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CATEGORICAL STRONG SHAPE THEORY

by Mikhail A. BATANIN

RESUME. Le but de l'article est une interprétation catégorique de la théorie de la forme forte. Pour cela, on construit une bicatégorie spéciale, appelée bicatégorie homotopiquement cohérente des distributeurs simpliciaux. On montre que la théorie de la forme forte d'un foncteur simplicial K peut être caractérisée comme une extension à droite, dans cette bicatégorie, d'un distributeur simplicial associé à K le long de lui-même.

Cette caractérisation permet d'obtenir des propriétés générales de la théorie de la forme forte, et on obtient une équivalence entre différentes approche de cette théorie.

Introduction

The homotopy theory of procategories appeared for the first time in T.Porter's work on the stability problem for topological spaces [39]. Strong shape category for topological spaces was introduced by D.A. Edwards and H.Hastings in [23] and at about the same time by F.W.Bauer [4].

A more geometric construction of strong shape, based on the use of homotopy coherent natural transformations, and hence, closely related to Porter's initial ideas, was suggested by J.T.Lisica and S.Mardešić [35]. At the present time there are a number of constructions, which however, lead to equivalent results [2,11,12,22,26,27,40].

On the other hand, ordinary shape theory has been interpreted from a categorical point of view by D.Bourn and J.M.Cordier [8,16], and the problem of finding a similar interpretation for strong shape theory was formulated [9,19]. Such an interpretation is of special interest, because it is connected directly with Grothendieck's plan sketched out in [25]. Having in view this application, D.Bourn, J.-M.Cordier and T.Porter developed some categorical machinery, which they called "homotopy coherent category theory" [9,15,13,14,17,18,19]. Closely related ideas can be found also in [10,20,29,28,38,44].

Our present article is devoted to a partial solution of this problem. We can not say that we have obtained the complete solution, because some very interesting questions should be clarified. For example, we do not discuss here exact squares and morphisms between strong shape theories. We mention some others in the following overview of our work.

In section 1 we recall the Bourn-Cordier axiomatics of shape theory and some related categorical constructions. Technically our approach is based essentially on a combination of the above mentioned Bourn-Cordier-Porter theory and the author's construction of the homotopy coherent category of a monad [2]. We give here some definitions and results from these theories.

In section 2 we demonstrate that some basic shape constructions can be defined in an arbitrary bicategory. We consider some axioms, which characterize a shape theory of an arrow K up to isomorphism as the right extension of K along itself.

In some bicategories, such an extension may not exist. But it is possible, that it exists in some extended bicategory. Depending on the choice of this bicategory, we thus obtain various type of shape theories. As an example we consider ordinary categorical shape theory.

The 2-category $\boldsymbol{Cat}$ of categories may be included in the following chain of embeddings of bicategories:

$$\boldsymbol{Cat} \subset \boldsymbol{pro}(\boldsymbol{Cat}) \subset \boldsymbol{Dist} \ ,$$

where $\boldsymbol{pro}(\boldsymbol{Cat})$ and $\boldsymbol{Dist}$ are the bicategories of procategories and distributors respectively (see example 2 in section 2 for the definition). The bicategory $\boldsymbol{pro}(\boldsymbol{Cat})$ contains the shape theories of Čech type. We thus obtain some classification of the types of possible shape theories. We consider also the bicategory $\boldsymbol{Pro}(\boldsymbol{Cat})$, which is a noncofiltered analogue of $\boldsymbol{pro}(\boldsymbol{Cat})$. Again we have some type of shape theories intermediate between Čech and general type theories. The existence of similar shape theories was observed in [8, Remarque p.182].

The most interesting examples arise, when $K : A \to B$ has a left adjoint L in $\boldsymbol{Pro}(\boldsymbol{Cat})$ or in $\boldsymbol{pro}(\boldsymbol{Cat})$. In this case $L(X)$ plays the role of a "resolution" of an object X. In some sense $Pro(B)$ contains the resolutions of all possible types and we call it the category of resolutions of B.

Our approach has again one advantage. It may be easily dualized to consider coshape theories as well. We develop this theory in a parallel way.

A goal of the next sections is to show that strong shape theory may be considered within the frame of the above bicategory approach.

In section 3 we construct the appropriate bicategory of distributors. We start from the simplicial enriched version of distributor theory [6] and substitute every category of simplicial natural transformations by the homotopy category of coherent transformations. The same process is applied to the composition functor and right and left extension functors. We thus obtain a biclosed bicategory $\boldsymbol{CHDist}$. As in the nonenriched situation we can associate with every simplicial functor K some simplicial distributor ϕ_K and, hence, consider a shape theory of this distributor in $\boldsymbol{CHDist}$. We call this theory a strong shape theory of the simplicial functor K. The theory developed in section 2 permits us to characterize this theory up to isomorphism.

Section 4 is devoted to the construction of a category $\mathcal{CPH}(B)$, which is a coherent analogue of $Pro(B)$ for a simplicial category B. We call this category the category of strong resolutions of B. We introduce a notion of strong K-resolution of an object X, which is a generalization of the notion of K-associated inverse system and study the properties of strongly K-continuous functors.

Here we give also the first examples of strong shape and coshape theories. A very interesting example of strong shape (coshape) category arises as the strong shape (coshape) category of the simplicial inclusion $i_c : Q_{cf} \subset Q_c$ ($i_f : Q_{cf} \subset Q_f$), where Q is a Quillen [41] simplicial closed model category and Q_c, Q_f, Q_{cf} are the subcategories of cofibrant, fibrant, and both fibrant and cofibrant objects of Q respectively. Then we prove, that the strong shape (coshape) category of i_c (i_f) is isomorphic to HoQ_c (HoQ_f) of [41] and hence is equivalent to HoQ.

Other examples are the coherent homotopy category of a monad [2] and various categories of homotopy homomorphisms [7].

The highly developed part of the known approaches to strong shape theory is based on the use of the appropriate homotopy category of inverse systems or similar concepts. We consider this type of strong shape constructions in section 5.

We introduce the notion of strong K-associated inverse system and develop the abstract categorical scheme for constructing the strong shape categories of this type. We show that this approach agrees with that of the authors mentioned above and with our general bicategory approach. The last part of section is devoted to proving the equivalence of the notions of strong K-associated inverse system and of Mardešić strong expansion [35].

This provides us with a number of examples of strong shape and coshape categories. Among them the coherent prohomotopy category and strong shape category of Lisica and Mardešić, $ho(pro - \mathcal{S})$ of Edwards and Hastings and strong shape and coshape categories of Cathey-Segal [12].

We cannot, however, develop the theory of strong shape in full analogy with the ordinary one. The reason for such a situation is the lack of an analogue of the Kleisli construction of a monad in the bicategory ***CHDist***. Indeed, in general the monad multiplication in ***CHDist*** is associative only up to homotopy and, hence, its Kleisli "category" is not an object of ***CHDist***.

This difficulty is considered in section 6. We show that our construction of coherent right and left extensions of a simplicial distributor along itself may be endowed with the structure of an A_∞-monoid in the sense of [3]. This allow us, by considering an appropriate Kleisli construction, to associate with every strong shape (coshape) theory some locally Kan simplicial category, whose homotopy category is initial strong shape (coshape) category. This result is closely related to the Dwyer-Kan hammock localization [21] and recent results of R.Schwänzl and R.Vogt

[42].

This construction shows, in addition, that it would be naturally to consider strong shape theories not in the bicategory ***CHDist*** but in some more general categorical concept, which may be called the A_∞-bigraph, which is a sort of A_∞-monoid in the category of simplicial bigraphs (see the concluding remark in section 5). We do not give a precise definition, because it is sufficiently complex and deserves separate consideration.

The other interesting question is the existence of limits and colimits in strong shape categories. As these categories by their nature are defined up to higher homotopies, we must consider only homotopy limits and colimits. So we need a definition of homotopy limit of A_∞-morphism between A_∞-graphs. This theme leads us to the question about the connections of our theory and A.Heller's approach to the genesis of homotopy theories [29]. These connections seem to be of great interest, because both these theories describe essentially the same mathematical concepts, but from different points of view.

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1 Preliminary discussion

We adopt the following notations: for a category B and two objects X, Y of B we denote by $B(X,Y)$ the set of morphisms from X to Y. If B is enriched in some monoidal category, then $\underline{B}(X,Y)$ is the value of the enriched *hom*-functor on X, Y. We denote by $\mathcal{S}$ the category of simplicial sets, and by Δ the cosimplicial object of $\mathcal{S}$ consisting of the standard simplicies $\Delta(n)$. We use also the notation Top for the category of all topological spaces, and $\mathcal{K}a$ for its subcategory of compactly generated spaces.

The category $\mathcal{K}a$ is a closed monoidal category and at the same time a simplicial category. We shall denote by $\underline{\mathcal{K}a}$ and by $\underline{\mathcal{K}a}_{\mathcal{S}}$ its internal and simplicial enriched hom-functors respectively. We use also the notation ANR for the category of absolute neighbourhood retracts.

The necessary basic category theory and enriched category theory can be found in [6,30,33]. The simplicial techniques we will need are in [10,24].

REMARK. Some words about the size (in the set theoretical sense) of the categories involved. In our paper we are taking the view of [30][2.6, 3.11 and 3.12] on the existence of functor categories, limits and colimits.

Let B be a simplicial category. We say, that two morphisms $f, g : X \to Y$ are homotopic if there is a 1-simplex F in $\underline{B}_1(X, Y)$, such that

$$d_0(F) = f \ , \ d_1(F) = g \ .$$

We call such a simplex a homotopy from f to g. If F, G are two homotopies from f to g we say, that they are homotopic provided there is a 2-simplex $\sigma \in \underline{B}_2(X, Y)$ such that

$$d_0(\sigma) = F \ , \ d_1(\sigma) = G \ ,$$

and $d_2(\sigma)$ is degenerate.

These relations are not equivalence relations unless B is locally Kan, that is $\underline{B}(X, Y)$ is a Kan simplicial set for any $X, Y \in ob(B)$, however we can associate with B its homotopy category $\pi(B)$ taking as $\pi(B)(X, Y)$ the set of connected components of $\underline{B}(X, Y\cdot)$.

We assume, that the reader is acquainted with the definition of bicategory of distributors, $\boldsymbol{Dist}$, [6,8,16] and of its basic properties.

To gain a better understanding of our approach we must recall some definitions of categorical shape theory.

In [8,16] the following definition of a shape theory is given.

Let $K : A \to B$ be a functor. Then a pair $(S, \boldsymbol{S}_K)$, where $\boldsymbol{S}_K$ is a category and $S : B \to \boldsymbol{S}_K$ is a functor, is said to be a shape theory for K if

1. $ob(\boldsymbol{S}_K) = ob(B)$ and $S(X) = X$,

2. if $s \in \boldsymbol{S}_K(X, K(Q))$, there is a unique $f \in B(X, K(Q))$ such that $S(f) = s$,

3. S is K-continuous functor.

It was shown that these three axioms are equivalent to the following ones:

1*. $\boldsymbol{S}_K$ is isomorphic to the Kleisli category of the monad (T, μ, η) in the bicategory of distributors ***Dist*** generated by an adjunction $\phi_S \dashv \phi^S$,

2*. $\eta \otimes 1_{\phi_K}$ is invertible,

3*. ϕ_S is a right extension of $\phi_S \otimes \phi_K$ along ϕ_K.

Our main idea was to separate the general bicategory properties of shape theory from those, which issue from the special feature of the bicategory of distributors. This leaded us to the following axiomatics.

We call a shape theory for K a monad (T, μ, η) in ***Dist*** with the following properties:

1** $\eta \otimes 1_{\phi_K}$ is invertible,

2** T is a right extension of $T \otimes \phi_K$ along ϕ_K.

If we have such a theory, we can consider the pair (S, Kl_T), where S is canonical functor to the Kleisli category and show that it is a shape theory for K in the sense of [8]. Conversely, the monad from axiom 1* satisfies our conditions if $(S, \boldsymbol{S}_K)$ is a shape theory for K. Below we shall see, that the existence of a multiplication, μ, is a consequence of the conditions 1**, 2**, and our axiomatics will take final form (see definition 2.6). We give a proof of the equivalence of the two axiom systems in proposition 2.5.

Our axiomatics may be considered in any bicategory and this is its main advantage. It only remains to us to find a bicategory appropriate for developing the strong shape theories.

For these purposes we use the homotopy coherent category theory of [15,19] and the construction of the homotopy coherent category of a monad from [2,14]. Below we recall some points of these theories.

We use in our work a general definition of homotopy limit given in [9], which generalizes the Bousfield-Kan notion of homotopy limit [10].

The following construction is extracted from [19]. Let A be a small $\mathcal{S}$-category. For objects X, Y of A form the bisimplicial set $\Psi(X, Y)$

defined by

$$\Psi(X,Y)_{n,*} = \coprod_{A_0,\ldots,A_n} \underline{A}(X,A_0) \times \underline{A}(A_0,A_1) \times \ldots \times \underline{A}(A_n,Y)$$

where

$$d_i : \Psi(X,Y)_{n,*} \to \Psi(X,Y)_{n-1,*}$$

is defined by composition in A,

$$\underline{A}(A_{i-1},A_i) \times \underline{A}(A_i,A_{i+1}) \to \underline{A}(A_{i-1},A_{i+1}) \ ,$$

and $s_i : \Psi(X,Y)_{n,*} \to \Psi(X,Y)_{n+1,*}$ by the canonical morphism

$$\Delta(0) \to \underline{A}(A_i,A_i) \ .$$

Set now $\check{A}(X,Y) = Diag(\Psi(X,Y))$.

DEFINITION 1.1 *Let B be a complete $\mathcal{S}$-category, with cotensorization $(-)^{(-)} : B \times \mathcal{S}^{op} \to B$,*

$$T : A^{op} \times A \to B$$

an $\mathcal{S}$-functor.

The simplicially coherent end of T will be the object $\oint_A T(A,A)$ of B defined by

$$\oint_A T(A,A) = \int_{A^{op}\times A} T(A',A'')^{\check{A}(A',A'')}$$

DEFINITION 1.2 *Let B be a cocomplete $\mathcal{S}$-category with tensorization $-\times- : B \times \mathcal{S} \to B$,*

$$T : A^{op} \times A \to B$$

an $\mathcal{S}$-functor.

The simplicially coherent coend of T will be the object $\oint^A T(A,A)$ of B defined by

$$\oint^A T(A,A) = \int^{A^{op}\times A} T(A',A'') \times \hat{A}(A',A'') \ ,$$

where $\hat{A}(A',A'') = Diag(\Psi(A'',A'))$.

As was established in [19], for coherent ends (coends), the cosimplicial (simplicial) replacement formula and a universal property similar to that of homotopy limits (colimits) are satisfied. We need also the following properties of coherent ends.

PROPOSITION 1.1 ([19]) *If a simplicial functor $T : A^{op} \times A \to \mathcal{S}$ is such, that $T(A', A'')$ is a Kan simplicial set (we shall say in this case, that T is locally Kan), then $\oint_A T(A, A)$ is a Kan simplicial set too.*

Recall now some definitions. Let A be an $\mathcal{S}$-category. Every $\mathcal{S}$-natural transformation between two $\mathcal{S}$-functors F, G from A to $\mathcal{S}$ is defined by a set of simplicial mappings from $\Delta(0)$ to $\underline{\mathcal{S}}(F(X), G(X))$, $X \in ob(A)$. Let $T_0(X) : F(X) \to G(X)$ be the corresponding simplicial map.

