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ON LIE ALGEBROID ACTIONS AND MORPHISMS

by Tahar MOKRI

Résumé: Si AG est l'algebroïde de Lie d'un groupoïde de Lie G, nous démontrons que toute action infinitésimale de AG sur une variété M, par des champs complets de vecteurs fondamentaux, se reléve en une unique action du groupoïde de Lie G sur M, si les $\alpha-$ fibres de G sont connexes et simplement connexes. Nous appliquons ensuite ce résultat pour intégrer des morphismes d'algébroïdes de Lie, lorsque la base commune de ces algebroïdes est compacte.

Introduction

It is known [12] that if a finite-dimensional Lie algebra $\mathcal{G}$ acts on a manifold M with complete infinitesimal generators, then this action arises from a unique action of the connected and simply connected Lie group G whose Lie algebra is $\mathcal{G}$, on the manifold M.

In the Lie algebroid case, the local integration of infinitesimal actions was studied by Pourreza in [5], and the local integration of Lie algebroid morphisms by Almeida and Kumpera in [2], following the earlier work of Pradines in [14].

In the local trivial case, the integration of Lie algebroid morphisms was dealt with in [8], and more recently Mackenzie and Xu have obtained in [9] a general version of the theorem 3.1. For a symplectic groupoid G on base B with α-connected fibers and with a complete symplectic realization (M, f) of the Poisson manifold B, Dazord in [4] and Xu in [19] have showed that the Lie algebroid action $\omega \longmapsto P(f_*\omega)$ integrates to a (global) action of the groupoid G on the manifold M, here the Lie algebroid AG of G is identified with the cotangent bundle T^*B, and P denotes the bundle isomorphism $P\colon T^*M \to TM$ induced by the symplectic structure on M. We prove here a global integration result of Lie algebroid actions when the actions are by complete infinitesimal generators. As application, we prove that any Lie algebroid morphism

$F : AG \longrightarrow AG'$, over the same base B, integrates to a unique base preserving Lie groupoid morphism $f : G \longrightarrow G'$, provided that the common base B is compact and the α-fibres of G are connected and simply connected, where α is the source map of G. This method of dealing with Lie algebroid morphisms applies only when the Lie algebroids are over a common compact base.

For a Lie groupoid G on base B, we denote by α_G, $\beta_G\colon G \longrightarrow B$ the source and the target maps of G, respectively. For $g \in G$, we denote by g^{-1} or by $i_G g$ the inverse of g, and for $b \in B$ we denote by 1_b^G the value of the identity map $1^G\colon B \longrightarrow G$ at b. We omit then the subscripts when there is no confusion in doing so.

If M and N are two manifolds, with a surjective submersion $\pi : M \to N$, we call M a π-*(simply) connected* manifold if for all $b \in B$ the fibres $\pi^{-1}(b)$ are (simply) connected subspaces of M. Lastly, $C(M)$ refers always to the module of smooth real valued maps on a manifold M, and all manifolds are assumed C^∞, real, Hausdorff and second countable.

1 Lie groupoids and Lie algebroids

We begin by recalling the notions of Lie groupoid actions and morphisms. Let G be a Lie groupoid on base B, and let $p\colon M \longrightarrow B$ be a smooth map, where M is a manifold. *A (left) action* of G on M is a smooth map $\Phi\colon G{*}M \longrightarrow M$, where $G{*}M = \{(g,m) \in G\times M \mid \alpha(g) = p(m)\}$, such that

(i) $p(\Phi_g(m)) = \beta(g)$, where $\Phi_g(m) = \Phi(g,m)$.

(ii) $\Phi_{gh}(m) = \Phi_g\Phi_h(m)$,

(iii) $\Phi_{1_{p(m)}}(m) = m$,

for all $g, h \in G$ and $m \in M$ which are suitably compatible. A right action of G on M is defined similarly.

We recall now the definition of a base preserving Lie groupoid morphism, see [6] for the general case. Let G and G' be two Lie groupoids with a common base B. A smooth map $f\colon G \longrightarrow G'$ is *a Lie groupoid morphism* if

(iv) $\alpha_{G'} \circ f = \alpha_G$,

(v) $\beta_{G'} \circ f = \beta_G$,

(vi) $f(gh) = f(g)f(h)$,

for all composable pair $(g, h) \in G \times G$, that is $\alpha_G(g) = \beta_G(h)$.

A Lie algebroid on base B is a vector bundle $q\colon A \longrightarrow B$ together with a map $a\colon A \longrightarrow TB$ of vector bundles over B, called the *anchor* of A, and an $\mathbb{R}$-bilinear, antisymmetric bracket of sections $[\ ,\ ]\colon \Gamma A \times \Gamma A \longrightarrow \Gamma A$, which obey the Jacobi identity, and satisfies the relations

(vii) $a[X\ ,\ Y] = [a(X)\ ,\ a(Y)]$,

(viii) $[X\ ,\ fY] = f[X\ ,\ Y] + (a(X)f)Y$,

for all $X, Y \in \Gamma A, f \in C(B)$. Here $a(X)(f)$ is the Lie derivative of f with respect to the vector field $a(X)$.

