos $\boldsymbol{T}$, $D = [[x|x^2 = 0]] \subset \mathbb{C}$, and $T(B)$ is the classical tangent bundle. Recall that the object $D$ is representable by the analytic scheme $D = (\{0\}, \mathcal{O}_{1,0}/z^2) \simeq (\{0\}, \mathbb{C}[\varepsilon])$, where $\mathbb{C}[\varepsilon] = \{a + b\varepsilon \text{ with } a, b \in \mathbb{C}, \text{ and } \varepsilon^2 = 0\}$ (see 4.2 below with $n = 0$).

**4.1 Observation.** *Let $x \in \mathbb{C}^n$, let $J_x \subset \mathcal{O}_{n,x}$ be any ideal, and let $(J_x, z^2) \subset \mathcal{O}_{n+1,(x,0)}$ be the ideal generated by the germs at $(x,0)$ of the functions of $J_x$ and the function $z^2$, $z \in \mathbb{C}$, considered as functions of $n + 1$ variables. Then, for all $[f]_{(x,0)} \in \mathcal{O}_{n+1,(x,0)}$ we have:*

$[f]_{(x,0)} \in (J_x, z^2)$ if and only if $[f(-,0)]_x \in J_x$ and $[\partial f/\partial z(-,0)]_x \in J_x$.

**Proof.** Write $[f]_{(x,0)} = [b_0]_x + [b_1]_x[z]_0 + [t]_{(x,0)}[z^2]_0$, with $[b_0]_x$, $[b_1]_x \in \mathcal{O}_{n,x}$ and $[t]_{(x,0)} \in \mathcal{O}_{n+1,(x,0)}$ (notice the abuse of notation). Since $f(-,0) = b_0$ and $\partial f/\partial_z(-,0) = b_1$, the result follows.

□

**4.2 Definition.** *Let $E = (E, \mathcal{O}_E)$ be any object in $\mathcal{H}$ determined by two coherent shaves of ideals $I$ and $J$ in an open subset $U$ of $\mathbb{C}^n$, $J \subset I$. Recall that $E = Zeros(I)$, and $\mathcal{O}_{E,x} = \mathcal{O}_{n,x}/J_x$ for $x \in E$. We define the object $(E, \mathcal{O}_E[\varepsilon])$ to be the $A$-ringed space with fibers*

$$\mathcal{O}_E[\varepsilon]_x = \mathcal{O}_{E,x}[\varepsilon] = \{[\sigma_0]_x + [\sigma_1]_x\varepsilon \text{ with } [\sigma_0]_x, [\sigma_1]_x \in \mathcal{O}_{E,x} \text{ and } \varepsilon^2 = 0\}$$

Let $\pi_x: \mathcal{O}_{n,x} \to \mathcal{O}_{E,x}$ be the quotient map. There is a morphism of analytic rings $\delta_x: \mathcal{O}_{n+1,(x,0)} \to \mathcal{O}_{E,x}[\varepsilon]$ defined by

$$\delta_x([f]_{(x,0)}) = \pi_x([f(-,0)]_x) + \pi_x([\partial f/\partial_z(-,0)]_x)\,\varepsilon$$

which identifies $\mathcal{O}_{E,x}[\varepsilon]$ with the quotient $\mathcal{O}_{n+1,(x,0)}/(J_x, z^2)$. This follows by 4.1 and shows that $(E, \mathcal{O}_E[\varepsilon])$ is an object in $\mathcal{H}$. Notice that $(J_x, z^2)$ is an ideal in $\mathcal{O}_{n+1}(U \times V)$, where $V$ is any open subset of $\mathbb{C}$ such that $0 \in V$. Moreover, by construction of coproducts of analytic rings and products in $\mathcal{H}$ ([6]), we have:

$$E \times D \simeq (E, \mathcal{O}_E[\varepsilon])$$

(we should be careful here with the identification $E \simeq E \times \{0\}$).