DEFINITION 1.3 *We will say that an $\mathcal{S}$-natural transformation T between two $\mathcal{S}$-functors from A to $\mathcal{S}$ is a level weak equivalence (level homotopy equivalence) provided $T_0(X)$ is a weak equivalence (homotopy equivalence) of simplicial sets for every object X of A.*

An important property of coherent ends is

PROPOSITION 1.2 ([19]) *If $S, T : A^{op} \times A \to \mathcal{S}$ are two locally Kan $\mathcal{S}$-functors and $\eta : S \to T$ is an $\mathcal{S}$-natural transformation which is a level homotopy equivalence, then*

$$\oint_A \eta(A.A) : \oint_A S(A, A) \to \oint_A T(A, A)$$

is a homotopy equivalence.

If we use weak equivalences instead homotopy equivalences, then the similar property is true for coherent coends, without the assumption that S, T are locally Kan.

Finally, we have to formulate a statement, which we call, following [19], the coherent Yoneda lemma.

PROPOSITION 1.3 (COHERENT YONEDA LEMMA [19]) *For each $\mathcal{S}$-functor*

$$F : A \to \mathcal{S}$$

there is an $\mathcal{S}$-natural homotopy equivalence

$$\eta_F : F(X) \to \oint_A \underline{\mathcal{S}}(\underline{A}(X, A), F(A)) .$$

The following construction was considered in [2] for a simplicial monad. Here we dualize it for the case of a simplicial comonad.

Let (L, μ, ϵ) be an $\mathcal{S}$-comonad on A. Then for every $X \in ob(A)$ one can define a simplicial object $L_*(X)$ of A putting

$$L_n(X) = L^{n+1}(X) \ , \ d_i = L^{n-i} \cdot \epsilon \cdot L^i \ , \ s_i = L^{n-i} \cdot \mu \cdot L^i \ . \qquad (1)$$

Suppose now, that A is $\mathcal{S}$-tensored and there exists the realization

$$L_\infty(X) = \textstyle\int^n L_n(X) \times \Delta(n) \ .$$

We thus obtain an $\mathcal{S}$-endofunctor on A which we shall denote by L_∞. The counit ϵ, considered as a simplicial morphism from $L_*(X)$ to a constant simplicial object X_*, generates after simplicial realization an $\mathcal{S}$-natural transformation

$$\epsilon_\infty : L_\infty \to I \ .$$

By applying the functor $\underline{A}(-, Y)$ to $L_*(Y)$ levelwise we obtain a cosimplicial object $\underline{A}(L_*(X), Y)$ in $\mathcal{S}$ and

$$\underline{A}(L_\infty(X), Y) = Tot(\underline{A}(L_*(X), Y)) \ ,$$

where Tot is Bousfield-Kan total space functor [10].

As was proved in [2], if $\underline{A}(L_*(X), Y))$ is a fibrant cosimplicial simplicial set in the sense of [10], then one can define a simplicial transformation

$$\mu_\infty : L_\infty \to L_\infty^2 \ ,$$

such that $(L_\infty, \mu_\infty, \epsilon_\infty)$ becomes a comonad on $\pi(A)$.

One special case of this construction is especially important for us. Let A be a simplicial category. Then, following [19], the simplicial set of coherent transformations between simplicial functors $F, G : A \to \mathcal{S}$ is given by

$$Coh(A, \mathcal{S})(F, G) = \oint_A \underline{\mathcal{S}}(F(A), G(A)) \ .$$

With every simplicial category A one can associate its discretization A^d. This is the $\mathcal{S}$-category with

- $ob(A^d) = ob(A)$,
- $\underline{A^d}(X,Y) = \emptyset$ if $X \neq Y$ and $\underline{A^d}(X,X) = \Delta(0)$.

For a simplicial category A let $\mathcal{F}(A,\mathcal{S})$ be the category of simplicial functors from A to $\mathcal{S}$ and their simplicial transformations. Let $i : A^d \to A$ be the inclusion functor. Then we have a pair of adjoints:

$$i^* : \mathcal{F}(A,\mathcal{S}) \to \mathcal{F}(A^d,\mathcal{S}) ,$$

$$Lan_i : \mathcal{F}(A^d,\mathcal{S}) \to \mathcal{F}(A,\mathcal{S}) .$$

Let (L,μ,ϵ) be an $\mathcal{S}$-comonad on $\mathcal{F}(A,\mathcal{S})$ generated by this adjunction. Then we have

PROPOSITION 1.4 *For every simplicial functors $F, G : A \to \mathcal{S}$ there is a natural isomorphism*

$$Coh(A,\mathcal{S})(F,G) \simeq \underline{\mathcal{F}(A,\mathcal{S})}(L_\infty(F),G).$$

PROOF. Let us give an explicit description of $L(F) = Lan_i \cdot i^*(F)$. Namely, for an object X of A we have

$$L(F)(X) = \coprod_{A_0 \in Ob(A)} F(A_0) \times \underline{A}(A_0,X).$$

There is an action

$$F(A_0) \times \underline{A}(A_0,X) \to F(X)$$

induced by the simpliciality of F. It generates the counit ϵ of L.

The comultiplication μ is defined on a summand $F(A_0) \times \underline{A}(A_0,X)$ by

$$F(A_0) \times \Delta(0) \times \underline{A}(A_0,X) \stackrel{id\times 1\times id}{\to} F(A_0) \times \underline{A}(A_0,A_0) \times \underline{A}(A_0,X),$$

where 1 is the canonical map $\Delta(0) \to \underline{A}(A_0,A_0)$ which "picks out the identity map"[19].

Now, we have

$$\underline{\mathcal{F}(A,\mathcal{S})}(L(F),G) \simeq \underline{\mathcal{F}(A,\mathcal{S})}(Lan_i \cdot i^*(F),G) \simeq$$

$$\simeq \underline{\mathcal{F}(A^d,\mathcal{S})}(i^*(F),i^*(G)) \simeq \prod_{A_0\in Ob(A)} \underline{\mathcal{S}}(F(A_0),G(A_0)).$$

But

$$\underline{\mathcal{F}(A,\mathcal{S})}(L_\infty(F),G) \simeq Tot(\underline{\mathcal{F}(A,\mathcal{S})}(L_*(F),G))$$

and

$$\underline{\mathcal{F}(A,\mathcal{S})}(L_p(F),G)) \simeq \prod_{A_p\in ob(A)} \underline{\mathcal{S}}(L_{p-1}(F)(A_p),G(A_p)) \simeq$$

$$\simeq \prod_{A_0,\ldots,A_p} \underline{\mathcal{S}}(\underline{A}(A_0,A_1)\times\ldots\times\underline{A}(A_{p-1},A_p),\underline{\mathcal{S}}(F(A_0),G(A_p))) = Y(T)^p \ ,$$

where $T(X,Y) = \underline{\mathcal{S}}(F(X),G(Y))$ and $Y(T)^*$ is a cosimplicial object from [19]. Hence, we have

$$\underline{\mathcal{F}(A,\mathcal{S})}(L_\infty(F),G) \simeq Tot(Y(T)^*) \simeq \oint_A T(A,A)$$

by the cosimplicial replacement formula from [19]. The last object is exactly $Coh(A,\mathcal{S})(F,G)$.

Q.E.D.

Finally, we have to say something about the procategories which we shall use in our work.

Let Λ be an arbitrary small $\mathcal{S}$-category, $X : \Lambda \to A$ be an $\mathcal{S}$-functor. We can associate with X the following simplicial functor $P_X : A \to \mathcal{S}$:

$$P_X(A) = colim_\lambda \underline{A}(X_\lambda, A) \ .$$

DEFINITION 1.4 *The category of resolutions of A is the $\mathcal{S}$-category $Pro(A)$, which has the simplicial functors from different small $\mathcal{S}$-categories to A as the objects, and the enriched hom-functor*

$$\underline{Pro(A)}(\{X_\lambda\},\{Y_\mu\}) = \int_A \underline{\mathcal{S}}(P_Y(A),P_X(A)) \ .$$

Let now Λ be a directed set. We call an inverse system in A over Λ an arbitrary functor $X : \Lambda' \to A$, where Λ' is the small category associated to Λ. We shall denote such a functor by $\{X_\lambda\}$ if it leads to no confusion.

DEFINITION 1.5 *The procategory pro(A) is the full simplicial subcategory of Pro(A) generated by the inverse systems in A over different directed sets.*

Using the enriched Yoneda lemma [30] it is not hard to show, that

$$\underline{Pro(A)}(\{X_\lambda\},\{Y_\mu\}) = lim_\mu colim_\lambda \underline{A}(X_\lambda, Y_\mu) \ .$$

More detailed information about $pro(A)$ may be found in [16,23,12]

There is an obvious dualization of the construction above. We thus obtain the categories of direct systems $inj(A)$ [23] and of coresolutions $Inj(A)$.

2 Shape theories in a bicategory

Let $\boldsymbol{D}$ be a bicategory with objects $A, B, C, \ldots$. For every A, B let $\boldsymbol{D}(A,B)$ be the category of arrows from A to B. Let $\otimes$ be the functor of composition in $\boldsymbol{D}$:

$$- \otimes - : \boldsymbol{D}(B,C) \times \boldsymbol{D}(A,B) \to \boldsymbol{D}(A,C) \ .$$

For two arrows $S, T \in ob(\boldsymbol{D}(A,B))$ we denote by $D(S,T)$ the set of morphisms in $\boldsymbol{D}(A,B)$ from S to T, that is the set of 2-cells. For each object A, let I_A or simply I, denote the identity arrow on A. These data are connected by a coherent system of canonical isomorphisms [5]. Usually, one does not mention these isomorphisms to shorten the notations. This is possible because of the coherence theorem [34].

For any object A of $\boldsymbol{D}$ the category $\boldsymbol{D}(A,A)$ is monoidal with respect to $\otimes$ and I_A.

DEFINITION 2.1 *A triple (T,μ,η), where T is an object of $\boldsymbol{D}$, $\mu \in D(T \otimes T, T)$ and $\eta \in D(I_A, T)$ is called a monad over A in $\boldsymbol{D}$, provided (T,μ,η) is a monoid in $\boldsymbol{D}(A,A)$. That is*

$$\mu(1 \otimes \eta) = \mu(\eta \otimes 1) = 1 \ , \ \mu(1 \otimes \mu) = \mu(\mu \otimes 1) \ .$$

Let $K : A \to B$ be an arrow, and $\rho \in (K \otimes T, K)$. We shall say that (K, ρ) is a right module over T (or simply a right T-module) if

$$\rho(1 \otimes \eta) = 1 \ , \ \rho(1 \otimes \mu) = \rho(\rho \otimes 1) \ .$$

The definition of left T-module is evident.

DEFINITION 2.2 *Let $K : A \to B$ be an arrow in $\boldsymbol{D}$. We shall say that in $\boldsymbol{D}$ there exists a right extension of $T \in \boldsymbol{D}(A, C)$ along K if the functor $D(- \otimes K, T)$ is representable.*

We shall denote a corresponding representing object (which is unique up to isomorphism) by $Ran(K, T)$. So we have a natural isomorphism

$$\alpha : D(S \otimes K, T) \to D(S, Ran(K, T)) \ .$$

If in $\boldsymbol{D}$ there exist right extensions of any $T \in \boldsymbol{D}(A, C)$, $C \in ob(\boldsymbol{D})$ along K then we say that $\boldsymbol{D}$ is closed on the right with respect to K. In this case, the functor $- \otimes K : \boldsymbol{D}(B, C) \to \boldsymbol{D}(A, C)$ has a right adjoint $Ran(K, -)$.

In the dual situation we give

DEFINITION 2.3 *We shall say that in $\boldsymbol{D}$ there exists a left extension of $T \in \boldsymbol{D}(C, B)$ along K if $D(K \otimes -, T)$ is representable.*

We shall denote a corresponding representing object by $Lan(K, T)$. As above, we have a natural isomorphism

$$\alpha : D(K \otimes S, T) \to D(S, Lan(K, T))$$

and the dual definition of closedness on the left with respect to K.

Thus we have that $\boldsymbol{D}$ is closed on the right (on the left) if it is closed on the right (on the left) with respect to any arrow K and $\boldsymbol{D}$ is biclosed if it is closed both on the right and on the left.

Let now there exist in $\boldsymbol{D}$ a right extension of some $S \in \boldsymbol{D}(A, C)$ and of K itself along K. Then we have the following isomorphisms

$$\alpha : D(Ran(K, S) \otimes K, S) \to D(Ran(K, S), Ran(K, S))$$

$$\alpha : D(K, K) \to D(I, Ran(K, K))$$
$$\alpha : D(Ran(K, K) \otimes Ran(K, K) \otimes K, K) \to$$
$$\to D(Ran(K, K) \otimes Ran(K, K), Ran(K, K)) \ .$$

Define the 2-cells

$$\rho : Ran(K, S) \otimes K \to S \ , \ \eta : I \to Ran(K, K)$$

by the formulas:

$$\rho = \alpha^{-1}(1_{Ran(K,S)}) \ , \ \eta = \alpha(1_K)$$

and

$$\mu : Ran(K, K) \otimes Ran(K, K) \to Ran(K, K)$$

by

$$\mu = \alpha(\rho(1 \otimes \rho)) \ ,$$

when $S = K$.

Dually, if in $\boldsymbol{D}$ there exist a left extension of $S \in \boldsymbol{D}(C, B)$ and of K along K then we have the following 2-cells

$$\rho : K \otimes Lan(K, S) \to S \ , \ \eta : I \to Lan(K, K)$$

and

$$\mu : Lan(K, K) \otimes Lan(K, K) \to Lan(K, K)$$

defined by

$$\rho = \alpha^{-1}(1_{Lan(K,S)}) \ , \ \eta = \alpha(1_K)$$
$$\mu = \alpha(\rho(\rho \otimes 1)) \ .$$

PROPOSITION 2.1 *The 2-cells μ and η define on $Ran(K, K)$ a structure of a monad over B in $\boldsymbol{D}$. Moreover, ρ defines on K a structure of a left module over $Ran(K, K)$.*

In the dual situation we obtain a monad structure on $Lan(K, K)$ over A and a right module structure on K over $Lan(K, K)$.

PROOF. Let A, B, C be the objects of $\boldsymbol{D}$, and let $X \in \boldsymbol{D}(A,B)$, $Y \in \boldsymbol{D}(A,C)$ and $Z, W \in \boldsymbol{D}(B,C)$. If in $\boldsymbol{D}$ there exists a right extension of Y along X then we have the following commutative diagram

$$\begin{array}{ccc}
D(W,Z) \times D(Z \otimes X, Y) & \xrightarrow{(-\otimes X)\times 1} & D(W \otimes X, Z \otimes X) \times D(Z \otimes X, C) \\
1 \times \alpha \downarrow & & \\
D(W,Z) \times D(Z, Ran(X,Y)) & & m \downarrow \\
m \downarrow & & \\
D(W, Ran(X,Y)) & \xrightarrow{\alpha^{-1}} & D(W \otimes X, Y)
\end{array}$$

where m is composition of 2-cells.

Thus for every $g \in D(Z \otimes X, Y)$ and $f \in D(W, Z)$ we have

$$\alpha(g(f \otimes 1)) = \alpha(g)(f) \ . \tag{2}$$

Taking $g = \rho : Ran(K,S) \otimes K \to S$ we have, that

$$\alpha^{-1}(f) = \rho(f \otimes 1_K) \tag{3}$$

for every $f : W \to Ran(K,S)$, if $Ran(K,S)$ does exist.