Let A be a Lie algebroid on B and let $f\colon M \longmapsto B$ be a smooth map. Then *an action* of A on M is an $\mathbb{R}$-linear map $X \longmapsto X^\dagger$, $\Gamma A \longrightarrow \chi(M)$ such that

(ix) $[X\ ,\ Y]^\dagger = [X^\dagger\ ,\ Y^\dagger]$, for $X, Y \in \Gamma A$,

(x) $(uX)^\dagger = (u \circ f)X^\dagger$ for $X \in \Gamma A,\ u \in C(B)$,

(xi) $Tf(X^\dagger(m)) = a(X)(f(m))$, for $X \in \Gamma A, m \in M$,

where $\chi(M)$ denotes the module of smooth vector fields on M.

The construction of the Lie algebroid of a Lie groupoid follows closely the construction of a Lie algebra of a Lie group; see [8] for a a full account. Let G be a Lie groupoid on base B, and let $T^\alpha G = Ker(T\alpha)$ be the vertical bundle along the fibres of α. Let $AG \longrightarrow B$ be the vector bundle pullback of the vector bundle $T^{\alpha}G$ accross the identity map $1 : B \longrightarrow G$. Notice that a section $X \in \Gamma AG$ is characterized by $X(b) \in T_{1_b}G_b$, $\forall\ b \in B$, where $G_b = \alpha^{-1}(b)$. Now take $X \in \Gamma AG$ and denote by $\overrightarrow{X}$ the right invariant vector field on G, defined by $\overrightarrow{X}(g) = TR_g X(\beta(g))$; the correspondence $X \longmapsto \overrightarrow{X}$ from ΓAG to the module of right invariant vector fields is a bijection; we equip ΓAG with the Lie algebra structure obtained by transferring the Lie algebra structure of the module of right invariant vector fields on G to ΓAG, via the bijection $X \longmapsto \overrightarrow{X}$. Namely, if $X,\ Y \in \Gamma AG$, we define $[X,\ Y] = [\overrightarrow{X},\ \overrightarrow{Y}] \circ 1$, and $a : AG \longrightarrow TB$, by $a(X_b) = T\beta X_b$. The vector bundle AG constructed above is called *the Lie algebroid of* G; we will denote it by $q_G : AG \longrightarrow B$, and its anchor map by $a_G : AG \longrightarrow TB$. If there is no confusion we will omit the subscripts. The pullback of $\overrightarrow{X}$ by the inversion map i of G, is denoted by $\overleftarrow{X}$ and is a left invariant vector field on G. Lastly, if (x_t) is the one parameter group of local diffeomorphisms which generates a right invariant vector field $\overrightarrow{X}$ on G, then $x_t(v) = x_t(1_{\beta(v)})v$, $\forall\ v \in G$, and we write $x_t(v) = Exp\,tX(\beta(v))v$, where $Exp\,tX(b) = x_t(1_b)$ [8].

A left action Φ of a Lie groupoid G with base B on $p : M \longrightarrow B$ induces an action $X \longmapsto X^{\dagger}_{\Phi}$ of the Lie algebroid AG of G on the map $p : M \longrightarrow B$, by the formula:

$$X^{\dagger}_{\Phi}(m) = T\left(g \longmapsto \Phi_g(m)\right) X(p(m)), \forall\ m \in M. \tag{1}$$

The vector field $X^{\dagger}$ is called *the fundamental vector field associated to* X or *the infinitesimal generator* of the action corresponding to the section $X \in \Gamma AG$.

Lastly, let A and A' be two Lie algebroids on a common base B, with anchor maps a and a', respectively, and let $F : A \longrightarrow A'$ be a vector bundle map. Then, F is a *Lie algebroid morphism* if the relations

(xii) $F[X\ ,\ Y] = [F(X)\ ,\ F(Y)]$,

(xiii) $a' \circ F = a$

hold, for all $X, Y \in \Gamma A$.

A base preserving Lie groupoid morphism $f : G \longrightarrow G'$ differentiates to a Lie algebroid morphism $Tf : AG \longrightarrow AG'$. The tangent linear map Tf, when restricted to AG, is sometimes called *the Lie functor* of f and is denoted by $A(f)$.

2 Integration of Lie Algebroid Actions

We assume here that there are given a Lie groupoid G on base B, a manifold M and a surjective submersion $p : M \longrightarrow B$. For $b \in B$, we denote by G_b and by G^b the α-fibre $\alpha^{-1}(b)$, and the β-fibre $\beta^{-1}(b)$, respectively. The following theorem is the main result of this section.