**4.3 Proposition.** *We refer to the notations in 4.1 and 4.2 above. Let $B$ be an open subset of a Banach space $C$, and let $g$ and $h$ be holomorphic functions, $g$, $h: U \times V \to B$. Notice that $\partial g/\partial_z$ and $\partial h/\partial_z$ are holomorphic functions $U \times V \to C$ (not $B$). We have:*

(a) $(g, g^*)|_{E\times D} = (h, h^*)|_{E\times D}$ *if and only if*

(b) $(g(-,0), g(-,0)^*)|_E = (h(-,0), h(-,0)^*)|_E$,
$(\partial g/\partial_z(-,0), \partial g/\partial_z(-,0)^*)|_E = (\partial h/\partial_z(-,0), \partial h/\partial_z(-,0)^*)|_E$

**Proof.** Let $x \in E$. It is clear that the statement (a) means that $g(x,0) = h(x,0)$ (say, $= p$) and $\delta_x \circ g^* = \delta_x \circ h^*$. That is, for all $[r]_p \in \mathcal{O}_{B,p}$,

$\delta_x([r \circ g]_{(x,0)}) = \delta_x([r \circ h]_{(x,0)})$. Then, by definition of $\delta_x$ it follows immediately that (a) is equivalent to:

(a1) $g(x,0) = h(x,0) = p$,

(a2) $\pi_x([r \circ g(-,0)]_x) = \pi_x([r \circ h(-,0)]x)$,

(a3) $\pi_x([\partial/_{\partial z}(r \circ g)(-,0)]_x) = \pi_x([\partial/_{\partial z}(r \circ h)(-,0)]_x)$.

Here, for $w \in V$, $\partial/_{\partial z}(r \circ g)(w,0) = Dr(g(w,0))(\partial g/_{\partial z}(w,0))$, and similarly for $h$.

On the other hand, (b) means that $g(x,0) = h(x,0) = p$, $\partial g/_{\partial z}(x,0) = \partial h/_{\partial z}(x,0)$, (say $= q \in C$), $\pi_x \circ g(-,0)^* = \pi_x \circ h(-,0)^*$ and $\pi_x \circ \partial g/_{\partial z}(-,0)^* = \pi_x \circ \partial h/_{\partial z}(-,0)^*$. Considering all $[r]_p \in \mathcal{O}_{B,p}$, $[s]_q \in \mathcal{O}_{C,q}$, (b) is equivalent then to:

(b1) $g(x,0) = h(x,0) = p$,

(b2) $\pi_x([r \circ g(-,0)]_x) = \pi_x([r \circ h(-,0)]_x)$,

(b3) $\partial g/_{\partial z}(x,0) = \partial h/_{\partial z}(x,0) = q$,

(b4) $\pi_x([s \circ \partial g/_{\partial z}(-,0)]_x) = \pi_x([s \circ \partial h/_{\partial z}(-,0)]_x)$,

**a) $\Rightarrow$ b).** b1) and b2) are the same that a1) and a2) respectively. Notice that if $r = \alpha \in C'$ is a linear form, $D\alpha(g(w,0)) = \alpha$. Thus $\partial/_{\partial z}(\alpha \circ g)(w,0) = \alpha(\partial g/_{\partial z}(w,0))$, therefore

$$[\partial/_{\partial z}(\alpha \circ g)(-,0)]_x = [\alpha \circ \partial g/_{\partial z}(-,0)]_x.$$

$$\text{Similarly} \quad [\partial/_{\partial z}(\alpha \circ h)(-,0)]_x = [\alpha \circ \partial h/_{\partial z}(-,0)]_x.$$

Assume (a3), and let $\alpha \in C'$ be a linear form. We have

$$\pi_x([\alpha \circ \partial g/_{\partial z}(-,0)]_x) = \pi_x([\alpha \circ \partial h/_{\partial z}(-,0)]_x).$$

$$\text{Thus } [\alpha \circ (\partial g/_{\partial z}(-,0)) - \alpha \circ (\partial h/_{\partial z}(-,0)]_x \in J_x.$$