Applying this formula for μ we thus obtain

$$\rho(\mu \otimes 1_K) = \alpha^{-1}(\mu) = \rho(1 \otimes \rho) \ , \tag{4}$$

where $1 = 1_{Ran(K,K)}$. Putting in (2) $g = \rho(1 \otimes \rho)$ and $f = 1 \otimes \mu$ or $f = \mu \otimes 1$ we have

$$\mu \otimes (1 \otimes \mu) = \alpha(\rho(1 \otimes \rho)(1 \otimes \mu \otimes 1_K)) \tag{5}$$

and

$$\mu \otimes (\mu \otimes 1) = \alpha(\rho(1 \otimes \rho)(\mu \otimes 1 \otimes 1_K)) \tag{6}$$

The formula (4) gives us the following equation

$$(1 \otimes \rho)(1 \otimes \mu \otimes 1_K) = (1 \otimes \rho)(1 \otimes 1 \otimes \rho) \ ,$$

hence

$$\rho(1 \otimes \rho)(1 \otimes 1 \otimes \rho) = \rho((1 \otimes \rho)(1 \otimes \mu \otimes 1_K) = \rho(\mu \otimes 1)(1 \otimes \mu \otimes 1_K) \ .$$

On the other hand

$$\rho(1 \otimes \rho)(1 \otimes 1 \otimes \rho) = \rho((\mu \otimes 1)(1 \otimes \mu \otimes 1_K) =$$

$$= \rho(1 \otimes \rho)(\mu \otimes 1 \otimes 1_K) = \rho(\mu \otimes 1)(\mu \otimes 1 \otimes 1_K) .$$

So by (5) and (6) we obtain

$$\mu(\mu \otimes 1) = \mu(1 \otimes \mu) . \tag{7}$$

Moreover, applying (3) to η we have

$$\rho(\eta \otimes 1_K) = 1_K . \tag{8}$$

This gives us two equations

$$\alpha(\rho(1 \otimes \rho)(\eta \otimes 1 \otimes 1_K)) = 1$$

$$\alpha(\rho(1 \otimes \rho)(1 \otimes \eta \otimes 1_K)) = 1 .$$

Putting again in (2) $g = \rho(1 \otimes \rho)$ and $f = 1 \otimes \eta$ or $f = \eta \otimes 1$ we have

$$\mu(\eta \otimes 1) = \alpha(\rho(1 \otimes \rho)(\eta \otimes 1 \otimes 1_K)) = 1 \tag{9}$$

and

$$\mu(1 \otimes \eta) = \alpha(\rho(1 \otimes \rho)(1 \otimes \eta \otimes 1_K)) = 1 . \tag{10}$$

Thus the formulas (4),(7),(8),(9),(10) give us the desired result.

Q.E.D.

COROLLARY 2.1.1 *Let S and T be two right (left) extensions of K along K, then the monads (T, μ_T, η_T) and (S, μ_S, η_S) obtained as in the proposition 2.1 are canonically isomorphic.*

By analogy with [8,16,33] we give the following

DEFINITION 2.4 *The monad $(Ran(K,K), \mu, \eta)$ is called the codensity monad for K. Dually, $(Lan(K,K), \mu, \eta)$ is called the density monad for K.*

Let $K : A \to B$ be an arrow in $\boldsymbol{D}$. Then K generates two functors: $- \otimes K$ and $K \otimes -$.

DEFINITION 2.5 *Let $T : B \to C$ be an arrow. We shall say that T is K-continuous if for any arrow $S : B \to C$ the functor $- \otimes K$ induces a bijection*

$$\delta : D(S,T) \to D(S \otimes K, T \otimes K) .$$

Dually, $T : C \to A$ is K-cocontinuous if for any arrow $S : C \to A$ the functor $K \otimes -$ induces a bijection

$$\delta : D(S,T) \to D(K \otimes S, K \otimes T) .$$

LEMMA 2.1 *If in $\boldsymbol{D}$ there exists $Ran(K, T\otimes K)$, then T is K-continuous if and only if the 2-cell*

$$r = \alpha(1_{T\otimes K}) : T \to Ran(K, T \otimes K)$$

is an isomorphism.

Dually, if $Lan(K, K \otimes T)$ exists then T is K-cocontinuous if and only if

$$r = \alpha(1_{K\otimes T}) : T \to Lan(K, K \otimes T)$$

is an isomorphism.

PROOF. Immediately from the definitions.

Q.E.D.

Let $K; A \to B$ be as above. The main definition of this section is

DEFINITION 2.6 *Let $T : B \to B$ be an arrow and let $\eta : I \to T$ be a 2-cell. We shall say that the pair (T, η) is a shape theory for K if the following two axioms are fulfilled:*

1. $\eta \otimes 1 : K \to T \otimes K$ is an isomorphism.

2. T is K-continuous.

Dually, let $T : A \to A$ be an arrow and let $\eta : I \to T$ be a 2-cell. We shall say that the pair (T, η) is a coshape theory for K if the following two axioms are fulfilled:

1. $1 \otimes \eta : K \to K \otimes T$ is an isomorphism.

2. T is K-cocontinuous.

PROPOSITION 2.2 *Suppose for an arrow $K \in \boldsymbol{D}(A,B)$ that there exists a*
(co)shape theory (T,η), then one can define 2-cells

$$\mu : T \otimes T \to T \ , \ \rho : T \otimes K \to K \ ,$$

$$(\rho : K \otimes T \to K) \ ,$$

such that (T,μ,η) is a monad for K, (K,ρ) is a left (right) T-module and ρ is an isomorphism.

PROOF. Define for a shape theory (T,η) :

$$\rho = (\eta \otimes 1)^{-1} \ , \ \mu = \delta^{-1}(1 \otimes \rho) \ .$$

The proof that ρ and μ have the necessary properties is analogous to that of the proposition 2.1.

For a coshape theory we have to define

$$\rho = (1 \otimes \eta)^{-1} \ , \ \mu = \delta^{-1}(\rho \otimes 1) \ .$$

Q.E.D.

Thus according to this proposition every (co)shape theory has a canonical monad structure. So we shall say that a triple (T,μ,η) is a (co)shape theory for K having in view that (T,η) is one and μ is its canonical multiplication.

PROPOSITION 2.3 *Let $K : A \to B$ be an arrow admitting a (co)shape theory (T,μ,η) and $S : B \to C$ $(S : C \to A)$ be K-(co)continuous, then there exists a 2-cell*

$$k : S \otimes T \to S \quad (k : T \otimes S \to S)$$

such that (S,k) is a right (left) T-module.

PROOF. Define

$$k = \delta^{-1}(1 \otimes \rho) \ (k = \delta^{-1}(\rho \otimes 1)) \ .$$

The remaining part of the proof is again analogous to that of the proposition 2.1.

Q.E.D.

THEOREM 2.1 *If in $\boldsymbol{D}$ there exists a right (left) extension of K along K, then the pair $(Ran(K,K),\eta)$ $((Lan(K,K),\eta))$, where η is the unit of a codensity (density) monad, is a shape (coshape) theory for K if and only if the 2-cell*

$$\rho : Ran(K,K)\otimes K \to K \quad \textit{is an isomorphism} \tag{11}$$

$$(\rho : K\otimes Ran(K,K) \to K \quad \textit{is an isomorphism})\ . \tag{12}$$

Moreover, if there exists a shape (coshape) theory (T,η) for K then T is a right (left) extension of K along K and (T,μ,η) is isomorphic to the codensity (density) monad for K.

DEFINITION 2.7 *We shall say that an arrow $K\in \boldsymbol{D}(A,B)$ is (co)formal in $\boldsymbol{D}$ if a right (left) extension of K along K exists and satisfies condition (11)((12)).*

Thus we can reformulate the theorem:
for an arrow K there exists a (co)shape theory if and only if K is (co)formal, and this (co)shape theory is multiplicatively isomorphic to $(Ran(K,K),\mu,\eta)$ $((Lan(K,K),\mu,\eta))$.

PROOF. We will prove the theorem for shape theory. The proof for a coshape theory is dual.

Let $Ran(K,K)$ exist and let (11) be satisfied. Then from proposition 2.1 we have that $\eta\otimes 1_K$ is an isomorphism.

Furthermore, formula (3) gives us the equality $\alpha^{-1}(f)=\rho(\delta)(f)$ for every $f\in D(S,Ran(K,K))$. As α^{-1} and ρ are isomorphisms so δ is also and thus $Ran(K,K)$ is K-continuous. Hence $(Ran(K,K),\eta)$ is a shape theory for K.

On the other hand, if $(Ran(K,K),\eta)$ is a shape theory for K, then ρ is an isomorphism because $\rho(\eta\otimes 1)=1$ and $\eta\otimes 1$ is an isomorphism by definition.

Let us prove the second part of the theorem. Indeed, let (T,η) be a shape theory for K, then for every S we have the natural isomorphisms:

$$D(S\otimes K,K)\xrightarrow{\eta\otimes 1_K} D(S\otimes K,T\otimes K)\xrightarrow{\delta^{-1}} D(S,T)\ .$$

Thus T is a right extension of K along K.

Q.E.D.

EXAMPLES. Below we give some examples of bicategories and shape theories.

1. Let $\boldsymbol{Cat}$ be the 2-category of categories, functors and natural transformations. In this bicategory a right extension of K along K is a right Kan extension $Ran_K K$, whereas a left extension is not $Lan_K K$. It is indeed a left extension of K along K in bicategory $\boldsymbol{Cat}^{op}$, which has as objects the categories and

$$\boldsymbol{Cat}^{op}(A,B) = \boldsymbol{Cat}(B,A) \ , \ Cat^{op}(S,T) = Cat(T,S) \ .$$

2. Let now $\boldsymbol{pro}(\boldsymbol{Cat})$ be the following bicategory. The objects of $\boldsymbol{pro}(\boldsymbol{Cat})$ are the categories, the arrows from A to B are the functors from A to $pro(B)$ and 2-cells are their natural transformations. The composition of arrows is defined by

$$A \xrightarrow{S} pro(B) \xrightarrow{pro(T)} pro(pro(C)) \xrightarrow{\sim} pro(C) \ .$$

There is an obvious full embedding

$$p : \boldsymbol{Cat} \to \boldsymbol{pro}(\boldsymbol{Cat}) \ ,$$

which is a homomorphism of bicategories.

There is a dual construction for the category of direct systems. Denote the corresponding bicategory by $\boldsymbol{inj}(\boldsymbol{Cat}^{op})$. Then we have a homomorphism and full embedding of bicategories

$$p : \boldsymbol{Cat}^{op} \to \boldsymbol{inj}(\boldsymbol{Cat}^{op}) \ .$$

3. The similar process leads to the bicategories $\boldsymbol{Pro}(\boldsymbol{Cat})$ and $\boldsymbol{Inj}(\boldsymbol{Cat})$.

4. Finally, let $\boldsymbol{Dist}$ be the bicategory of distributors [6,8,16]. There are the following homomorphisms of bicategories, which are the full embeddings

$$\boldsymbol{Cat} \xrightarrow{p} \boldsymbol{pro}(\boldsymbol{Cat}) \xrightarrow{i} \boldsymbol{Pro}(\boldsymbol{Cat}) \xrightarrow{\Phi_*} \boldsymbol{Dist} \ ,$$

where for a functor $L : A \to Pro(B)$ the distributor $\Phi_*(L) = \phi_L$ is

$$\phi_L(X,Y) = pro(B)(X, L(Y)) = \lim_{\lambda} B(X, \{Y_\lambda\}) \ , \{Y_\lambda\} = L(Y) \ .$$

The composition $\Phi_* \cdot i \cdot p$ is the embedding of Benabou, $\boldsymbol{Cat} \subset \boldsymbol{Dist}$, and is denoted by Φ_* as well [6].

There are also the dual embeddings [6]

$$\boldsymbol{Cat}^{op} \to \boldsymbol{inj}(\boldsymbol{Cat}^{op}) \xrightarrow{i} \boldsymbol{Inj}(\boldsymbol{Cat}^{op}) \xrightarrow{\Phi^*} \boldsymbol{Dist} \ ,$$

defined for a functor $L : A \to Inj(B)$ by

$$\phi^L(X,Y) = inj(B)(L(X), Y) = \lim_{\lambda} B(\{X_\lambda\}, Y) \ , \ \{X_\lambda\} = L(X).$$

Via the homomorphisms Φ_* , Φ^* one usually defines a shape (coshape) category for a functor $K : A \to B$ as the Kleisli category of the codensity monad for ϕ_K (of the density monad for ϕ^K) in $\boldsymbol{Dist}$ [8,16]. Remark however that a homomorphism of bicategories may not preserve right (left) extensions and so the shape (coshape) theories. For example, Φ_* preserves only the pointwise right Kan extensions [8].

5. In any bicategory if K has a left adjoint L with unit $\phi : I \to K \otimes L$ and counit $\psi : L \otimes K \to I$, then $Ran(K,K)$ and $Lan(L,L)$ may be calculated as $K \otimes L$. It is easy to see that $(K \otimes L, \phi)$ is a shape theory for K and a coshape theory for L provided ψ is isomorphism.

As an example of such a situation we can consider the Mardešić shape theory of topological spaces [36]. This theory corresponds to the full embedding of the homotopy categories $\pi(ANR) \to \pi(Top)$. As it was noted in [16], a full subcategory $i : A \subset B$ is dense in the sense of [36] if and only if i has a left adjoint in $\boldsymbol{pro}(\boldsymbol{Cat})$. This left adjoint for $X \in ob(B)$ is given by an A-expansion of X.

In the situation of adjunction a shape (coshape) theory has some very nice properties. For example, the image of such a theory under any homomorphism of bicategories is again a shape theory.

Another situation, related closely with the properties of adjoints is described in the following proposition:

PROPOSITION 2.4 *Let K be an arrow from A to B and let*

$$S : B \to C \ , \ T : C \to B$$

$$(S : A \to C \ , \ T : C \to A)$$

be a pair of adjoints with unit $\phi : I \to T \otimes S$ and counit $\psi : S \otimes T \to I$, such that $\phi \otimes 1_K$ $(1_K \otimes \phi)$ is an isomorphism and S is K-continuous (T is K-cocontinuous).

Then $(T \otimes S, \phi)$ is a shape (coshape) theory for K and the monad $(T \otimes S, \mu, \phi)$, generated by the adjunction, is isomorphic to the codensity (density) monad of K.

PROOF. The proof for a shape theory follows from the fact that $T \otimes S$ is K-continuous as T is a right adjoint. Thus $(T \otimes S, \phi)$ is a shape theory for K. It is not hard to check from the definition of adjointness that μ and the multiplication in the codensity monad coincide.

Q.E.D.

This proposition allows us to prove the equivalence of our axiomatics for a shape theory and that of [8].

PROPOSITION 2.5 *If $(S, \boldsymbol{S_K})$ is a shape theory of a functor K in the sense of [8](see section 1), then $\boldsymbol{S_K}$ is isomorphic to the Kleisli category of the codensity monad in **Dist** of ϕ_K and S is the corresponding canonical functor.*

*Conversely, if (T, μ, η) is a shape theory for ϕ_K in **Dist** , then (S, Kl_T) is a shape theory in the Bourn-Cordier sense.*

PROOF. The first part of the proposition is a corollary of 2.4. The second follows from the lemma 2.1 and the remark on the pages 174-175 of [8], that $\phi_S \otimes r$ is invertible if and only if r is invertible, where r is the 2-cell from the lemma 2.1.

Q.E.D.

3 Homotopy coherent bicategory of simplicial distributors

This section is devoted to the construction of a bicategory of distributors convenient for the development of the strong shape theories.