Theorem 2.1 *Let $X \longmapsto X^\dagger$ be an action of the Lie algebroid AG on the manifold M, by complete infinitesimal generators. Assume that the module of right invariant vector fields on G is composed of complete vector fields (hence, the module of left invariant vector fields is also composed of complete vector fields). Then, if the Lie groupoid G is α-connected and α-simply connected there exists a unique left action Φ of G on $p : M \longrightarrow B$, with*

$$X_\Phi^\dagger = X^\dagger, \ \forall\, X \in \Gamma AG. \qquad (2)$$

Since p is a surjective submersion, the subspace $G * M = \{(g,m) \in G \times M, \mid \alpha(g) = p(m)\}$ is a submanifold of $G \times M$. Let Δ be the subbundle of $T(G * M) = TG * TM$ generated by the set of pairs of vector fields of the form $\left(\overleftarrow{X}, X^\dagger\right)$, with $X \in \Gamma AG$. The subbundle Δ is an involutive differentiable distribution of rank $dimG - dimB$ on $G * M$, therefore Δ is integrable by the Frobenius theorem. Let $\mathcal{F}$ be the corresponding foliation on $G*M$, and let $\mathcal{S}_{(g,m)}$ be the leaf through $(g,m) \in G * M$. We denote the leaf through $(1^G_{p(m)},\ m)$ simply by $\mathcal{S}_{(1,m)}$.

Lemma 2.2 *For all $(g',m') \in \mathcal{S}_{(g,m)}$, the relation $\beta(g') = \beta(g)$ holds.*

Proof: For $(g',m') \in \mathcal{S}(g,m)$, there exist p vector fields $(\overleftarrow{X_1}, X_1^\dagger), \ldots, (\overleftarrow{X_p}, X_p^\dagger)$, tangent to the foliation $\mathcal{F}$ such that if x_t^i and ξ_t^i are the flows of $\overleftarrow{X_i}$ and of $X_i^\dagger$, respectively, then [17] (see also [16] or [11]).

$$(g',m') = \left(x_{t_1}^1 \circ \cdots \circ x_{t_p}^p(g), \xi_{t_1}^1 \circ \cdots \circ \xi_{t_p}^p(m)\right). \qquad (3)$$

It follows that $\beta(g') = \beta(g)$, since the maps x_t^i are right translations, for $i = 1, 2, \dots, p$. $\square$

For $h \in G$, let l_h be the left translation defined in $G^{\alpha(h)}$, by $l_h(g) = hg$; if we extend the map l_h to $G^{\alpha(h)} * M$, by setting $l_h(g, m) = (hg, m)$, then l_h is defined in the whole leaf $\mathcal{S}_{(g,m)}$ by 2.2, provided that $\alpha(h) = \beta(g)$. In this case we have:

Lemma 2.3 *For all $(g, m) \in G * M$ with $\beta(g) = \alpha(h)$, the following relation*

$$l_h \mathcal{S}_{(g,m)} = \mathcal{S}_{(hg,m)} \tag{4}$$

holds.

Proof: The conclusion is clear, since the first factor of the distribution Δ is invariant under left translations. $\square$

Lemma 2.4 *For all $b \in B$, there exists an open connected neighbourhood V^b of 1_b in the β fibre G^b, and for all $m \in p^{-1}(b)$ there exists an open connected neighbourhood V_m of $(1, m)$ in $\mathcal{S}_{(1,m)}$, such that if P denotes the projection $G \times M \longrightarrow G$ of $G \times M$ onto the first factor G then*

(i) $P_m \colon V_m \longrightarrow V^b$ is a diffeomorphism, where P_m is the restriction of P to the leaf $\mathcal{S}_{(1,m)}$; furthermore, V_m is the connected component of $(1, m)$ in $P_m^{-1}(V^b)$;

(ii) the union $V = \cup_{b \in B} V^b$ is an open neighbourhood of the base B in G.

Proof: Fix $b_0 \in B$, and let $X^1, X^2, \dots, X^p$ be p left invariant vector fields on G, such that for g in a connected neighbourhood U_{b_0} of 1_{b_0} in G, the vector tangents $X^i(g), i = 1, \dots, p$ generate the tangent space $T_g G^{\beta(g)}$. Furthermore, it is always possible to assume that U_{b_0} is α-saturated, that is $U_{b_0} = \alpha^{-1}(\alpha(U_{b_0})$ [1]. Let $Y^i = (X^i)^\dagger$ be the corresponding fundamental vector field on M, for $i = 1, 2 \dots p$. We denote by x_t^i and by y_t^i, for $i = 1, 2, \ldots, p$, the (global) flows of X^i and Y^i, respectively. Consider the maps $f_{b_0}\colon \mathbb{R}^p \longrightarrow G^{b_0}$, and $f\colon \mathbb{R}^p \times \overline{U}_{b_0} \longrightarrow G$ defined, respectively, by

$$f_{b_0}(t_1, t_2, \ldots, t_p) = x_{t_1}^1 \circ x_{t_2}^2 \circ \cdots \circ x_{t_p}^p(1_{b_0}), \tag{5}$$

and by

$$f(t_1, t_2, \ldots, t_p, b) = x_{t_1}^1 \circ x_{t_2}^2 \circ \cdots \circ x_{t_p}^p(1_b), \tag{6}$$

where $\overline{U}_{b_0} = \alpha(U_{b_0})$.