Since $J_x \subset I_x$, the value at $x$ of any germ in $J_x$ is $0$. Thus $\alpha(\partial g/_{\partial z}(x,0)) = \alpha(\partial h/_{\partial z}(x,0))$ (for all $\alpha \in C'$). It follows by the Hahn-Banach theorem that $\partial g/_{\partial z}(x,0) = \partial h/_{\partial z}(x,0) = q \in C$. This shows (b3). Moreover, it follows now by the lemma 2.6, that

$$[s \circ (\partial g/_{\partial z}(-,0)) - s \circ (\partial h/_{\partial z}(-,0)]_x \in J_x$$

for all $[s]_q \in \mathcal{O}_{C,q}$. This shows (b4), which completes the proof of b).

**b) $\Rightarrow$ a).** Let $\alpha \in C'$, from b4), arguing exactly as before we have $\pi_x([\partial/_{\partial z}(\alpha \circ g)(-,0)]_x) = \pi_x([\partial/_{\partial z}(\alpha \circ h)(-,0)]_x)$. On the other hand by b2) $\pi_x([(\alpha \circ g)(-,0)]_x) = \pi_x([(\alpha \circ h)(-,0)]_x)$ These two equalities mean (by definition of $\delta_x$) that $\delta_x([\alpha \circ g]_{(x,0)}) = \delta_x([\alpha \circ h]_{(x,0)})$, that is $[\alpha \circ g - \alpha \circ h]_{(x,0)} \in (J_x, z^2)$. Then, again by lemma 2.6 applied this time to the ideal $(J_x, z^2)$, it follows $[r \circ g - r \circ h]_{(x,0)} \in (J_x, z^2)$, that is $\delta_x([r \circ g]_{(x,0)}) = \delta_x([r \circ h]_{(x,0)})$ (for all $[r]_p \in \mathcal{O}_{B,p}$). This finishes the proof of a).

□

We are now in condition to prove that Banach spaces in the topos become "objects of line type". That is:

**4.4 Theorem.** *Let $B$ be an open subset of a complex Banach space $C$. Then,*

$$(jB)^D \simeq jB \times jC \text{ in } \boldsymbol{T}.$$

**Proof.** We have to show that for each $E \in \mathcal{H}$, there is a natural (in $E$) bijection:

$$[E, (jB)^D] \to [E, jB \times jC].$$

a) Let $\xi$ be an arrow, $\xi: E \to (jB)^D$. That is, $\xi$ is an arrow $E \times D \to jB$ in $\boldsymbol{T}$. Then $\xi$ is a morphism of $A$-ringed spaces, $\xi: (E, \mathcal{O}_E[\varepsilon]) \to (B, \mathcal{O}_B)$ (see 4.2) which has local extensions. For each $x \in E$, there is an open subset $U$ of $\mathbb{C}^n$ such that $x \in U$, an open subset $V$ of $\mathbb{C}$ such that $0 \in V$ and an holomorphic function $g: U \times V \to B$ such that $(g, g^*)|_{E' \times D} = \xi|_{E' \times D}$, where $E' = U \cap E$. In this way we have an open covering $E = \bigcup E'$, and, for each $E'$, a pair of morphisms

$$(g(-,0), g(-,0)^*)|_{E'}: E' \to (B, \mathcal{O}_B),$$

$$(\partial g/_{\partial z}(-,0), \partial g/_{\partial z}(-,0)^*)|_{E'}: E' \to (C, \mathcal{O}_C).$$

Suppose that we have $E''$ in this covering with an holomorphic function $h$, $(h, h^*)|_{E'' \times D} = \xi|_{E'' \times D}$. It follows that $(g, g^*)|_{(E' \cap E'') \times D} = (h, h^*)|_{(E' \cap E'') \times D}$, and thus, by proposition 4.3 (on the object $E' \cap E''$) we have

$$(g(-,0), g(-,0)^*)|_{E' \cap E''} = (h(-,0), h(-,0)^*)|_{E' \cap E''},$$

$$(\partial g/_{\partial z}(-,0), \partial g/_{\partial z}(-,0)^*)|_{E'\cap E''} = (\partial h/_{\partial z}(-,0), \partial h/_{\partial z}(-,0)^*)|_{E'\cap E''}.$$