We start from the bicategory of simplicial distributors $\boldsymbol{SDist}$. The objects of $\boldsymbol{SDist}$ are the simplicial categories. If A and B are two $\mathcal{S}$-categories then $\boldsymbol{SDist}(A,B)$ is the $\mathcal{S}$-category of $\mathcal{S}$-distributors from A to B, that is the $\mathcal{S}$-category of $\mathcal{S}$-functors from $B^{op} \times A$ to $\mathcal{S}$. For two simplicial distributors S and T, let $SDist(S,T)$ be the simplicial set of $\mathcal{S}$-natural transformations from S to T. The composition of the distributors is given by the coend

$$S \otimes T(X,Y) = \textstyle\int^{C} S(X,Z) \times T(Z,Y) \ .$$

The identity arrow $I \in \boldsymbol{SDist}(A,A)$ is given by the enriched *hom*-functor

$$\underline{A} : A^{op} \times A \to \mathcal{S} \ .$$

In addition, we have a right extension $Ran(S,T)$ of T along S for any $S \in \boldsymbol{SDist}(A,B)$ and $T \in \boldsymbol{SDist}(A,C)$ given by the end [6,8,16]

$$Ran(S,T)(X,Y) = \textstyle\int_{A} \underline{\mathcal{S}}(S(X,Z),T(X,Z)) \ .$$

Similarly, for $S \in \boldsymbol{SDist}(B,A)$ and $T \in \boldsymbol{SDist}(C,A)$

$$Lan(S,T)(X,Y) = \textstyle\int_{A} \underline{\mathcal{S}}(S(Z,X),T(Z,Y)) \ ,$$

and we have the $\mathcal{S}$-natural isomorphisms

$$SDist(S \otimes K, T) \simeq SDist(S, Ran(K,T)) \ ,$$

$$SDist(K \otimes S, T) \simeq SDist(S, Lan(K,T)) \ ,$$

For two simplicial categories A, B define the following $\mathcal{S}$-endofunctors on $\boldsymbol{SDist}(A,B)$.

If K is an $\mathcal{S}$-distributor from A to B then put:

$$L'K(X,Y) = \coprod_{Z \in Ob(A)} K(X,Z) \times \underline{A}(Z,Y)$$

$$L''K(X,Y) = \coprod_{Z \in Ob(B)} \underline{B}(X,Z) \times K(Z,Y) \ .$$

$$L'''K(X,Y) = \coprod_{Z_0 \in Ob(B), Z_1 \in Ob(A)} \underline{B}(X,Z_0) \times K(Z_0,Z_1) \times \underline{A}(Z_1,Y)$$

It is evident, that L', L'', L''' are $\mathcal{S}$-endofunctors on $\boldsymbol{SDist}(A,B)$. Furthermore, L', L'', L''' have $\mathcal{S}$-comonad structures. More precisely, let

$$\eta' : \coprod_{Z \in Ob(A)} K(X,Z) \times \underline{A}(Z,Y) \to K(X,Y)$$

be defined by the action

$$K(X,Z) \times \underline{A}(Z,Y) \to K(X,Y) \ ,$$

and

$$\rho' : \coprod_{Z \in Ob(A)} K(X,Z) \times \underline{A}(Z,Y) \to \coprod_{Z_0, Z_1 \in Ob(A)} K(X,Z_0) \times \underline{A}(Z_0,Z_1) \times \underline{A}(Z_1,Y)$$

be defined on the summand $K(X,Z) \times \underline{A}(Z,Y)$ by the morphism

$$K(X,Z) \times \Delta(0) \times \underline{A}(Z,Y) \to K(X,Z) \times \underline{A}(Z,Z) \times \underline{A}(Z,Y) \ ,$$

induced by $\Delta(0) \to \underline{A}(Z,Z)$. It is not hard to check that L' is a comonad with η as a counit and ρ as a comultiplication. The comonad structures on L'' and L''' are defined analogously.

LEMMA 3.1 *Let A^d be the discretization of an $\mathcal{S}$-category A, and let $i : A^d \to A$ be the corresponding inclusion functor. Then we have a pair of adjoints:*

$$(1 \times i)^* : \boldsymbol{SDist}(A,B) \to \boldsymbol{SDist}(A^d,B) \ ,$$

$$Lan_{1 \times i} : \boldsymbol{SDist}(A^d,B) \to \boldsymbol{SDist}(A,B) \ ,$$

where $(1 \times i)^*$ is the restriction functor. Then (L', ρ', η') is induced by the adjunction above.

Analogously, (L'', ρ'', η'') and (L''', ρ''', η''') are induced by the adjunctions generated by the inclusions

$$j^{op} \times 1 : (B^{op})^d \times A \to B^{op} \times A$$

and

$$j^{op} \times i : (B^{op} \times A)^d \to B^{op} \times A$$

respectively.

PROOF. Proof is immediate from the definitions.

Q.E.D.

LEMMA 3.2 *For the monads (L', ρ', η') , (L'', ρ'', η'') , (L''', ρ''', η''') the natural transformations η'_∞ , η''_∞ , η'''_∞ are level homotopy equivalences.*

PROOF. The proof for L' is standard via the existence of the simplicial deformation retraction of simplicial objects

$$(1 \times i)^*(L'_*(K)) \to (1 \times i)^*(K_*)$$

induced by augmentation $\eta' : L'_0(K) \to K$, where K_* is a constant simplicial object.

Indeed, the pair of adjoints $(1 \times i)^*$ and $Lan_{1\times i}$ induces a simplicial monad $M' = ((1 \times i)^* \cdot Lan_{1\times i}, \mu, \epsilon)$ on $\boldsymbol{SDist}(A^d, B)$. Moreover, the simplicial distributor $(1 \times i)^*(K)$ has an obvious M'-algebra structure. Hence, we can consider May's bar construction $B_*(M', M', (1 \times i)^*(K))$. It is easy to verify now that there is a natural isomorphism of simplicial objects

$$B_*(M', M', (1 \times i)^*(K)) \simeq (1 \times i)^*(L'_*(K))$$

and so the existence of the desired deformation retraction follows from a well known resolvent property of the bar-construction [37,38]. This deformation retraction gives us the necessary homotopy equivalence after simplicial realization.

The proof for L'' and L''' is analogous.

Q.E.D.

DEFINITION 3.1 *For two simplicial distributors R, T from A to B define the simplicial set of coherent transformations from R to T by*

$$\underline{CHDist}(R,T) = \oint_{B^{op}\times A} \underline{S}(R(X,Y), S|T(X,Y)|)$$

and the set of homotopy classes of coherent transformation by

$$CHDist(R,T) = \pi_0(\underline{CHDist}(R,T)) \ ,$$

where $S(-)$ and $|-|$ are the singular complex functor and geometric realization functor respectively.

REMARK. We use in this definition the functor $S|-|$ to avoid the difficulties associated with a possible "wrong" homotopy type of the coherent end $\oint_A T(A,A)$ if some $T(A,B)$ are not fibrant.

It is evident that $\underline{CHDist}(-,-)$ is a simplicial functor and so we can consider $CHDist$ as an endodistributor T_{AB} on $\pi(\boldsymbol{SDist}(A,B))$

$$T_{AB} : (\pi(\boldsymbol{SDist}(A,B))^{op} \times \pi(\boldsymbol{SDist}(A,B)) \to \mathcal{S}et \ .$$

PROPOSITION 3.1 *There are the 2-cells of endodistributors on $\pi(\boldsymbol{SDist}(A,B))$*

$$m : T_{AB} \otimes T_{AB} \to T_{AB} \ , \ e : I \to T_{AB} \ ,$$

which define T_{AB} as a monad in $\boldsymbol{Dist}$.

Moreover, the canonical functor P from $\pi(\boldsymbol{SDist}(A,B))$ to the Kleisli category of T_{AB} is the localization functor at the class of the level weak equivalences of $\mathcal{S}$-distributors.

PROOF. Note, that by proposition 1.4 and lemma 3.1 we have an $\mathcal{S}$-natural isomorphism

$$\underline{CHDist}(R,T) \simeq SDist(L'''_{\infty}R, S|T|) \ .$$

This allows us to define $e : I \to T_{AB}$ by the mappings induced by η'''_{∞} and canonical inclusion of a simplicial set to the singular complex of its realization.

Furthermore, the cosimplicial simplicial set

$$
\begin{aligned}
SDist(L_*'''R, S|T|) &= \textstyle\int_{B^{op}\times A}\underline{S}(L_*'''R(X,Z), S|T(X,Z)|) \\
&\simeq \textstyle\int_{B^{op}\times A}\underline{\mathcal{K}a}_{\mathcal{S}}(L_*'''|R(X,Z)|, |T(X,Z)|) \\
&\simeq \textstyle\oint_{B^{op}\times A}(\underline{\mathcal{K}a}_{\mathcal{S}}(|R(X,Z)|, |T(X,Z)|)
\end{aligned}
\tag{13}
$$

and hence, is fibrant [2,19]. On the other hand this gives us a homotopy associative and homotopy unitary composition on T_{AB} (see [2,14] for the definition of this composition for simplicial transformations between simplicial diagrams in $\mathcal{K}a$).

Furthermore, (13) shows, that the canonical functor

$$\pi(\boldsymbol{SDist}(A,B)) \to Kl_{T_{AB}}$$

inverts the level weak equivalences of simplicial distributors, because

$$SDist(L_\infty'''R, S|T|) \simeq Tot(SDist(L_*'''R, S|T|)).$$

Let now $F : \pi(\boldsymbol{SDist}(A,B)) \to C$ be a functor inverting the level weak equivalences. Then define $G : Kl_{T_{AB}} \to C$ on objects by $G(T) = F(T)$. A morphism $f : R \to T$ in $Kl_{T_{AB}}$ is specified as a homotopy class of a morphism $\phi : L_\infty'''R \to S|T|$. Thus we can take $G(f)$ as a morphism fitting commutatively into the diagram:

$$
\begin{array}{ccc}
F(L_\infty'''R) & \overset{F(\phi)}{\to} & F(S|T|) \\
F(\eta_\infty''') \downarrow \simeq & & \uparrow \simeq \\
F(R) & \overset{G(f)}{\to} & F(T)
\end{array}
$$

It is evident that we thus defined $Kl_{T_{AB}}$ as the localization of $\pi(\boldsymbol{SDist}(A,B))$

at the class of the level weak equivalences.

Q.E.D.

DEFINITION 3.2 *Let A, B be two simplicial categories. Define the category*

$$\boldsymbol{CHDist}(A,B) = Kl_{T_{AB}}\ .$$

So the objects of $\boldsymbol{CHDist}(A,B)$ are all simplicial distributors from A to B, and the morphisms are the homotopy classes of their coherent transformations. Remark that this definition of coherent transformations differs from that of coherent transformation of simplicial functors from [19]. It is not hard to verify, however, that if T is a locally Kan simplicial distributor, that is $T(X,Y)$ is a Kan simplicial set for every X,Y, then there is a level homotopy equivalence

$$\underline{CHDist}(R,T) \sim Coh(B^{op} \times A, \mathcal{S})(R,T) \ .$$

DEFINITION 3.3 *Let $T : A \to B$ and $R : B \to C$ be two simplicial distributors. Define their coherent composition as*

$$T \otimes_H R(X,Y) = \oint^B T(X,Z) \times R(Z,Y) \ .$$

Let $T : A \to B$ and $R : A \to C$ be two simplicial distributors. We shall call the right coherent extension of R along T the following simplicial distributor, $CHRan(T,R) : B \to C$,

$$CHRan(T,R)(X,Y) = \oint_A \underline{\mathcal{S}}(T(Y,Z), S|R(X,Z)|) \ .$$

Analogously, the left coherent extension of $T : B \to A$ along $R : C \to A$ is $CHLan(R,T) : B \to C$

$$CHLan(R,T)(X,Y) = \oint_A \underline{\mathcal{S}}(R(Z,X), S|T(Z.Y)|) \ .$$

LEMMA 3.3 *There are the following $\mathcal{S}$-natural isomorphisms*

$$L'_\infty(L''_\infty(T)) \simeq L''_\infty(L'_\infty(T)) \simeq L'''_\infty(T) \tag{14}$$

$$T \otimes_H S \simeq L'_\infty(T) \otimes S \simeq T \otimes L''_\infty(S) \tag{15}$$

$$L'_\infty(S \otimes T) \simeq S \otimes L'_\infty(T) \tag{16}$$

$$L''_\infty(S \otimes T) \simeq L''_\infty(S) \otimes T \tag{17}$$

$$CHRan(T,R) \simeq Ran(L'_\infty(T), S|R|) \tag{18}$$

$$CHLan(R,T) \simeq Lan(L''_\infty(R), S|T|) \ . \tag{19}$$

PROOF. Let $E_{*,*,*}(X,Y)$ be a trisimplicial set defined by

$$E_{p,q,r}(X,Y) = L'_p(L''_q(T))(X,Y)_r$$

with the obvious face and degeneracy maps. Then $L'_\infty(L''_\infty(T))(X,Y)$ is isomorphic to $diag(diag_{23}E_{*,*,*}(X,Y))$, where $diag_{23}$ is the diagonal functor with respect to two last variables. On the other hand $L'''_\infty(T)(X,Y)$ is isomorphic to $diag(diag_{12}E_{*,*,*}(X,Y))$, where $diag_{12}$ is the diagonal functor with respect to two first variables. This implies the isomorphism (14).

The isomorphism (15) follows readily from the coend formulas for $\otimes$ and $\otimes_H$ and simplicial replacement formula for coherent coends. Analogously, the end formulas for the simplicial *Ran* (*Lan*) and *CHRan* (*CHLan*) and cosimplicial replacement formula for coherent ends lead us to the isomorphisms (18,19).

Finally, the isomorphism (16) follows from the following calculation

$$\int^p L'_p(S\otimes T)(X,Y)\times\Delta(p)\simeq$$

$$\simeq\int^p(\coprod_{Z_0,\ldots,Z_p}(\int^Z S(X,Z)\times T(Z,Z_0))\times\underline{A}(Z_0,Z_1)\times\ldots\times\underline{A}(Z_p,Y))\times\Delta(p)\simeq$$

$$\simeq\int^Z(S(X,Z)\times(\int^p(\coprod_{Z_0,\ldots,Z_p}T(Z,Z_0))\times\underline{A}(Z_0,Z_1)\times\ldots\times\underline{A}(Z_p,Y))\times\Delta(p))\simeq$$

$$\simeq S\otimes L'_\infty(T)(X,Y)\ .$$

The proof of (17) is analogous.

Q.E.D.

The main theorem of this section is

THEOREM 3.1 *The simplicial bifunctor* $-\otimes_H-$ *induces a bifunctor*

$$-\otimes_h-:\boldsymbol{CHDist}(B,C)\times\boldsymbol{CHDist}(A,B)\to\boldsymbol{CHDist}(A,C)\ .$$

There are natural isomorphisms

$$(S\otimes_h R)\otimes_h T\simeq S\otimes_h(R\otimes_h T)\ ,$$

$$T \otimes_h I \simeq T \simeq I \otimes_h T \ ,$$

which satisfy the coherency conditions.

Moreover, we have the following natural isomorphisms

$$CHDist(T \otimes_h K, R) \simeq CHDist(T, CHRan(K, R)) \ ,$$

$$CHDist(K \otimes_h T, R) \simeq CHDist(T, CHLan(K, R)) \ ,$$

So we have a biclosed bicategory ***CHDist*** *with the simplicial categories as the objects, the simplicial distributors as the arrows , the homotopy classes of their coherent transformations as 2-cells and $\otimes_h$ as the composition of arrows. We shall call this bicategory the homotopy coherent bicategory of simplicial distributors.*

PROOF. To prove the first statement of the theorem it is sufficient to remark that $\otimes_H$, being a coherent coend, preserves level weak equivalences of simplicial distributors [10,19].