The map f_{b_0} is étale at the origin 0 of $\mathbb{R}^p$, (the β-fibre G^{b_0} is equipped with the induced topology); so there is an open connected neighbourhood W_{b_0} of the origin in $\mathbb{R}^p$, and an open connected neighbourhood V^{b_0} of 1_{b_0} in G^{b_0}, such that $f_{b_0}\colon W_{b_0} \longrightarrow V^{b_0}$ is a diffeomorphism. It follows that, for $m \in M$ with $p(m) = b_0$, the map

$$F_m(t_1, t_2, \ldots, t_p) = (x_{t_1}^1 \circ x_{t_2}^2 \circ \cdots \circ x_{t_p}^p(1_{p(m)}), y_{t_1}^1 \circ y_{t_2}^2 \circ \cdots \circ y_{t_p}^p(m)) \tag{7}$$

is a diffeomorphism, from W_{b_0} onto an open connected neighbourhood $V_m = F_m(W_{b_0})$ of $(1, m)$ in $\mathcal{S}_{(1,m)}$ (equipped with its own leaf topology). Now if $P_{m|\,V_m}$ denotes the restriction of the projection P_m to V_m, then $P_{m|\,V_m} = f_{b_0} \circ F_m^{-1}\colon V_m \longrightarrow V^{b_0}$ is a diffeomorphism. V_m contains $(1, m)$ and is open in $P_m^{-1}(V^b)$, and since $P_m\colon V_m \longrightarrow V^b$ is a diffeomorphism V_m is closed in $P_m^{-1}(V^b)$. It follows that V_m is the connected component of $(1, m)$ in $P_m^{-1}(V^b)$.

We prove now the second assertion of the lemma. The left invariant vector fields $X^1, X^2, \ldots X^p$ are nowhere tangent to the base 1_B; it follows then easily that the map f defined by the relation (6) is étale at $(0, b_0) \in \mathbb{R}^p \times \overline{U_{b_0}}$. Hence, we can select an open neighbourhood W'_{b_0} of the origin in $\mathbb{R}^p$, and an open neighbourhood U'_{b_0} of b_0 in $\overline{U_{b_0}}$, such that $f\colon W'_{b_0} \times U'_{b_0} \longrightarrow A(b_0) = f(W'_{b_0} \times U'_{b_0})$ is a diffeomorphism. By shrinking, if necessary, W'_{b_0} we can assume that $W'_{b_0} \subset W_{b_0}$; hence, $A(b_0)$ is an open neighbourhood of 1_{b_0} in G, contained in V. Since this construction is valid for all $b_0 \in B$, the subset V is open in G. $\square$.

Lemma 2.5 *Under the notations of 2.4, the projection $P_m\colon \mathcal{S}_{(1,m)} \longrightarrow G^{p(m)}$ is a covering map, and since the fibres of G are α simply connected, P_m is a diffeomorphism.*

Proof: We start by proving that the map P_m is étale. If $(g,x) \in \mathcal{S}_{(1,m)}$, then $\mathcal{S}_{(g,x)} = \mathcal{S}_{(1,m)}$, and hence

$$T_{(g,x)}\mathcal{S}_{(1,m)} = T_{(g,x)}\mathcal{S}_{(g,x)} = \left\{ \left(\overleftarrow{X}\ (g)\ ,\ X^{\dagger}(x) \right),\ \mid X \in \Gamma AG \right\}.$$

It is clear now that the projection on the first factor

$$T_{(g,x)}P_m : T_{(g,x)}\mathcal{S}_{(1,m)} \longrightarrow T_g G^{p(m)}$$

is surjective. By a dimension counting argument $T_{(g,x)}P_m$ is a bijective map.

The map P_m is surjective: since the distribution D generated by the family of left invariant vector fields on G is completely integrable, with the β-fibres as the leaves of the corresponding foliation, for $g \in G^{p(m)}$ there exist p left invariant vector fields $X^1, X^2, \ldots, X^p$ on G, such that $g = x^1_{t_1} \circ x^2_{t_2} \circ \cdots \circ x^p_{t_p}(1_{p(m)})$, where x^i_t is the flow of X^i, for $i = 1, 2 \ldots p$. If $Y^i = X^{i\dagger}$, then $(g, y^1_{t_1} \circ y^2_{t_2} \circ \cdots \circ y^p_{t_p}(1_{p(m)}) \in \mathcal{S}_m$, where y^i_t is the flow of $Y^i, i = 1, 2, \ldots, p$.