This shows there is a pair of morphisms of $A$-ringed spaces

$$\psi: (E, \mathcal{O}_E) \to (B, \mathcal{O}_B), \quad \beta: (E, \mathcal{O}_E) \to (C, \mathcal{O}_C),$$

unique such that for each $E'$ in the covering, it holds that

$$\psi|_{E'} = (g(-,0), g(-,0)^*)|_{E'} \text{ and } \beta|_{E'} = (\partial g/_{\partial z}(-,0), \partial g/_{\partial z}(-,0)^*)|_{E'}.$$

By a similar argument it follows that $\psi$ and $\beta$ do not depend on the covering. Clearly they have local extensions, thus, they are actually arrows in the topos, which determine an arrow $(\psi, \beta): E \to jB \times jC$ in $\boldsymbol{T}$. This defines the function $[E, (jB)^D] \to [E, jB \times jC]$. We have to prove now that it is a bijection.

b) (inyectivity). Suppose that we have two arrows $\xi_1$ and $\xi_2: E \to (jB)^D$ in $\boldsymbol{T}$ which determine the same pair $(\psi, \beta)$. $\xi_1$ and $\xi_2$ correspond to arrows $E \times D \to jB$, that is, morphisms of $A$-Ringed spaces $(E, \mathcal{O}_E[\varepsilon]) \to (B, \mathcal{O}_B)$ with local extensions. For each $x \in E$, let $(g, g^*)$ and $(h, h^*)$ be a local extensions of $\xi_1$ and $\xi_2$ around $(x, 0)$ respectively. We can assume they are defined in a same open subsets $U$ of $\mathbb{C}^n$, $x \in U$, $V$ of $\mathbb{C}$, $0 \in V$, $g$, $h: U \times V \to B$, $(g, g^*)|_{E'\times D} = \xi_1|_{E'\times D}$, $(h, h^*)|_{E'\times D} = \xi_2|_{E'\times D}$, where $E' = U \cap E$. Since $\xi_1$ and $\xi_2$ determine the same pair $(\psi, \beta)$, it follows that $(g(-,0), g(-,0)^*)|_{E'} = (h(-,0), h(-,0)^*)|_{E'}$ and $(\partial g/_{\partial z}(-,0), \partial g/_{\partial z}(-,0)^*)|_{E'} = (\partial h/_{\partial z}(-,0), \partial h/_{\partial z}(-,0)^*)|_{E'}$. Then, by proposition 4.3, we have $(g, g^*)|_{E'\times D} = (h, h^*)|_{E'\times D}$, thus $\xi_1|_{E'\times D} = \xi_2|_{E'\times D}$. And this for each $E'$ on a covering. It follows that $\xi_1 = \xi_2$.

c) (suryectivity). Let $\psi: E \to jB$, $\beta: E \to jC$ in $\boldsymbol{T}$, that is

$$\psi: (E, \mathcal{O}_E) \to (B, \mathcal{O}_B), \quad \beta: (E, \mathcal{O}_E) \to (C, \mathcal{O}_C),$$

morphisms of $A$-ringed spaces with local extensions. For each $x \in E$, let $(g_0, g_0^*)$ be a local extension of $\psi$ around $x$, and let $(g_1, g_1^*)$ be a local extension of $\beta$ around $x$, where $g_0: U \to B$ and $g_1: U \to C$ are holomorphic functions and $U$ is an open subset of $\mathbb{C}^n$ such that $x \in U$. Let $g: U \times \mathbb{C} \to C$ be the function defined by: $g(w, z) = g_0(w) + z g_1(w)$. Thus, $g$ is holomorphic and $g(x, 0) \in B$. It follows that there exists an open subset $W$ of $\mathbb{C}^n$ such that $x \in W$ and an open subset $V$ of $\mathbb{C}$