The coend formula gives us the associativity condition as well. The isomorphism $T \otimes_h I \simeq T$ is obtained as follows

$$T \otimes_h I \simeq L'_\infty(T) \otimes I \simeq L'_\infty(T) \overset{\eta'_\infty}{\simeq} T \ ,$$

because η'_∞ is a level homotopy equivalence and hence is an isomorphism in ***CHDist***. The proof for $I \otimes_h T$ is analogous. The coherency conditions may be checked immediately.

Finally, using lemma 3.3 we obtain

$$CHDist(T \otimes_h K, R) \simeq SDist(L'''_\infty(T \otimes_h K), S|R|) \simeq$$

$$\simeq SDist(L''_\infty(L'_\infty(T) \otimes L'_\infty(K)), S|R|) \simeq$$

$$\simeq SDist(L'''_\infty(T) \otimes L'_\infty(K), S|R|) \simeq SDist(L'''_\infty(T), Ran(L'_\infty(K), S|R|)) \simeq$$

$$\simeq CHDist(T, CHRan(K, R)) \ .$$

The proof for $CHLan$ is analogous.

Q.E.D.

Let $K : A \to B$ be a simplicial functor between two $\mathcal{S}$-categories. Then we can define two simplicial distributors $\phi_K : A \to B$ and $\phi^K : B \to A$.

$$\phi_K(x, y) = \underline{B}(x, K(y)) \ , \ \phi^K(x, y) = \underline{B}(K(x), y) \ .$$

PROPOSITION 3.2 *There are the natural isomorphisms in* $\boldsymbol{CHDist}$

$$\phi_K \otimes_h \phi_L \simeq \phi_{K \cdot L} \ , \ \phi^K \otimes_h \phi^L \simeq \phi^{L \cdot K} \ ,$$

and weak equivalences of simplicial sets

$$(T \otimes_h \phi_K)(x,y) \simeq T(x, K(y)) \ , \ (\phi^K \otimes_h T)(x,y) \simeq T(K(x), y) \ ,$$

for every simplicial distributor T.

PROOF.

$$(T \otimes_h \phi_K)(x,y) \simeq (L'_\infty(T) \otimes \phi_K)(x,y) \simeq L'_\infty(T)(x, K(y)) \ .$$

The last simplicial set is weakly equivalent to $T(x, K(y))$ via η'_∞.

Q.E.D.

PROPOSITION 3.3 *Let* $K, L : A \to B$ *be two simplicial functors such that* ϕ_L *is a locally Kan simplicial distributor. Then there is a natural homotopy equivalence*

$$Coh(A,B)(K,L) \sim \underline{CHDist}(\phi_K, \phi_L) \ .$$

PROOF. The proof is an easy consequence of the coherent end formula, coherent Yoneda lemma and the fact that for a Kan simplicial set k the natural inclusion of k to $S|k|$ is a homotopy equivalence.

Q.E.D.

Let $K : A \to B$ be a simplicial functor.

DEFINITION 3.4 *We shall say that a simplicial distributor*

$$T : B \to C \ \ (T : C \to B)$$

is strongly K*-continuous (strongly* K*-cocontinuous) provided* T *is* ϕ_K*-continuous (*ϕ^K*-cocontinuous) in* $\boldsymbol{CHDist}$.

A simplicial functor

$$T : B \to C$$

is strongly K*-continuous (strongly* K*-cocontinuous) if* ϕ_T *is a strongly* K*-continuous distributor (*ϕ^T *is a strongly* K*-cocontinuous distributor).*

Finally, we can give the definition of a strong shape (coshape) theory for a simplicial functor $K : A \to B$. In analogy with the nonenriched (or rather trivially enriched) situation we define

DEFINITION 3.5 *A strong shape (coshape) theory for K will be a simplicial distributor $T : B \to B$ together with a coherent transformation $\eta : I \to T$ such that:*

- η induces a weak equivalence of simplicial sets

$$\underline{B}(X, K(Y)) \to T(X, K(Y))$$

$$(\underline{B}(K(X), Y) \to T(K(X), Y)\ )$$

for every $X \in ob(B)$, $Y \in ob(A)$,

- T is strongly K-continuous (strongly K-cocontinuous).

Thus a strong shape (coshape) theory for K is a shape (coshape) theory for $\phi_K(\phi^K)$ in $\boldsymbol{CHDist}$. If a strong shape (coshape) theory exists for K then, according to theorem 2.1 the canonical morphism

$$CHRan(\phi_K, \phi_K)\otimes_h\phi_K \to \phi_K$$

$$(\phi^K\otimes_h CHLan(\phi^K, \phi^K) \to \phi^K)$$

is an isomorphism and we will say that K is strongly formal (strongly coformal) functor.

We thus obtain

THEOREM 3.2 *For a simplicial functor $K : A \to B$ there exists a strong shape (coshape) theory if and only if K is strongly formal (strongly coformal) and this theory is isomorphic to the codensity (density) monad of ϕ_K (ϕ^K) in $\boldsymbol{CHDist}$.*

DEFINITION 3.6 *We say that a simplicial functor $K : A \to B$ is a weak full embedding if it induces a weak equivalence*

$$\underline{A}(X, Y) \to \underline{B}(K(X), K(Y))$$

for every $X, Y \in ob(A)$.

PROPOSITION 3.4 *Every weak full embedding is strongly formal and strongly coformal.*

PROOF. We have the following chain of homotopy equivalences and isomorphisms

$$CHRan(\phi_K, \phi_K)\otimes_h \phi_K(X,Y) \to CHRan(\phi_K, \phi_K)(X, K(Y)) \simeq$$

$$\simeq \oint_A \underline{Ka}_{\mathcal{S}}(|\underline{B}(K(Y), K(Z))|, |\underline{B}(X, K(Z))|) \to$$

$$\to \oint_A \underline{Ka}_{\mathcal{S}}(|\underline{A}(Y,Z)|, |\underline{B}(X, K(Z))|) \simeq \oint_A \underline{S}(\underline{A}(Y,Z), S|\underline{B}(X, K(Z))|).$$

The last is homotopy equivalent by the coherent Yoneda lemma to $S|\underline{B}(X, K(Y))|$, which is weakly equivalent to $\phi_K(X,Y)$.

The proof of strong coformality is analogous.

Q.E.D.

Remark now that for any monad (T, μ, η) in ***CHDist*** over a simplicial category B, we can construct a monad $(\pi(T), \pi(\mu), \pi(\eta))$ in ***Dist*** over πB by passing to the set of connected components. We shall call this monad a homotopy monad of (T, μ, η). We can apply this to the codensity(density) monad in ***CHDist*** and we thus obtain the following definition.

DEFINITION 3.7 *Let $K : A \to B$ be a strongly formal (coformal) functor. Then we will call the Kleisli category of the homotopy monad of its codensity (density) monad in **CHDist** the strong shape (coshape) category for K. We will denote this category by Ssh_K (Csh_K).*

The canonical functor

$$Ss_K : \pi B \to Ssh_K (Cs_K : \pi B \to Csh_K)$$

will be called the strong shape (coshape) functor.

The basic examples of strong shape (coshape) categories will be considered in the next sections. Here we remark only one important property of strong shape categories, namely the existence of a comparison functor between strong and weak shape categories.

PROPOSITION 3.5 *Let $K : A \to B$ be a strongly formal (coformal) functor. Then $\pi K : \pi A \to \pi B$ is formal (coformal) and the canonical functor*

$$W : Ssh_K \to \boldsymbol{S}_{\pi K}$$

fits commutatively into the diagram in **Cat** *:*

$$\begin{tikzcd} & \pi B \arrow[dl, "Ss_K"'] \arrow[dr, "S"] & \\ Ssh_K \arrow[rr] & & \boldsymbol{S}_{\pi K} \end{tikzcd}$$

A similar diagram exists for the coshape theories.

PROOF. The proof is immediate.

Q.E.D.

4 Strong shape theory and strong resolutions

Let B be a simplicial category. Let Λ, M be two small simplicial categories and let

$$X : \Lambda \to B, Y : M \to B$$

be two simplicial functors. We shall denote such functors by $\{X_\lambda\}, \{Y_\mu\}$ respectively.

We can associate with these functors two simplicial functors from B to $\mathcal{S}$:

$$P(Z) = hocolim_\lambda \underline{B}(X_\lambda, Z) \ , Q(Z) = hocolim_\mu \underline{B}(Y_\mu, Z) \ .$$

Then define

$$\underline{\mathcal{CPH}(B)}(\{X_\lambda\}, \{Y_\mu\}) = \oint_B \underline{\mathcal{S}}(Q(Z), S|P(Z)|) \ .$$

As in proposition 3.1 we establish that on $\underline{\mathcal{CPH}(B)}$, there is a natural multiplication, which is associative up to homotopy and has a homotopy unity. Thus we can consider a category $\mathcal{CPH}(B)$ with the simplicial functors of the type $X : \Lambda \to B$, $Y : M \to B$ as objects and with

$$\mathcal{CPH}(B)(\{X_\lambda\},\{Y_\mu\}) = \pi_0(\underline{\mathcal{CPH}(B)})(\{X_\lambda\},\{Y_\mu\})$$

as the set of morphisms. The composition is induced by the multiplication on $\underline{\mathcal{CPH}(B)}$ and the identity is given by the identity natural transformation of $\{X_\lambda\}$.

DEFINITION 4.1 *We call $\mathcal{CPH}(B)$ the category of strong resolutions of B.*

LEMMA 4.1 *There is a homotopy equivalence:*

$$\underline{\mathcal{CPH}(B)}(\{X_\lambda\},\{Y_\mu\}) \sim S(holim_\mu|hocolim_\lambda \underline{B}(X_\lambda, Y_\mu)|) \ .$$

If Λ , M are trivially simplicial enriched and cofiltered then

$$\underline{\mathcal{CPH}(B)}(\{X_\lambda\},\{Y_\mu\}) \sim S(holim_\mu|colim_\lambda \underline{B}(X_\lambda, Y_\mu)|) \ .$$

If in addition, $\underline{B}(X_\lambda, Y_\mu)$ is Kan for every λ , μ then

$$\underline{\mathcal{CPH}(B)}(\{X_\lambda\},\{Y_\mu\}) \sim holim_\mu colim_\lambda \underline{B}(X_\lambda, Y_\mu) \ .$$

PROOF.

$$\underline{\mathcal{CPH}(B)}(\{X_\lambda\},\{Y_\mu\}) = \oint_B \underline{S}(Q(Z), S|P(Z)|) =$$

$$= \oint_B \underline{S}(hocolim_\mu \underline{B}(Y_\mu, Z), S|P(Z)|)$$

But $\oint_B \underline{S}(\underline{B}(Y_\mu, Z), S|P(Z)|)$ is Kan [19] and by the coherent Yoneda lemma is homotopy equivalent to $S|P(Y_\mu)|$. Hence

$$holim_\mu \oint_B \underline{S}(\underline{B}(Y_\mu, Z), S|P(Z)|) \simeq S(holim_\mu|hocolim_\lambda \underline{B}(X_\lambda, Y_\mu)|).$$

The second equivalence follows from the fact that in the cofiltered case the homotopy colimit is weakly equivalent to the colimit [10].

The third homotopy equivalence follows immediately as the cofiltered limit of Kan simplicial sets is a Kan simplicial set.

Q.E.D.

DEFINITION 4.2 *Let $K : A \to B$ be a simplicial functor, $\{X_\lambda\}$ be an object of $\mathcal{CPH}(B)$. Let X be an object of B and let p be a 0-simplex of $hocone(X, \{X_\lambda\})$ [9]. We denote it by $p : X \to \{X_\lambda\}$ and call it a homotopy coherent cone over X.*

We say that the pair $(\{X_\lambda\}, p)$ is a strong K-resolution of X if the natural map

$$p^* : hocolim_\lambda \underline{B}(X_\lambda, K(P)) \to \underline{B}(X, K(P)) \tag{20}$$

induced by p, is a weak equivalence for every $P \in ob(A)$.

THEOREM 4.1 *Let $K : A \to B$ be a simplicial functor. Suppose that for an object X of B there is an object $P = \{P_\lambda\}$ of $\mathcal{CPH}(A)$ together with a homotopy coherent cone $p : X \to K(P) = \{K(P_\lambda)\}$ such that $(K(P), p)$ is a strong K-resolution of X. Similarly, let $K(Q) = (\{K(Q_\mu)\}, q)$ be a strong K-resolution of Y.*

Then there is a homotopy equivalence

$$CHRan(\phi_K, \phi_K)(X, Y) \sim \underline{\mathcal{CPH}}(B)(K(P), K(Q))$$

if K is strongly formal.

PROOF. We have the following chain of isomorphisms and homotopy equivalences:

$$CHRan(\phi_K, \phi_K)(X, Y) = \oint_A \underline{S}(\underline{B}(Y, K(P)), S|\underline{B}(X, K(P))|) \simeq$$

$$\simeq \oint_A \underline{Ka}_{\mathcal{S}}(|\underline{B}(Y, K(P))|, |\underline{B}(X, K(P))| \sim$$

$$\sim holim_\mu \oint_A \underline{Ka}_{\mathcal{S}}(|\underline{B}(K(Q_\mu), K(P))|, |\underline{B}(X, K(P))| \simeq$$

$$\simeq holim_\mu CHRan(\phi_K, \phi_K)(X, K(Q_\mu)).$$

From the strong formality of K, we have a homotopy equivalence

$$holim_\mu CHRan(\phi_K, \phi_K)(X, K(Q_\mu)) \sim holim_\mu S|\underline{B}(X, K(Q_\mu)| \sim$$

$$\sim holim_\mu S|hocolim_\lambda \underline{B}(K(P_\lambda), K(Q_\mu)| \ .$$

Q.E.D.

The following results are the homotopy coherent analogues of some shape-theoretic results [16,36] and yields some more understanding of the notion of strong K-continuous functor.

PROPOSITION 4.1 *Let $K : A \to B$ be a simplicial functor, $T : B \to C$ be a strongly K-continuous and locally Kan distributor. Let $(\{Y_\mu\}, q)$ be a strong K-resolution of Y. Then q induces a natural homotopy equivalence*

$$T(X,Y) \to holim_\mu T(X,Y_\mu)$$

Let $L : B \to C$ be a strongly K-continuous functor such that ϕ_L is locally Kan. Then q induces a natural homotopy equivalence

$$L(Y) \to holim_\mu L(Y_\mu)$$

if this homotopy limit exists in C.

PROOF.

$$CHRan(\phi_K, \phi_T \otimes_h \phi_K)(X,Y) \sim \oint_A \underline{S}(\underline{B}(Y, K(a)), S|T(X, K(a))|) \sim$$

$$\sim holim_\mu \oint_A \underline{S}(\underline{B}(Y_\mu, K(a)), S|T(X, K(a))|) \sim$$

$$\sim holim_\mu CHRan(\phi_K, T \otimes_h \phi_K)(X, Y_\mu) \ .$$

As T is strong K-continuous and locally Kan, the last object is homotopy equivalent to $holim_\mu T(X,Y_\mu)$ by lemma 2.1.

The remaining part of the proposition is evident.

Q.E.D.