It is enough now to prove that any $g \in G^{p(m)}$ has an open neighbourhood U_g in $G^{p(m)}$, such that if C denotes a connected component of $P_m^{-1}(U_g)$ in $\mathcal{S}_{(1,m)}$, then $P_m \colon C \longrightarrow U_g$ is a diffeomorphism. In 2.4 we have constructed, for all $m \in M$, an open connected neighbourhood V_m of $(1,m)$ in $\mathcal{S}_{(1,m)}$, and an open neighbourhood $V^{p(m)}$ of $1_{p(m)}$ in $G^{p(m)}$, such that $P_m \colon V_m \longrightarrow V^{p(m)}$ is a diffeomorphism, and $V = \cup_{b \in B} V^b$ is open in G. By using the smoothness of the map $(h,g) \longmapsto h^{-1}g$, defined for $\beta(h) = \beta(g)$, we establish easily the existence of an open connected neighbourhood U of 1_b in G, such that $U^{-1}U \subset V$, for all $b \in B$.

For $g \in G^{p(m)}$, let U_g be the connected open neighbourhood gU of g in $G^{p(m)}$, where U is the open neighbourhood of $1_{p(m)}$, with $U^{-1}U \subset V$. Let C be a connected component of $P_m^{-1}(U_g)$, and fix $(h,x) \in C$. In particular $h = gl$ for some $l \in U$; therefore $\beta(h) = \beta(g) = p(m)$. It follows from $(h,x) \in G * M$, that $\alpha(h) = p(x)$. Now we have $(h,x) \in \mathcal{S}_{(1,m)} = \mathcal{S}_{(h,x)} = l_h \mathcal{S}_{(1,x)}$, by 2.3, and the map $l_h \colon \mathcal{S}_{(1,x)} \longrightarrow \mathcal{S}_{(1,m)}$ is a diffeomorphism. We have $h^{-1}gU = l^{-1}U \subset U^{-1}U \cap G^{p(x)} = V^{p(x)}$; in particular, $h^{-1}gU$ is a connected neighbourhood of $(1_{p(x)})$ in $V^{p(x)}$, and $l_h \colon P_x^{-1}(h^{-1}gU) \longrightarrow P_m^{-1}(gU)$ is a diffeomorphism. It follows that $C = l_h(K)$, where K is the component of $(1,x)$ in $P_x^{-1}(h^{-1}gU)$. Since, from 2.4(i), $P_x \colon K \longrightarrow h^{-1}gU$ is a diffeomorphism, and since $P_{m \mid C} =$

$l_h \circ P_x \circ l_h^{-1}$, the result follows. $\square$.

Proof of the theorem 2.1: Let $(g, m) \in G * M$, then $ig \in G^{p(m)}$. Let $\Phi_g(m)$ be the unique element in M such that $(ig, \Phi_g(m)) \in \mathcal{S}_{(1,m)}$. We start by proving that Φ is a smooth map from $G * M$ to M. The distribution D on G generated by the family of left invariant vector fields on G is completely integrable with the β fibres as leaves. Let $p = dimG - dimB$ be its rank.

For $g_0 \in G$, there exists an open neighbourhood U of g_0 in G, and p left invariant vector fields $X^1, X^2, \ldots, X^p$ in G, such that $X^1, X^2, \ldots, X^p$ generate the distribution D in U, and the brackets $[X^i\ ,\ X^j], i, j \in \{1, 2, \ldots, p\}$, vanish identically in U [11]. Let $Y^i = (X^i)^\dagger$, and let y_t^i be the (global) flow of Y^i, for $i = 1, 2, \ldots, p$. The open set U can always be taken in such a way that $U = \beta^{-1}(\beta(U))$ [1]. Let x_t^i be the flow of X^i, for $i = 1, 2, \ldots, p$. Since $U \cap G^b = G^b$ for all $b \in \beta(U)$, and since G is β-connected, the distribution D induces a distribution D_U on U, whose leaves of the corresponding foliation are the β-fibres G^b, with $b \in \beta(U)$. It follows that, for $b_0 = \beta(g_0)$,

$$g_0 = x_{s_1}^1 \circ x_{s_2}^2 \cdots \circ x_{s_p}^p(1_{b_0}),$$

for some real numbers $s_1, s_2, \ldots, s_p$. Let now $F\colon \mathbb{R}^p \times V \longrightarrow G$, be the map defined by

$$F(t_1, t_2, \ldots, t_p, b) = x_{t_1}^1 \circ x_{t_2}^2 \cdots \circ x_{t_p}^p(1_b),$$