such that $0 \in V$ and $g(W \times V) \subset B$. Consider $g: W \times V \to B$ and the morphism of $A$-Ringed spaces $(g, g^*)|_{E' \times D}: E' \times D \to (B, \mathcal{O}_B)$, where $E' = W \cap E$. Notice that $g_0 = g(-,0)$ and $g_1 = \partial g/_{\partial z}(-,0)$. We have an open covering $E = \bigcup E'$, and for each set $E'$ in this covering, a morphism $(g, g^*)|_{E' \times D}: E' \times D \to (B, \mathcal{O}_B)$. Exactly in the same way as before in this proof, it is straightforward to check that these morphisms are compatible in the intersections of the covering (use proposition 4.3). Thus, this determines a morphism of $A$-ringed spaces $\xi: E \times D \to (B, \mathcal{O}_B)$ unique such that for each $E'$ in the covering, the restriction $\xi|_{E' \times D} = (g, g^*)|_{E' \times D}$. It is clear that $\xi$ has local extensions and thus it defines an arrow $\xi: E \times D \to jB$ in $\boldsymbol{T}$, that is, $\xi: E \to (jB)^D$. It is immediate also that $\xi$ determines (by the construction defined in a) above) the pair $(\psi, \beta)$.

Finally, it is straightforward to check the naturality in $E$ of this correspondence.

□

Given any open set $B$ in a Banach Space $C$, the tangent bundle $TB \to B$ of $B$ is, as expected, just the product $B \times C$ with the first projection as ground morphism. That is, we have $(TB \to B) = (B \times C \to B)$, in particular, $TB = B \times C$. Putting together theorems 3.2 and 4.4 it follows then that the functor $j$ preserves the construction of the tangent bundle. That is, $(jB)^D \simeq j(TB)$. We shall see now that the functor $j$ is also compatible with the construction of the Derivative map. First, we make an observation to fix the notation.

**4.5 Notation.** Let $B_1$ and $B_2$ be open subsets of complex Banach spaces $C_1$ and $C_2$ respectively, and let $f$ be an holomorphic function, $f: B_1 \to B_2$. Consider the arrow $jf: jB_1 \to jB_2$ and the induced arrow $(jf)^D: (jB_1)^D \to (jB_2)^D$. By 4.4 we have $(jB_1)^D \simeq jB_1 \times jC_1$ and $(jB_2)^D \simeq jB_2 \times jC_2$. Thus, we have a corresponding arrow $L: jB_1 \times jC_1 \to jB_2 \times jC2$.

Let $F: B_1 \times C_1 \to C_2$ be given by $F(p, v) = Df(p)(v)$. Given $p \in B_1$, the function $Df(p): C_1 \to C_2$ is a linear and continuous map, and thus it is holomorphic. Given $v \in C_1$, the function $p \mapsto Df(p)(v)$ is holomorphic ([8], 7.18). Thus, $F$ is separately holomorphic. It follows ([8], 8.10) that $F$ is holomorphic. This determines an holomorphic map $G: B_1 \times C_1 \to B_2 \times C_2$, $G(p, v) = (f(p),\ Df(p)(v)) = (f(p), F(p, v))$. Thus, $G = (f \circ \pi, F)$, where $\pi$ is the first projection $\pi: B_1 \times C_1 \to B_1$. Now, since $j$ preserves products we have $jG: jB_1 \times jC_1 \to jB_2 \times jC_2$.

Clearly $jG = (jf \circ \pi, jF)$ $(\pi = j\pi)$.