Let now $K : A \to B$ be a weak full embedding. Suppose that for each object X of B, there is an object $P(X) = \{P_\lambda\}$ of $\mathcal{CPH}(A)$ together with a homotopy coherent cone $p : X \to K(P(X)) = \{K(P_\lambda)\}$ such that $(K(P(X)), p)$ is a strong K-resolution of X. Then there is a natural weak equivalence:

$$\underline{B}(X, K(-)) \leftarrow hocolim_\lambda \underline{A}(P_\lambda, -)$$

and hence an isomorphism

$$\mathcal{CPH}(B)(X, K(-)) \simeq \mathcal{CPH}(A)(P(X), -) .$$

Thus the theorem 4.1 and the arguments of [16, p.57] give us

THEOREM 4.2 *The correspondence $X \to P(X)$ provides a functor*

$$P : \pi(B) \to \mathcal{CPH}(A).$$

The strong shape category of the functor K is isomorphic to the category C defined as follows:

- *$ob(C) = ob(B)$*
- *for $X, Y \in ob(B)$*

$$C(X, Y) = \mathcal{CPH}(A)(P(X), P(Y))$$

Under the conditions of theorem 4.2, we can give the following characterization of K-continuous functors.

THEOREM 4.3 *Let C be a simplicial category with small homotopy limits. Let $T : B \to C$ be a simplicial functor such that $\phi_{(T\cdot K)}$ is locally Kan distributor.*

Then T is strong K-continuous if and only if for any object X of B and any strong K-resolution $(\{K(P_\lambda)\}, p)$ of X, we have a homotopy equivalence

$$p_* : T(X) \to holim_\lambda T(K(P_\lambda)). \tag{21}$$

PROOF. Proposition 4.1 gives us the proof in one direction.

If (21) is satisfied, then the strong K-continuity of T follows easily from the coherent Yoneda lemma and lemma 2.1:

$$CHRan(\phi_K, \phi_T \otimes_h \phi_K)(X, Y) \sim \oint_A \underline{S}(\underline{B}(Y, K(a)), S|\underline{C}(X, TK(a))|) \sim$$

$$\sim holim_\mu \oint_A \underline{S}(\underline{B}(K(P_\mu, K(a)), S|\underline{C}(X, TK(a))|) \sim$$

$$\sim holim_\mu S|\underline{C}(X, TK(P_\mu))| \sim \underline{C}(X, holim_\mu TK(P_\mu)) \sim \phi_T(X, Y)$$

Q.E.D.

There is an evident dualization of the constructions above for the case of strong coshape theory. Thus we have a category of strong coresolutions $\mathcal{CIH}(B)$ and an isomorphism:

$$\mathcal{CIH}(B)(\{X_\lambda\},\{Y_\mu\}) \simeq \pi(holim_\lambda S|hocolim_\mu \underline{B}(X_\lambda, Y_\mu)|) \ ,$$

the notion of strong K-coresolution and dual versions of all the results above. We omit the details as they should be clear.

Now we are able to give some examples of strong shape and strong coshape theories.

EXAMPLES.

1. Let $K : A \to B$ be a simplicial functor, having a left simplicial adjoint L. Then

$$CHRan(\phi_K, \phi_K) \simeq \phi_{K \cdot L}$$
$$CHLan(\phi^L, \phi^L) \simeq \phi^{L \cdot K} \ ,$$

hence

$$Ssh_K \simeq \pi(Kl_T) \ ,$$

if K is strongly formal and

$$Csh_L \simeq \pi(Kl_R)$$

if L is strongly coformal, where (T, μ, η) and (R, ρ, ϵ) are the simplicial monad and comonad respectively generated by the adjunction.

Thus in this case the phenomenon of coherence does not play any role.

2. Suppose we are in the situation of example 1, and let $im(A)$ be a full image of A in B and $k : im(B) \to B$ be corresponding simplicial inclusion.

PROPOSITION 4.2 *Let $\eta : X \to T^*(X)$ be a natural augmentation considered as a cone over X. Then the pair $(T^*(X), \eta)$ is a strong k-resolution of X.*

If $\underline{B}(X, T^*(Y))$ is a fibrant cosimplicial simplicial set for every $X, Y \in ob(B)$ (i.e. $T = (T, \mu, \eta)$ is a fibrant monad in the terminology of [2]) then Ssh_k is isomorphic to the coherent homotopy category $CHT_\infty - B$ [2].

PROOF. Let $Z \in ob(A)$. Then we have the following weak equivalence

$$hocolim_n \underline{B}(T^{n+1}(X), K(Z)) \sim Diag(\underline{A}(L(T^*(X)), Z)) \ .$$

But $L(T^*(X))$ is contractible to the constant cosimplicial object $L(X)$ and hence,

$$Diag(\underline{A}(L(T^*(X)), Z) \sim \underline{A}(L(X), Z) \simeq \underline{B}(X, K(Z)) \ .$$

The second part of the proposition follows from the strong k-continuity of

$$CHRan(\phi_k, \phi_k)$$

and proposition 4.1.

Q.E.D.

REMARK. There is an evident dual version of this proposition for the case of the comonad (R, ρ, ϵ). We obtain thus a characterization of the homotopy coherent category of this comonad as a strong coshape category.

2^a) As a corollary of this proposition we obtain that the category of coherent diagrams and their coherent transformations [2,15,19,44] may be considered as a strong shape or strong coshape category depending on the choice of an adjoint to the restriction functor. In particular, the categories $\boldsymbol{CHDist}(A, B)$ are examples of strong shape and strong coshape categories.

2^b) Let R be a commutative ring with unit. Consider the Bousfield-Kan monad (R, μ, ϵ) on the category of simplicial sets [10,2]. Let k be an inclusion of the full subcategory generated by the R-algebras in $\mathcal{S}$. Then we obtain a homotopy equivalence

$$CHRan(\phi_k, \phi_k)(X, Y) \sim \underline{\mathcal{S}}(X, R_\infty(Y)) \ ,$$

where $R_\infty(Y)$ is the Bousfield-Kan R-completion of Y [10].

2^c) Finally, a very important example arises when one considers a category of algebras of some simplicial monad (T, μ, η) and its subcategory of free algebras. Then the simplicial resolution of a T-algebra X

given by the bar-construction $B_*(T,T,X)$ [37,38] is a strong k-coresolution of X.

The resulting strong coshape category is isomorphic to the coherent homotopy category of T-algebras [3]. Depending on the choice of T, it presents the various theories of homotopy homomorphisms [7], for example, the $A_\infty-$ and E_∞-morphisms [3,7,31].

3. Let Q be a Quillen [41] simplicial closed model category and Q_c, Q_f, Q_{cf} the subcategories of cofibrant, fibrant, and fibrant-cofibrant objects of Q respectively. Let us show that the strong shape category of the simplicial inclusion $i_c : Q_{cf} \subset Q_c$ (strong coshape category of $i_f : Q_{cf} \subset Q_f$), is isomorphic to HoQ_c (respectively to HoQ_f) of [41] and hence, is equivalent to HoQ.

Indeed, for $X \in ob(Q_c)$ let $p : X \to P_X$ be a trivial cofibration with $P_X \in ob(Q_{cf})$. Then for every fibrant and cofibrant Z, we have a homotopy equivalence

$$p^* : \underline{Q}(X,Z) \leftarrow \underline{Q}(P_X,Z)$$

and hence (P_X, p) is a strong i_c-resolution of X. The conclusion follows now from theorem 4.2 and [41, Theorem 1]. In the dual case the proof is similar.

5 Inverse system approach

DEFINITION 5.1 *We call the coherent prohomotopy category of a simplicial category B the full subcategory $CPH(B)$ of the category of strong resolutions of B generated by the functors*

$$\{X_\lambda\} = (X_\lambda, p_{\lambda\lambda'}, \Lambda)$$

for which Λ is the category associated with a directed set. We call such functors inverse systems.

Recall, that a map between inverse systems in $pro(B)$ is said to be a levelwise homotopy equivalence provided it may be represented by some natural transformation, which is a level homotopy equivalence.

THEOREM 5.1 *Let B be a locally Kan finitely $\mathcal{S}$-cotensored complete $\mathcal{S}$-category. Then $CPH(B)$ is the localization of $\pi(pro(B))$ with respect to levelwise homotopy equivalences. The full subcategory of $CPH(B)$ generated by the inverse systems over cofinite directed sets is isomorphic to the coherent prohomotopy category for B introduced in [2].*

PROOF. Via the lemma 4.1, for every two inverse systems over cofinite directed sets we have the isomorphisms

$$CPH(B)(\{X_\lambda\},\{Y_\mu\}) \simeq \pi(holim_\mu colim_\lambda \underline{B}(X_\lambda, Y_\mu)) \simeq$$

$$\simeq \pi(pro(B))(\{X_\lambda\}, Ran_\infty(\{Y_\mu\}))\ ,$$

where Ran_∞ is a monad on $\pi(pro(B))$ from [2]. Thus the full subcategory of $CPH(B)$ generated by the inverse systems over cofinite directed sets is isomorphic to the Kleisli category of Ran_∞ and so the second statement is proved.

To finish the proof we have to substitute every inverse system by an isomorphic system over a cofinite set [36] and to repeat the proof of theorem 6.2 from [2].

Q.E.D.

COROLLARY 5.1.1 *The category $CPH(Top)$ is isomorphic to the Lisica-Mardešić coherent prohomotopy category $CPH - Top$ [32].*

REMARK. The formula

$$CPH - Top(\{X_\lambda\},\{Y_\mu\}) \simeq \pi(holim_\mu colim_\lambda \underline{Top}(X_\lambda, Y_\mu))$$

was proven for the first time by J.-M.Cordier [13].

THEOREM 5.2 *Let B be a simplicial closed model Quillen category, and suppose that every object of B is cofibrant. We assume, in addition, that the condition $\mathcal{N}$ from [23, p.45] is satisfied. Then $CPH(B)$ is isomorphic to $Ho(pro(B))$ of Edwards-Hastings.*

PROOF. Let us show that $CPH(B)$ is the localization of $\pi(pro(B))$ at the class of trivial cofibrations.

Let us associate with every inverse system X a fibrant inverse system over a cofinite directed set together with a trivial cofibration

$$X \to Ex^{\infty}(X)$$

(see the construction of Ex^{∞} in [23]). Then we have the following isomorphisms

$$\underline{pro(B)}(X, Ex^{\infty}(Y)) \simeq lim_{\mu} colim_{\lambda} \underline{B}(X_{\lambda}, Ex^{\infty}(Y)_{\mu}) \simeq$$

$$\simeq lim_{\mu} Ex^{\infty}(colim_{\lambda} \underline{B}(X_{\lambda}, Y_{\mu})) ,$$

as all X_{λ} are cofibrant, Λ is directed and M is cofinite. But the last simplicial set is none other then

$$holim_{\mu}^{EH} colim_{\lambda} \underline{B}(X_{\lambda}, Y_{\mu}) ,$$

where $holim_{\mu}^{EH}$ is Edwards-Hastings homotopy limit in $\mathcal{S}$ [23].

As was shown in [18], for an inverse system $\{X_{\lambda}\}$ in $\mathcal{K}a$, there is a natural homotopy equivalence

$$holim_{\lambda}(\{X_{\lambda}\}) \sim holim_{\lambda}^{EH}(\{X_{\lambda}\}) .$$

Then we have

$$CPH(B)(\{X_{\lambda}\}, \{Y_{\mu}\}) \simeq \pi_0(S(holim_{\mu} |colim_{\lambda} \underline{B}(X_{\lambda}, Y_{\mu})|)) \simeq$$

$$\simeq \pi_0(S(holim_{\mu}^{EH} |colim_{\lambda} \underline{B}(X_{\lambda}, Y_{\mu})|)) \simeq \pi_0(holim_{\mu}^{EH} S(|colim_{\lambda} \underline{B}(X_{\lambda}, Y_{\mu})|)) \simeq$$

$$\simeq \pi_0(holim_{\mu}^{EH} colim_{\lambda} \underline{B}(X_{\lambda}, Y_{\mu})) \simeq \pi(pro(B)(X, Ex^{\infty}(Y)))$$

Thus every morphism $f : X \to Y$ in $CPH(B)$ may be specified as a morphism in $\pi(pro(B))$ from X to $Ex^{\infty}(Y)$. If now f is a trivial cofibration, then this formula shows that it induces a bijection

$$CPH(B)(X, Z) \leftarrow CPH(B)(Y, Z)$$

for every Z, because $Ex^{\infty}(Z)$ is fibrant.

The proof may be completed now as in proposition 3.1.

Q.E.D.

DEFINITION 5.2 *Let $K : A \to B$ be a simplicial functor, $X \in ob(B)$. An inverse system $\{X_\lambda\}$ is strongly K-associated to X if there is a cone $p : X \to \{X_\lambda\}$ (i.e. a morphism in $pro(B)$), such that for every $P \in ob(A)$:*

$$p^* : colim_\lambda \underline{B}(X_\lambda, K(P)) \to \underline{B}(X, K(P)) \tag{22}$$

is a weak equivalence.

It is evident now that the obvious analogues of theorems 4.1, 4.2, 4.3 and proposition 4.1 remain true after the passage from $\mathcal{CPH}(B)$ to $CPH(B)$.

EXAMPLES.

1. For a simplicial category B, let K be inclusion $K : B \to pro(B)$, then Ssh_K is isomorphic to $CPH(B)$.

Similarly, $CIH(B)$ is isomorphic to the strong coshape category of $K : B \to inj(B)$.

2. Let $M \subset C$ be a admissible pair of categories in the sense of Cathey-Segal [12] and let K be the corresponding inclusion functor.

PROPOSITION 5.1 *Assume that M satisfies the conditions of Theorem 4.1 of [12]. Then the strong shape category of K is isomorphic to $ho_M C$ of [12].*

PROOF. Let X be an object of C and let $i : X \to \bar{X}$ be a trivial cofibration with fibrant $\bar{X}$.

As was shown in Theorem 4.1 of [12], every fibrant object of C is the limit of an inverse system over M. Let

$$\bar{X} = lim_\lambda M_\lambda \ .$$

Then i induces a cone $X \to \{M_\lambda\}$. We have, in addition, for every object P of M a homotopy equivalence

$$i^* : \underline{C}(X, P) \leftarrow \underline{C}(\bar{X}, P)$$

and an isomorphism (see the continuity condition [36])

$$\underline{C}(\bar{X}, P) \simeq colim_\lambda \underline{C}(M_\lambda, P) \ .$$

Then $(\{M_\lambda\}, p)$ is strongly K-associated to X. The conclusion follows now from theorem 4.2 and the fact that $\{M_\lambda\}$ is a fibrant inverse system [12, Remark 4.2].

Q.E.D.

This proposition provides us with a number of examples of strong shape categories (see [12]).

REMARK. Using this proposition one can prove the isomorphism of Ssh_K and $ho_M C$ without any restriction on M. But this demands some supplementary work and we shall do it in a further article.

We are going now to prove the equivalence of our notion of strong K-associated inverse system and Mardešić's notion of strong expansion [35].

Let A be a full simplicial subcategory of B and K be the inclusion functor.

If $p : X \to Z$ is a morphism in B, then it induces a simplicial map

$$\underline{B}(Z,Y) \to \underline{B}(X,Y) \ ,$$

and we denote by $\sigma \cdot p$ a simplex of $\underline{B}_n(X,Y)$, which corresponds to a simplex $\sigma \in \underline{B}_n(Z,Y)$ under $\underline{B}(p,1)$.

DEFINITION 5.3 *Let* $\{X_\lambda\} = (X_\lambda, p_{\lambda\lambda'}, \Lambda)$ *be an inverse system in* B *and let* $p = \{p_\lambda\} : X \to \{X_\lambda\}$ *be a morphism in* $pro(B)$.