where V is an open neighbourhood of b_0 in B, contained in $\beta(U)$. A direct computation of the tangent linear map TF of F, using the relations $[X^i\ ,\ X^j] = 0$ for $i, j = 1, 2, \ldots p$, shows that

$$T_{(s_1,s_2,\ldots,s_p,b_0)}F(c_1, c_2, \ldots, c_p, X_{b_0}) = c_1X^1(g_0) + c_2X^2(g_0) + \cdots + c_pX^p(g_0) + T(1)X_{b_0}, \tag{8}$$

where $c_1, c_2, \ldots, c_p$ are p real numbers. Hence, if

$$T_{(s_1,s_2,\ldots,s_p)}F(c_1, c_2, \ldots, c_p, X_b) = 0,$$

then $X_{1_b} = 0$, by applying $T\beta$ to the left and the right hand side of (8), and then $c_1 = c_2 = \ldots c_p = 0$. Now, by a dimensional counting argument, the map F is étale at the point $(s_1, s_2, \dots s_p, b_0)$, and its image contains g_0. So, there exists an open neighbourhood I of $(s_1, s_2, \dots s_p)$ in in $\mathbb{R}^p$, and and an open neighbourhood V' of b_0 in B, contained in V, such that that F is a diffeomorphism from $I \times V'$ to an open neighbourhood U' of g_0, contained in U.

For $(g_0, m_0) \in G * M$, let

$$W = \{(g, m) \in G * M \mid g \in U'\},$$

then W is an open neighbourhood of (g_o, m_0) in $G*M$. For $(g, m) \in W$, we have

$$\Phi_g(m) = y^1_{t_1} \circ y^2_{t_2} \circ \cdots \circ y^p_{t_p}(m),$$

with $(t_1, t_2, \dots, t_p) = p_1(F^{-1}(g))$, where p_1 is the projection of $\mathbb{R}^p \times B$ onto the first factor $\mathbb{R}^p$. Since the smoothness is a local property, $\Phi \colon G * M \longrightarrow M$ is a smooth map.

We prove now that Φ satisfies the relations (i), (ii) and (iii) in the definition of a Lie groupoid action, see §1.

(i) From $(ig, \Phi_g(m)) \in G * M$, we obtain

$$p\left(\Phi_g(m)\right) = \beta(g), \ \forall \ (g, m) \in G * M.$$

(ii) Let (h, g) be a composable pair in $G \times G$ and take $m \in M$ such that $\alpha(g) = p(m)$, then $\Phi_{hg}(m)$ is the unique element in M with

$$(i(hg), \ \Phi_{hg}(m)) \in \mathcal{S}_{(1,m)}.$$

If $y = \Phi_g(m)$, then $(ig, y) \in \mathcal{S}_{(1,m)}$ so $\mathcal{S}_{(1,m)} = \mathcal{S}_{(ig,y)} = l_{ig}\mathcal{S}_{(1,y)}$, by (4). Since $(igih, \Phi_{hg}(m)) \in \mathcal{S}_{(1,m)} = l_{ig}\mathcal{S}_{(1,y)}$, we deduce that $(ih, \Phi_{hg}(m)) \in \mathcal{S}_{(1,y)}$, then we have necessarily

$$\Phi_{hg}(m) = \Phi_h(y) = \Phi_h(\Phi_g(m)).$$

(iii) It follows from $(1_{p(m)}, \ m) \in \mathcal{S}_{(1,m)}$, that $\Phi_{1_{p(m)}}(m) = m, \ \forall \ m \in M$.

We have now proved that the map Φ is a left action of G on M. The curve $\left(iExptX(p(m)), \ \Phi_{ExptX(p(m))}(m)\right)$ lies in $\mathcal{S}_{(1,m)}$, for all $X \in \Gamma AG$, and all $m \in M$. By differentiating this curve with respect to t at $t = 0$, we get

$$\left.\frac{d}{dt}\right|_{t=0} \Phi_{ExptX(p(m))}(m) = X^{\dagger}(m), \forall\ X \in \Gamma AG,\ \forall\ m \in M,$$

and that proves that $X_{\Phi}^{\dagger} = X^{\dagger},\ \forall\ X \in \Gamma AG.$

The uniqueness: assume the existence of a left action Ψ of G on M such that:

$$\left.\frac{d}{dt}\right|_{t=0} \Psi_{ExptX(p(m))}(m) = X^{\dagger}(m),\ \forall\ X \in \Gamma AV,\ \forall\ m \in M.$$