**4.6 Theorem.** *Let $B$ be an open subset of a complex Banach space $C$. Then:*

a) $(jB)^D \simeq j(TB)$ in $\boldsymbol{T}$,

b) *Under this isomorphism, given any holomorphic function $f: B_1 \to B_2$, the derivative map $TB_1 \to TB_2$ corresponds to the arrow*

$$(jf)^D : (jB_1)^D \to (jB_2)^D \quad \text{in } \boldsymbol{T}.$$

**Proof.** The first part is just theorems 3.2 and 4.4 together. The second part, with the notation defined in 4.5, means explicitly that the equation $L = jG$ holds. We shall prove this now. Let $E \in \mathcal{H}$, $\psi: E \to jB_1$ and $\beta: E \to jC_1$ in $\boldsymbol{T}$. We have to prove that

$$(jG) \circ (\psi, \beta) = L \circ (\psi, \beta) : E \to jB_2 \times jC_2.$$

Clearly $(jG) \circ (\psi, \beta) = (jf \circ \psi, jF \circ (\psi, \beta))$. For each $x \in E$, let $(g_0, g_0^*)$ be a local extension of $\psi$ around $x$, and let $(g_1, g_1^*)$ be a local extension of $\beta$ around $x$, where $g_0 : U \to B$ and $g_1 : U \to C$ are holomorphic functions and $U$ is an open subset of $\mathbb{C}^n$ such that $x \in U$. Under the isomorphism $jB_1 \times jC1 \simeq (jB_1)^D$ the pair $(\psi, \beta)$ corresponds to some $\xi$, where $\xi: E \to (jB_1)^D$, that is, $\xi: E \times D \to jB_1$. Let $(g, g^*)$ be a local extension of $\xi$ around $(x, 0)$, where $g(w, z) = g_0(w) + z g_1(w)$. It follows that $(f \circ g, (f \circ g)^*)$ is a local extension of $jf \circ \xi$ around $(x, 0)$. Thus, under the isomorphism $(jB_2)^D \simeq jB_2 \times jC_2$, the map $jf \circ \xi$ corresponds to a pair $(\alpha, \delta)$, where $\alpha: E \to jB_2$ and $\delta: E \to jC_2$ are arrows in $\boldsymbol{T}$, and $L \circ (\psi, \beta) = (\alpha, \delta)$. Moreover, $((f \circ g)(-, 0), (f \circ g)(-, 0)^*)$ is a local extension of $\alpha$ around $x$, and $(\partial/\partial_z (f \circ g)(-, 0), \partial/\partial_z (f \circ g)(-, 0)^*)$ is a local extension of $\delta$ around $x$. But we have $f(g(w, 0) = f(g_o(w))$, thus, $(f \circ g)(-, 0) = f \circ g_0$. Moreover,

$$\begin{aligned} \partial/\partial_z (f \circ g)(w, 0) &= Df(g(w, 0))(\partial g/\partial_z)(w, 0) \\ &= Df(g_0(w))(g_1(w)) = F(g_0(w), g_1(w)). \end{aligned}$$

Thus, it follows that $(\partial/\partial_z (f \circ g)(-, 0) = F \circ (g_0, g_1)$. So $(f \circ g_0, (f \circ g_0)^*)$ is a local extension of $\alpha$ around $x$, and $(F \circ (g_0, g_1), F \circ (g_0, g_1))^*)$ is a local extension of $\delta$ around $x$. On the other hand, since $(g_0, g_0^*)$ is a local extension of $\psi$ around $x$, it follows that $(f \circ g_0, (f \circ g_0)^*)$ is a local extension of $jf \circ \psi$ around $x$. Thus, $\alpha$ and $jf \circ \psi$ have the same local extensions, and therefore $\alpha = jf \circ \psi$. Finally, since $(g_0, g_0^*)$ is a local extension of $\psi$ around $x$, and $(g_1, g_1^*)$ is a local extension of $\beta$ around $x$, we have that $((g_0, g_1), (g_0, g_1)^*)$ is a local extension of $(\psi, \beta)$ around $x$. It follows that $(F \circ (g_0, g_1), (F \circ (g_0, g_1))^*)$ is a local extension of $jF \circ (\psi, \beta)$ around $x$. Thus, $\delta$ and $jF \circ (\psi, \beta)$ have the same local extensions, and therefore $\delta = jF \circ (\psi, \beta)$. We have shown then that $L \circ (\psi, \beta) = (\alpha, \delta) = (jf \circ \psi, jF \circ (\psi, \beta)) = (jG) \circ (\psi, \beta)$.

$\square$

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