Then we say that $p : X \to \{X_\lambda\}$ *is a strong expansion of* X *provided the following two conditions are satisfied:*

(SM1). For every object P *of* A *and every morphism* $f : X \to P$, *there exists a* $\lambda \in \Lambda$ *and a morphism* $h : X_\lambda \to P$ *such that*

$$h \cdot p_\lambda \ \textit{is homotopic to} \ f \ .$$

(SM2). For every $\lambda \in \Lambda$, $P \in obA$, *morphisms* $f_0, f_1 : X_\lambda \to P$ *and homotopy* $F \in \underline{B}_1(X, P)$, *such that*

$$d_0(F) = f_0 \cdot p_\lambda \ , \ d_1(F) = f_1 \cdot p_\lambda \ ,$$

there exist a $\lambda' \geq \lambda$ *and a homotopy* $H \in \underline{B}_1(X_{\lambda'}, P)$ *such that*

$$d_0(H) = f_0 \cdot p_{\lambda\lambda'} \quad , d_1(H) = f_1 \cdot p_{\lambda\lambda'} \ ,$$

and $H \cdot p_{\lambda'}$ *is homotopic to* F.

If all X_λ *are objects of* A *we say that* p *is a strong* A*-expansion.*

It is obvious that for the inclusion $ANR \subset Top$, this definition is exactly Mardešić's definition of strong expansion [35, Definition 1.1].

THEOREM 5.3 *Let* B *be finitely cotensored and suppose that* A *is closed with respect to this cotensorization. Let* ϕ_K *be a locally Kan distributor then for* $p : X \to \{X_\lambda\}$ *the following are equivalent:*

- p *is a strong expansion of* X,

- $(\{X_\lambda\}, p)$ *is strongly* K*-associated to* X.

PROOF. Let p be a strong expansion of X. Let us show that for every $P \in ob(A)$, the map p^* from the condition (22) induces an isomorphism of all homotopy groups for any choice of base point.

Indeed, this is obvious for π_0 [35, Remark 1.2].

Denote by $q_\lambda, q_{\lambda\lambda'}$ the simplicial mappings induced by application of the functor $\underline{B}(-, P)$ to $p_\lambda, p_{\lambda\lambda'}$ respectively. As A is finitely $\mathcal{S}$-cotensored, then (SM1) implies the following property:
for any finite simplicial set k and simplicial map $r : k \to \underline{B}(X, P)$ there exist a $\lambda \in \Lambda$ and a mapping $r_\lambda : k \to \underline{B}(X_\lambda, P)$ such that $q_\lambda \cdot r_\lambda$ is homotopic to r.

Let y be a vertex of $colim_\lambda \underline{B}(X_\lambda, P)$, which we take as a base point. Let $x = p^*(y)$. Let $\partial\Delta(n)$ be the boundary of $\Delta(n)$ with base point $*$ and let $r : \partial\Delta(n) \to \underline{B}(X, P)$ be a base point preserving map. Then from the above property we conclude that there exist a $\lambda \in \Lambda$ and a mapping $r_\lambda : \partial\Delta(n) \to \underline{B}(X_\lambda, P)$ such that $q_\lambda \cdot r_\lambda$ is homotopic to r. Without loss of generality we can assume that in $\underline{B}(X_\lambda, P)$ there is a base point y_λ, such that $q_\lambda(y_\lambda) = x$, but, of course, r_λ may not preserve the base points.

However, a homotopy joining r and $q_\lambda \cdot r_\lambda$ gives us a 1-simplex F such that:

$$d_0(F) = x \ , \ d_1(F) = q_\lambda \cdot r_\lambda(*) \ .$$

Thus, by (SM2) there exist a $\lambda' \geq \lambda$ and 1-simplex H in $\underline{B}(X_{\lambda'}, P)$ such that:

$$d_0(H) = q_{\lambda\lambda'}(y_\lambda) = y_{\lambda'} \ , \ d_1(H) = q_{\lambda\lambda'} \cdot r_\lambda(*) \ , \tag{23}$$

and

$$q_{\lambda'}(H) \ \text{is equivalent to} \ F \tag{24}$$

in the fundamental groupoid of $\underline{B}(X, P)$.

Now, (23) implies that $q_{\lambda\lambda'} \cdot r_\lambda$ defines some element

$$\phi \in \pi_{n-1}(\underline{B}(X_{\lambda'}, P), y_{\lambda'}) \ ,$$

whereas (24) shows that $\pi_{n-1}(q_{\lambda'})(\phi)$ is equal to the element representing by r in $\pi_{n-1}((\underline{B}(X, P)), x)$, because $\underline{B}(X, P)$ and $\underline{B}(X_{\lambda'}, P)$ are Kan. Hence, p^* is surjective in homotopies.

Let $i : \partial\Delta(n) \to \Delta(n)$ be canonical inclusion. We are going to show that p satisfies the following generalization of the condition (SM2):

(SM2*) For every commutative diagram

$$\begin{array}{ccc} \partial\Delta(n) & \xrightarrow{\phi} & \underline{B}(X_\lambda, P) \\ \downarrow i & & \downarrow q_\lambda \\ \Delta(n) & \xrightarrow{\psi} & \underline{B}(X, P) \end{array} \tag{25}$$

there exist a $\lambda' \geq \lambda$ and a map

$$\psi' : \Delta(n) \to \underline{B}(X_{\lambda'}, P)$$

such that in the diagram

$$\begin{tikzcd} \partial\Delta(n) \arrow[rr, "\phi"] \arrow[dr] \arrow[dd, "i"'] & \arrow[d, phantom, "\boldsymbol{A}"] & \underline{B}(X_\lambda, P) \arrow[dl, "q_{\lambda\lambda'}"'] \arrow[dd, "q_\lambda"] \\ \arrow[r, phantom, "\boldsymbol{B}"] & \underline{B}(X_{\lambda'}, P) \arrow[dr, "q_{\lambda'}"] \arrow[d, phantom, "\boldsymbol{D}"] \arrow[r, phantom, "\boldsymbol{C}"] & \\ \Delta(n) \arrow[ur, "\psi'"] \arrow[rr, "\psi"] & & \underline{B}(X, P) \end{tikzcd} \tag{26}$$

the following relations hold:

- the triangles $\boldsymbol{A}$, $\boldsymbol{B}$, $\boldsymbol{C}$ commute,

- there exists a homotopy G, which makes $\boldsymbol{D}$ homotopy commutative and fits commutatively into the diagram

$$\begin{array}{ccc} \partial\Delta(n)\times\Delta(1) & \xrightarrow{proj.} & \partial\Delta(n) \\ \downarrow\ i\times 1 & & \downarrow\ i \\ \Delta(n)\times\Delta(1) & \xrightarrow{G} & \underline{B}(X,P)\ . \end{array} \qquad (27)$$

Indeed, denote

$$(\mathcal{E}_\lambda, e_{\lambda\lambda'}, \Lambda) = \underline{S}(\Delta(n), \{\underline{B}(X_\lambda, P)\})\ ,\ (\mathcal{B}_\lambda, b_{\lambda\lambda'}, \Lambda) = \underline{S}(\partial\Delta(n), \{\underline{B}(X_\lambda, P)\})\ ,$$

$$\mathcal{E} = \underline{S}(\Delta(n), \underline{B}(X,P))\ ,\ \mathcal{B} = \underline{S}(\partial\Delta(n), \underline{B}(X,P))\ ,$$

and let

$$\rho_\lambda : \mathcal{E}_\lambda \to \mathcal{B}_\lambda\ ,\ \rho : \mathcal{E} \to \mathcal{B}\ ,$$

be the morphisms induced by i. These are Kan fibrations because $\underline{B}(X,P)$ and $\underline{B}(X_\lambda, P)$ are Kan simplicial sets.

As A is closed under finite cotensorisation, we have from (SM1) and (SM2) and Kan conditions for $\mathcal{B}$ and $\mathcal{B}_\lambda$:
- for every 0-simplex ξ of $\mathcal{B}$ there exist a $\lambda\in\Lambda$, 0-simplex ξ_λ of $\mathcal{B}_\lambda$ and 1-simplex σ of $\mathcal{B}$ such that

$$d_0(\sigma) = \xi\ ,\ d_1(\sigma) = b_\lambda(\xi_\lambda)$$

- for every pair of 0-simplices $\xi_0, \xi_1 \in \mathcal{B}_\lambda$ and 1-simplex $\sigma\in\mathcal{B}$, such that

$$d_0(\sigma) = b_\lambda(\xi_0)\ ,\ d_1(\sigma) = b_\lambda(\xi_1)\ ,$$

there exist a $\lambda'\geq\lambda$ and 1-simplex $\varpi\in\mathcal{B}_{\lambda'}$ such that

$$d_0(\varpi) = b_{\lambda\lambda'}(\xi_0)\ ,\ d_1(\varpi) = b_{\lambda\lambda'}(\xi_1)\ ,$$

and there exists a 2-simplex $\Sigma\in\mathcal{B}$ such that

$$d_0(\Sigma) = \sigma\ ,\ d_1(\Sigma) = b_{\lambda'}(\varpi)$$

and $d_2(\Sigma)$ is degenerate.
The same is true for $e : \{\mathcal{E}_\lambda\} \to \mathcal{E}$.

Returning to the diagram (25) we see that ψ and ϕ are 0-simplices in $\mathcal{E}$ and $\mathcal{B}$ respectively and $\rho(\psi) = b_\lambda(\phi)$. From the above property of e we conclude that there exist a $\lambda' \geq \lambda$, a 0-simplex $\psi_{\lambda'} \in \mathcal{E}_{\lambda'}$ and 1-simplex $\sigma \in \mathcal{E}$ such that

$$d_1(\sigma) = \psi \ , \ d_0(\sigma) = e_{\lambda'}(\psi_{\lambda'}) \ .$$

Thus

$$d_1(\rho(\sigma)) = \rho(\psi) = b_\lambda(\phi) = b_{\lambda'} \cdot b_{\lambda\lambda'}(\phi) \ ,$$
$$d_0(\rho(\sigma)) = \rho(e_{\lambda'}(\psi_{\lambda'})) = b_{\lambda'}(\rho_{\lambda'}(\psi_{\lambda'})) \ .$$

Then there exist a $\lambda'' \geq \lambda'$, 1-simplex σ' in $\mathcal{B}_{\lambda''}$ such that

$$d_1(\sigma') = b_{\lambda''\lambda'} \cdot b_{\lambda'\lambda}(\phi) = b_{\lambda''\lambda}(\phi) \ ,$$

$$d_0(\sigma') = b_{\lambda''\lambda'} \cdot \rho_{\lambda'}(\psi_{\lambda'}) = \rho_{\lambda''} \cdot e_{\lambda''\lambda'}(\psi_{\lambda'}) \ , \tag{28}$$

and 2-simplex Σ in $\mathcal{B}$ such that

$$d_0(\Sigma) = \rho(\sigma) \ , \ d_1(\Sigma) = b_{\lambda''}(\sigma') \ , \ d_2(\Sigma) = s_0(b_\lambda(\phi)) \tag{29}$$

The relation (28) gives us a commutative diagram:

$$\begin{array}{ccc} \Delta(0) & \xrightarrow{e_{\lambda''\lambda'}(\psi_{\lambda'})} & \mathcal{E}_{\lambda''} \\ d^0 \downarrow & & \rho_{\lambda''} \downarrow \\ \Delta(1) & \xrightarrow{\sigma'} & \mathcal{B}_{\lambda''}. \end{array}$$

As $\rho_{\lambda''}$ is a Kan fibration, there exists a 1-simplex σ'' in $\mathcal{E}_{\lambda''}$ such that

$$\rho_{\lambda''}(\sigma'') = \sigma' \ , \ d_0(\sigma'') = e_{\lambda''\lambda'}(\psi_{\lambda'}) \ .$$

Put

$$\psi' = d_1(\sigma'') \ .$$

On the other hand

$$\rho(e_{\lambda''}(\sigma_{\lambda''})) = b_{\lambda''}(\rho_{\lambda''}(\sigma'')) = b_{\lambda''}(\sigma') \ ,$$

and so from (29) and the equality $d_0(\sigma) = d_0(e_{\lambda''}(\sigma''))$ we have the following diagram:

$$\begin{array}{ccc} \Lambda^2(2) & \xrightarrow{\sigma\cup e_{\lambda''}(\sigma'')} & \mathcal{E} \\ i\downarrow & & \rho\downarrow \\ \Delta(2) & \xrightarrow{\Sigma} & \mathcal{B}. \end{array}$$

Let H be a solution of corresponding left lifting problem. Then put

$$G = d_2(H) \ .$$

We thus have

$$\rho(G) = \rho(d_2(H)) = s_0(b_\lambda(\phi)) \tag{30}$$

$$d_0(G) = d_0(d_2(H)) = d_1(\sigma) = \psi \ , \tag{31}$$

$$d_1(G) = d_1(d_2(H)) = d_1(e_{\lambda''}(\sigma'')) = e_{\lambda''}(\psi) \ . \tag{32}$$

$$\rho_{\lambda''}(\psi') = \rho_{\lambda''}(d_1(\sigma'')) = d_1(\sigma') = b_{\lambda''\lambda}(\phi) \ . \tag{33}$$

It is easy to verify, using (30,31,32 and 33), that ψ' and G, considered as map $\Delta(n) \to \underline{B}(X_{\lambda''}, P)$ and homotopy $\Delta(n) \times \Delta(1) \to \underline{B}(X, P)$ respectively, satisfy the conditions (26,27).

Let now $r_\lambda : \partial\Delta(n) \to \underline{B}(X_\lambda, P)$ preserves base point and let $q_\lambda \cdot r_\lambda$ represent the trivial element in $\pi_{n-1}((\underline{B}(X, P)), x)$. Thus, by (SM2*) there exists a $\lambda' \geq \lambda$ such that $q_{\lambda'\lambda} \cdot r_\lambda$ represents the trivial element in $\pi_{n-1}((\underline{B}(X_{\lambda'}, P)), y_{\lambda'})$ and hence, p^* is injective in homotopies.

Finally, if $(\{X_\lambda\}, p)$ is strong K-associated to X, then p^* induces a bijection of π_0 and an equivalence of fundamental groupoids. The result follows now readily from the well known description of the fundamental groupoid of a Kan simplicial set [24].

Q.E.D.

REMARK. The proof of this theorem is an abstract simplicial version of the proof by Mardešićof his product theorem and the main lemma on strong expansions [35, Theorem 2.1, lemmas 3.1, 3.2].

COROLLARY 5.3.1 *Let $K : ANR \to Top$ be the inclusion functor. Then the Lisica-Mardešić strong shape category of topological spaces is isomorphic to Ssh_K.*

PROOF. This follows from the theorem 4.2, which generalizes thus the main theorem of [35, Theorem 4.2] asserting that strong expansions are coherent expansions in the sense of [32].

Q.E.D.

6 Strong shape categories and function complexes

This section is based on a theory developed by author in [3], so we recall some definitions and results from that paper.

DEFINITION 6.1 *Let B be an $\mathcal{S}$-category with a bifunctor*

$$\otimes_B : B \times B \to B \ ,$$

an object I_B of B and isomorphisms

$$a_{X,Y,Z} : (X \otimes_B Y) \otimes_B Z \to X \otimes_B (Y \otimes_B Z),$$

$$l_X : I_B \otimes_B X \to X, r_X : X \otimes_B I_B \to X,$$

which make B a monoidal category [30]. We shall say that B is a monoidal $\mathcal{S}$-category if $\otimes_B$ is an $\mathcal{S}$-functor and a, l, r are $\mathcal{S}$-natural.