For any $(h, m) \in G * M$, let

$$d_{(h,m)} = \left\{(hig,\ \Psi_g(m)),\ g \in G_{p(m)}\right\}.$$

Since G is α-connected, $d_{(h,m)}$ is a connected submanifold of $G * M$ of dimension $dimG - dimB$. For any $X \in \Gamma AG$, the integral curve $\big(hiExptX(p(m)),\ \Psi_{ExptX(p(m))}(m)\big)$ lies in $d_{(h,m)}$. It follows, by a differentiation process that

$$\Delta_{(h,m)} \subseteq T_{(h,m)}d_{(h,m)},$$

and then, by a dimensional argument, $\Delta_{(h,m)} = T_{(h,m)}d_{(h,m)}$; that is, $d_{(h,m)}$ is an integral manifold for the distribution Δ. Therefore, $(hig, \Psi_g(m)) \in \mathcal{S}_{(h,m)}$. By the lemma 2.3, we have $(ig, \Psi_g(m)) \in \mathcal{S}_{(1,m)}$. Since $\Phi_g(m)$ is the unique element of M such that $(ig, \Phi_g(m)) \in \mathcal{S}_{(1,m)}$, we have $\Phi = \Psi$ $\square$.

Remark 2.6 *The module of left (or right) invariant vector fields on G is composed of complete vector fields if B is compact [7](see also [1]).*

Remark 2.7 The conclusion on the uniqueness subsistes if we assume that Ψ is only defined in an open neighbourhood W of M in $G * M$, the manifold M is then identified with the submanifold $\left\{(1_{p(m)}, m)|\ m \in M\right\}$ of $G * M$. The properties (i), (ii) and (iii) defining a Lie groupoid action being satisfied locally by Ψ, that is Ψ is *a local action* of G on M.

Remark 2.8 Replacing the left invariant vector field $\overleftarrow{X}$ by the right invariant $\overrightarrow{X}$ in the expression of the distribution Δ, yields the existence of a unique right action of G on $p : M \longrightarrow B$ such that the induced Lie algebroid action is the given Lie algebroid action.

3 Integration of morphisms

We assume here that there are given two Lie groupoids G and G' on a common compact base B, such that G is α-simply connected and α-connected. We denote by by $G * G'$ the submanifold $\{(g, g') \in G \times G' \mid \alpha_G(g) = \beta_{G'}(g')\}$ of $G \times G'$. The following theorem is the main result of this section. It was pointed out to me by K. Mackenzie that he has obtained with P. Xu a general version in [9].

Theorem 3.1 *If $F : AG \longrightarrow AG'$, be a base preserving Lie algebroid morphism, there exists then a unique Lie groupoid morphism $f : G \longrightarrow G'$, over the base B, such that $Af = F$.*

For $X \in \Gamma AG$, let $X^\dagger$ be the right invariant vector field $\overrightarrow{F(X)}$ on G'. We check easily that the map $\longrightarrow X^\dagger$ is an action of AG on $\beta : G' \longrightarrow B$, by complete infinitesimal generators, since B is compact. Now since G is α-connected and α-simply connected, there exists a unique left action Φ of G on $\beta : G' \longrightarrow B$, such that $X_\Phi^\dagger = X^\dagger$, by the theorem 2.1.

Proposition 3.2 *For all $(g, g') \in G * G'$ the relation*

$$\Phi_g(g') = \Phi_g(1^{G'}_{\alpha(g)})g \tag{9}$$

holds.

Lemma 3.3 *1. The relation $\alpha\Phi_g(g') = \alpha(g')$ holds, for all $(g, g') \in G * G$*

2. if $g = g_1.g_2 \ldots g_p$, and if $\Phi_{g_i}(g') = \Phi_{g_i}(1^{G'}_{a_i})g', \forall\ g' \in G^{\alpha(g_i)}$, and for all $i = 1, 2, \ldots p$, with $a_i = \alpha(g_i)$, then

(a) $\Phi_g(1^{G'}_{\alpha g}) = \Phi_{g_1}(1^{G'}_{a_1}).\Phi_{g_2}(1^{G'}_{a_2}). \ldots \Phi_{g_p}(1^{G'}_{\alpha g})$, and

(b) $\Phi_g(g') = \Phi_g(1^{G'}_{\alpha g})g'$, *for all* g' *such that* $\beta g' = \alpha g$.

Proof: 1) For any $X \in \Gamma AG$, we have

$$T\alpha X^{\dagger}(g') = 0 = T\left(g \longmapsto \alpha\Phi_g(g')\right) X(\beta(g')), \forall\ g' \in G'.$$

Since the fibre $G_{\beta(g')}$ is connected, the map $g \longrightarrow \Phi_g(g')$ is constant on $G_{\beta(g')}$. It follows that

$$\alpha\left(\Phi_g(g')\right) = \alpha\left(\Phi_{1_b}(g')\right) = \alpha(g'),$$

where $b = 1^{G'}_{\beta(g')} = 1^{G'}_{\alpha(g)}$.

2) We prove simultaneously the assertions (a) and (b) by induction on the number of elements g_i involved in the decomposition of g. Assume that

(i) $\Phi_{g_2 \cdots g_p}(1^{G'}_{\alpha(g)}) = \Phi_{g_2}(1^{G'}_{a_2}) \ldots \Phi_{g_p}(1^{G'}_{a_p})$, and

(ii) $\Phi_{g_2 \ldots g_p}(g') = \Phi_{g_2 \cdots g_p}(1^{G'}_{\alpha(g')})g'$,

then we get the relation (a) by applying Φ_{g_1} to the left and the right hand side of (i); we get the relation (b) by applying Φ_{g_1} to (ii), and by using the relation (a). □

Proof of the propositon 3.2: Let $(g, g') \in G * G'$; since the α fibres of G are connected, there exist, [1], p sections $X^1, X^2, \ldots, X^p$ of the Lie algebroid AG such that

$$g = x^1_{t_1} \circ x^2_{t_2} \circ \ldots \circ x^p_{t_p}(1^G_{\beta(g')}),$$

where x^i_t denotes the (global) flow of the right invariant vector field $\overrightarrow{X^i}$. As the maps x^i_t are left translations, one can write

$$g = x^1_{t_1}(1^G_{a_1}).x^2_{t_2}(1^G_{a_2}). \ldots .x^p_{t_p}(1^G_{a_p}),$$

where each a_i is in B and depends on the numbers $t_1, t_2, \ldots t_p$. The vector fields $X^{i\dagger}$, $i = 1, 2, \ldots, p$ are right invariant, by construction; hence [6]

$$\Phi_{x^i_t(1^G_{\beta(h)})}(h) = \Phi_{x^i_t(1^G_{\beta(h)})}(1^{G'}_{\beta(h)})h, \tag{10}$$

for $i = 1, 2 \ldots, p$ and for all $h \in G'$. The conclusion now follows by applying the lemma 3.3, with $g_i = x^i_{t_i}(1^G_{a_i})$. □

Proof of the theorem 3.1: Let $f : G \longrightarrow G'$ be the map defined by

$$f(g) = \Phi_g(1^{G'}_{\alpha(g)}), \forall\, g \in G, \tag{11}$$

then

- $\alpha'(f(g)) = \alpha'\left(\Phi_g(1^{G'}_{\alpha(g)})\right) = \alpha(g),\ \forall\, g \in G$, by 3.3;
- $\beta'(f(g)) = \beta(g),\ \forall\, g \in G$, since Φ is a left action.
- $f(g_1 g_2) = f(g_1) f(g_2)$, by a straightforward calculation, using the proposition 3.2.

It follows from the above relations that, the map f is a Lie groupoid morphism. On the other hand, for any $X \in \Gamma AG$, we have

$$TfX(b) = \left.\frac{d}{dt}\right|_{t=0} f(Exp\ tX(b)) = X^\dagger(1_b) = F(X)(b), \forall\, b \in B; \tag{12}$$

that is, $Af = F$.
Assume the existence of another Lie groupoid morphism $h : G \longrightarrow G'$, such that $Ah = Af = F$. Let Ψ be the map

$$\Psi_g(g') = h(g)g', \tag{13}$$

defined for all $(g, g') \in G * G'$. One can check then easily that the map Ψ, defined by the relation (10), is a left action of G on $\beta : G' \longrightarrow B$, such that

$$X^\dagger_\Psi = X^\dagger_\Phi,\ \forall\, X \in \Gamma AG.$$

By the theorem 2.1, $\Psi = \Phi$, hence $f = h$. □

Corollary 3.4 *Let $f\colon G \longrightarrow G'$ a Lie groupoid morphism. If*

(1) $Af\colon G \longrightarrow G'$ is an injection then f is an immersion.

(2) If Af is a Lie algebroid isomorphism, and if the α'- fibres of G' are connected and simply connected, then $f\colon G \longrightarrow G'$ is a Lie groupoid isomorphism.

Proof: (1) Let X_g be a vector tangent to G such that $Tf(X_g) = 0$. It follows that $T\alpha X_g = 0$, and

$$TR_{f(g)}^{-1}Tf(X_g) = Tf(TR_g^{-1}X_g) = Af(TR_g^{-1}X_g) = 0,$$

which implies $X_g = 0$.

(2) Since the inverse $(Af)^{-1}\colon AG' \longrightarrow AG$ of the Lie algebroid isomorphism Af is a Lie algebroid isomorphism, there exists a unique Lie groupoid morphism $f'\colon G' \longrightarrow G$ such that $Af' = (Af)^{-1}$. From the relation $A(f \circ f') = A(f) \circ Af' = id_{AG'}$, and from 3.1 we deduce that $f \circ f' = id_{G'}$, and similarly $f' \circ f = id_G$ $\square$.

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