Let $\mathcal{S}_f$ be a simplicial category, which has as objects sequences of simplicial sets and as enriched *hom*-functor

$$\underline{\mathcal{S}}_f(X,Y) = \underline{\mathcal{S}}((X_0, X_1, \ldots), (Y_0, Y_1, \ldots)) = \prod_{i=0}^{\infty} \underline{\mathcal{S}}(X_i, Y_i) \ ,$$

with the obvious definition of composition.

In $\mathcal{S}_f$ there is the following tensor product:

$$(X \otimes_{\mathcal{S}_f} Y)_j = \coprod_{k \geq 0} \coprod_{j_1 + \ldots + j_k = j} X_{j_1} \times \ldots \times X_{j_k} \times Y_k.$$

Let, in addition, $I_{\mathcal{S}_f}$ be the family $(\emptyset, \Delta(0), \emptyset, \ldots)$.

LEMMA 6.1 ([3]) *There are $\mathcal{S}$-natural isomorphisms a, r, l, which together with $\otimes_{\mathcal{S}_f}$ and $I_{\mathcal{S}_f}$ make $\mathcal{S}_f$ a monoidal $\mathcal{S}$-category. We call it the category of simplicial families.*

REMARK. A similar tensor structure on the category of the families of chain complexes was considered by V.Smirnov in [43].

DEFINITION 6.2 *An $\mathcal{S}$-operad $\mathcal{E}$ is a monoid $(\mathcal{E}, \gamma, \eta)$ [33] in $\mathcal{S}_f$ with multiplication*

$$\gamma : \mathcal{E} \otimes_{\mathcal{S}_f} \mathcal{E} \to \mathcal{E}$$

and unit

$$\eta : I \to \mathcal{E} \ .$$

A morphism of $\mathcal{S}$-operads is a morphism of monoids in $\mathcal{S}_f$.

Let now B be a monoidal $\mathcal{S}$-category with enriched *hom*-functor $\underline{B}$, tensor product $\otimes_B$ and unit object I_B. For $X \in Ob(B)$ put

$$X^j = \underbrace{X \otimes_B \ldots \otimes_B X}_{j} \ , \ j \neq 0 \ , X^0 = I_B \ .$$

Then we have an $\mathcal{S}$-functor :

$$\overline{\underline{B}}_f \ : \ B^{op} \times B \to \mathcal{S}_f \ ,$$

$$\overline{\underline{B}}_f(X,Y)_j = \underline{B}(X^j, Y) \ .$$

LEMMA 6.2 ([3]) *For each object X of K the family $\overline{\underline{B}}_f(X,X)$ has a natural structures of an $\mathcal{S}$-operad.*

Let $\mathcal{E}$ be an $\mathcal{S}$-operad .

DEFINITION 6.3 *An $\mathcal{E}$-algebra in B is an object X of B together with a morphism of $\mathcal{S}$-operads*

$$\eta : \mathcal{E} \to \overline{\underline{B}}_f(X,X).$$

So our definition of algebra of operad is the same as in [37].

DEFINITION 6.4 *We call an $\mathcal{S}$-operad $\mathcal{E}$ an A_∞-operad provided all $\mathcal{E}_n$ are weakly contractible. An $\mathcal{E}$-algebra X in B is said to be an A_∞-monoid in B if $\mathcal{E}$ is an A_∞-operad.*

EXAMPLE. Every monoid in B is an A_∞-monoid.

We do not give the definition of the A_∞-morphisms between A_∞-monoids, because it demands too many place. For a detailed discussion of this subject see [3, 7].

Let B be a category with finite colimits and let $X^{*,*}$ be a bicosimplicial object in B . Then we can construct a cosimplicial object $\nabla(X) = \nabla(X^{*,*})$ as follows [2] :

$$\nabla(X)^0 = X^{0,0}$$

$\nabla(X)^{n+1}$ is colimit of the diagram

$$\begin{array}{ccccc} \ldots \rightarrow & X^{p,q+1} & & & \\ & \uparrow d_2^0 & & & \\ & X^{p,q} & \overset{d_1^{p+1}}{\rightarrow} & X^{p+1,q} & \\ & & & \uparrow & \ldots \end{array}$$

$$p + q = n.$$

See also [2] for the definition of cofaces and codegeneracies .

REMARK. As is noted in [19], $\nabla(X)$ is none other then a left Kan extension along the ordinal sum functor. As the construction of this type was developed for the first time in [1] , ∇ is called in [19] the Artin-Mazur codiagonal functor.

Let now B be a monoidal category with finite colimits . Then we can define a tensor product $\otimes_c$ of cosimplicial objects in B by the formula

$$X^* \otimes_c Y^* = \nabla((X \hat{\otimes}_B Y)^{*,*}) ,$$

where $(X\hat{\otimes}_B Y)^{*,*}$ is the following bicosimplicial object:

$$(X\hat{\otimes}_B Y)^{p,q} = X^p \otimes_B Y^q$$

with obvious cofaces and codegeneracies [2].

PROPOSITION 6.1 ([2,3]) *If $\otimes_B$ commutes with colimits, then the category cB of cosimplicial objects of B is monoidal with respect to the tensor product defined above and with constant cosimplicial object $I^n = I_B$ as the unit object.*

Consider two examples.

1. For a simplicial category B the $\mathcal{S}$-category $\boldsymbol{SDist}(B,B)$ is a monoidal $\mathcal{S}$-category. If we apply the realization functor to each simplicial *hom*-set in B, then we can consider B as a $\mathcal{K}a$-category and hence, construct a monoidal category of topological endodistributors on B, which we shall denote by $\boldsymbol{TDist}(B,B)$.

We can also introduce a simplicial structure in $\boldsymbol{TDist}(B,B)$ by applying the singular complex functor to each *hom*-space in $\boldsymbol{TDist}(B,B)$. It is obvious, that the fibrewise singular complex functor provides us with a monoidal $\mathcal{S}$-functor

$$S : \boldsymbol{TDist}(B,B) \to \boldsymbol{SDist}(B,B).$$

This functor is not strict monoidal, however, as we have only a natural morphism

$$S(X) \otimes S(Y) \to S(X \otimes Y) ,$$

which is not an isomorphism in general.

Remark also that the categories $\boldsymbol{SDist}(B,B)$ and $\boldsymbol{TDist}(B,B)$ are complete and simplicially cotensored. So for every cosimplicial object T^* in these categories the total object

$$Tot(T^*) = \textstyle\int_n (T^n)^{\Delta(n)}$$

is defined.

2. The category $Gr(C)$ of $\mathcal{S}$-graphs over the fixed set of objects C is the following $\mathcal{S}$-category. The objects of $Gr(C)$ are the functions :

$$G : C \times C \to Ob(\mathcal{S}) \ .$$

The enriched *hom*-functor is

$$\underline{Gr(C)}(F,G) = \prod_{a,b \in C} \underline{\mathcal{S}}(F(a,b), G(a,b)) \ .$$

The monoidal structure is given by

$$(G \otimes_{Gr(C)} F)(a,b) = \coprod_{c \in C} G(a,c) \times F(c,b)$$

and

$$I_{Gr(C)}(a,b) = \Delta(0) \text{ if } a = b \ , \ I_{Gr(C)}(a,b) = \emptyset \text{ if } a \neq b \ .$$

Remark that the category of monoids with respect to the monoidal structure above on $Gr(C)$ is isomorphic to the category of $\mathcal{S}$-categories with fixed set of objects C and $\mathcal{S}$-functors between them, which are identities on C .

DEFINITION 6.5 *We shall say that an $\mathcal{S}$-graph G is an A_∞-graph provided it has a structure of an A_∞-monoid in $Gr(C)$. We shall say also that G is locally Kan if $G(a,b)$ is a Kan simplicial set for every $a, b \in C$.*

Every A_∞-graph G over C generates a category $\pi(G)$, which we call the homotopy category of G. It is defined by

- *$ob(\pi(G)) = C$*
- *the set of morphisms from X to Y is $\pi(G(X,Y))$,*
- *the composition and identities are induced by A_∞-structure on G.*

The following proposition is proved in [3].

PROPOSITION 6.2 *Let (T^*, μ^*, η^*) be a cosimplicial monoid in $\boldsymbol{cTDist}(B,B)$ Then the simplicial endodistributor*

$$Tot(S(T^*))$$

has a natural A_∞-monoid structure in $\boldsymbol{SDist}(B,B)$. Moreover, the unit η^* induces a morphism of A_∞-monoids

$$\eta : I \to Tot(S(T^*)) \ .$$

We shall use this proposition to obtain the following theorem:

THEOREM 6.1 *For any simplicial distributor $K : A \to B$ the simplicial distributor $CHRan(K,K)$ $(CHLan(K,K))$ has a natural A_∞-monoid structure in $\boldsymbol{SDist}(B,B)$ (in $\boldsymbol{SDist}(A,A)$). Moreover, the unit of the codensity (density) monad of K is a morphism of A_∞-algebras.*

PROOF. Let us prove this theorem for the case of right extension. For this we use the isomorphism (18). We have

$$CHRan(K,K) \simeq Tot(Ran(L'_*(K), S|K|))$$

But

$$Ran(L'_*(K), S|K|)(X,Y) = \int_A \underline{S}(L_*(K)(Y,Z), S|K|(X,Z)) \simeq$$

$$\simeq \int_A \underline{Ka}_{\mathcal{S}}(|L'_*(K)(Y,Z)|, |K|(X,Z)) \simeq$$

$$\simeq S(\int_A \underline{Ka}(|L'_*(K)(Y,Z)|, |K|(X,Z))) = S(Ran(L'_*|K|, |K|)(X,Y))) \ ,$$

where Ran and L' in the last formula are the topological analogues in the topological bicategory $\boldsymbol{TDist}$ of the corresponding functors in $\boldsymbol{SDist}$. (Their definitions are evident and repeat the definitions of simplicial Ran and L'.)

Thus the theorem will follow from the proposition 6.2 if we prove that for every topological distributor $K : A \to B$ the cosimplicial distributor

$$Ran(L'_*(K), K)$$

is a monoid in $c\boldsymbol{TDist}(B,B)$.

Remark firstly that for any three distributors $E, F, K : B \to B$, we have a natural morphism

$$M : Ran(E,F) \otimes Ran(K,E) \to Ran(K,F) \ ,$$

$$M = \alpha(\rho(1_{Ran(E,F)} \otimes \rho)) \ .$$

M is associative and unitary with respect to $\eta : I \to Ran(X,X)$. It is sufficient to repeat the proof of proposition 2.1.

Let now $R : A \to C$ and $T : A \to D$ be two topological distributors. For $X \in ob(D)$ and $Z \in ob(A)$ let

$$\xi : T(X,Z) \to L'T(X,Z) = \coprod_A T(X,A) \times \underline{A}(A,Z)$$

be the morphism induced by

$$T(X,Z) \times \Delta(0) \to T(X,Z) \times \underline{A}(Z,Z) \ .$$

For $X \in ob(D)$ and $Y \in ob(C)$ we have the following isomorphisms

$$Ran(L'R, L'T)(X,Y) = \int_A \underline{\mathcal{K}a}(L'R(Y,A), L'T(X,A)) \simeq$$

$$\simeq \int_A \underline{\mathcal{K}a}(\coprod_{A'} R(Y,A') \times \underline{A}(A',A), L'T(X,A)) \simeq$$

$$\simeq \prod_{A'} \int_A \underline{\mathcal{K}a}(\underline{A}(A',A), \underline{\mathcal{K}a}(R(Y,A'), L'T(X,A))) \simeq$$

$$\overset{Yoneda}{\simeq} \prod_{A'} \underline{\mathcal{K}a}(R(Y,A'), L'T(X,A')) \ .$$

Define a natural transformation

$$L : Ran(R,T) \to Ran(L'R, L'T)$$

by the composition

$$Ran(R,T)(X,Y) = \int_A \underline{\mathcal{K}a}(R(Y,A), T(X,A)) \to$$

$$\to \prod_A \underline{\mathcal{K}a}(R(Y,A), T(X,A)) \overset{\underline{\mathcal{K}a}(1,\xi)}{\to} \prod_A \underline{\mathcal{K}a}(R(Y,A), L'T(X,A)) \ .$$

Let

$$L^p : Ran(R,T) \to Ran(L'_p R, L'_p T) \ , \ p \geq 0 \ ,$$

be the $(p+1)$-iteration of L.

Then the following composition

$$Ran(L'_p(K), K) \otimes Ran(L'_q(K), K) \overset{1 \otimes L^p}{\to} Ran(L'_p(K), K) \otimes$$

$$\otimes Ran(L'_p L'_q(K), L'_p(K)) \overset{M}{\to} Ran(L'_p L'_q(K), K) \to$$

$$\overset{Ran(s_{q-1},1)}{\to} Ran(L'_{p+q}(K), K) \ ,$$

defines a morphism

$$\mu^* : Ran(L'_*(K), K) \otimes_c Ran(L'_*(K), K) \to Ran(L'_*(K), K) \ .$$

It may be verified immediately that μ^* and $\eta^* : I \to Ran(L'_*(K), K)$, induced by $\alpha(\eta')$, give the needed monoid structure on $Ran(L'_*(K), K)$.

Q.E.D.

REMARK. It is straightforward, but rather long, to prove that in ***CHDist*** the multiplication on $CHRan(K, K)$ $(CHLan(K, K))$ induced by the A_∞-monoid structure above coincides with the canonical multiplication in the codensity (density) monad.

COROLLARY 6.1.1 *Let $K : A \to B$ be a strongly formal (strongly coformal) simplicial functor between simplicial categories.*

Then the assignment

$$X, Y \in ob(B) \mapsto CHRan(\phi_K, \phi_K)(X, Y)$$

$$(X, Y \in ob(B) \mapsto CHLan(\phi^K, \phi^K)(X, Y))$$

generates a locally Kan A_∞-graph SSH_K (CSH_K), over $ob(B)$, such that its homotopy category is isomorphic to Ssh_K (Csh_K). The unit of the codensity monad generates an A_∞-monoid morphism

$$P : B \to SSH_K \ (P : B \to CSH_K) ,$$

such that $\pi(P)$ is the canonical strong shape (strong coshape) functor.

Moreover, for SSH_K (CSH_K), there exists a locally Kan $\mathcal{S}$-category, which is isomorphic to SSH_K (CSH_K) as A_∞-graph (see [3, Section 2] for the definition of this isomorphism).

PROOF. This is an immediate corollary of Theorem 2.4 from [3].

Q.E.D.

REMARK 1. In general, the simplicial category constructed above is not locally small (see remark at the begining of section 1). But if K and $X, Y \in ob(B)$ satisfy the conditions of theorem 4.1, then $SSH_K(X, Y)$ is homotopy equivalent to some small simplicial set, and so Ssh_K is locally small. This situation is well known in ordinary shape theory

[16], and is discussed in detail for the strong shape theory of topological spaces in [26].

As is noted in [26], one of the advantages of the approach of [4,26] is the possibility to deal with an individual strong shape function and thus to use strictly commutative diagrams. The corollary above explains why this approach is possible.

CONCLUDING REMARK. Using the results of [3] one can show that for any simplicial categories A, B the assignment $R, T \mapsto \underline{CHDist}(R,T)$ provides a A_∞-graph, whose homotopy category is $\boldsymbol{CHDist}(A,B)$. Similarly, the coherent composition of distributors may be considered as an A_∞-mapping.

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