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P. B. JOHNSON

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# COHERENT SPACES CONSTRUCTIVELY

*by P. B. JOHNSON*[1]

**RESUME**. L'analyse du foncteur associant à chaque treillis distributif son *prime spectrum* fournit une caractérisation constructive de la catégorie des espaces et applications cohérents, en révélant comment les relations très étroites bien connues entre le comportement du foncteur et les propriétés structurales de la catégorie proviennent du *théorème de l'idéal premier pour les treillis locaux spatiaux*.

## 1. Introduction

The prime ideal theorem for distributive lattices is well known [2] to be equivalent to the Gödel-Henkin completeness theorem for coherent logic or, yet equivalent [5], the statement that every coherent locale is spatial. Constructively, however, the nature of the category of *coherent spaces* is at best nebulous. Somewhere in between (with regard to how much choice is assumed to be available) a detailed analysis of the functor assigning to each distributive lattice its space of points, the so-called *prime spectrum*, reveals how the reknown intimacy between the character of the functor, on the one hand, and the structural properties of the category, on the other, is impacted by *the prime ideal theorem for spatial frames*. The thrust is, apparently, to reduce some rather ill-defined logical notion (perhaps a completeness theorem comparable to, though weaker than, that of Gödel-Henkin, or a pointedness theorem for some class of toposes, in the sense of Deligne [6], Theorem 7.44) to extremely elementary properties of spaces which are to be developed here.

Some basic notation is now introduced, culminating with the careful specification of the entire *doctrine*, or environment, of each distributive

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lattice. Writing $\kappa X$ for the *sub-join-semilattice of compact elements* of the frame of opens $\Omega X$ of a locale or space $X$, a continuous map $X \xrightarrow{f} Y$ is *coherent*, provided, the corresponding frame homomorphism $f^{-1} : \Omega Y \longrightarrow \Omega X$ maps $\kappa Y$ into $\kappa X$. Familiarity is assumed with [5], chapter 2 and, in particular:

(*i*) The duality $\mathbf{Lat} \xrightarrow{E} \mathbf{COH}^{op}$ between the category of *distributive lattices*, and that of *coherent locales and maps*, given by $\Omega EA = Idl(A)$, and its adjoint $\mathbf{COH}^{op} \xrightarrow{\kappa} \mathbf{Lat}$;

(*ii*) The inclusion $\mathbf{COH} \xrightarrow{\subset} \mathbf{Loc}$ of coherent locales as a *non-full* reflective subcategory of the category of *all locales*, with reflection $\mathbf{Loc} \xrightarrow{R} \mathbf{COH}$ given by $RX = E\Omega X$, for each locale $X$; and

(*iii*) The functor $\mathbf{Loc} \xrightarrow{pt} \mathbf{Sp}$ given by restriction of locale data to the *spatial parts*, which serves as a co-reflection of **Loc** into its full subcategory of *sober spaces.*

A comprehensive devolution of this data in the presence of the axiom of choice may be found in [1]; whereas, proceeding constructively, liberties with regard to the specification of domain or codomain of functorial assignments shall be tolerated, as will the very same symbol denoting a given distributive lattice also be employed to denote its underlying set, and the jargon *lattice* invariably bear the connotation *distributive.*

Each lattice $A$ admits a canonical *kernel pair-coequalizer* presentation:

$$pres(A) \rightrightarrows FA \longrightarrow\!\!\!\!\rightarrow A;$$

the word problem rendered soluble upon identification of the free lattice $FA$ with the apparent lattice structure on the set $\mathbf{COH}(\mathbb{S}^A, \mathbb{S})$, where $\mathbb{S}$ is the *Sierpinski space*; all told, determining the family $pres(A)$ of all pairs of coherent maps $\mathbb{S}^A \rightrightarrows \mathbb{S}$ which agree in the quotient $FA \longrightarrow\!\!\!\!\rightarrow A$.

The entire doctrine of the lattice $A$ is completed with the natural locale maps:

$$\begin{array}{ccc} spec(A) & \xrightarrow{\subset} & \mathbb{S}^A \\ {\scriptstyle a}\downarrow & \nearrow{\scriptstyle b} & \\ EA & & \end{array}$$

where $EA \xrightarrow{b} \mathbb{S}^A$ is the regular monomorphism in (**COH**, preserved under the inclusion into) **Loc**, corresponding to the lattice presentation of $A$, that is, *$EA$ is the joint-equalizer of the familiy of pairs of coherent maps $pres(A)$,* while the functor *spec*, given by the composite $pt \circ E$, left adjoint when construed as taking values in $\mathbf{Sp}^{op}$, provides, for each of its object-values, the canonical regular monomorphism presentation $spec(A) \xrightarrow{\subset} \mathbb{S}^A$ in **Sp**, that is, $spec(A)$ consists of *precisely all those points of $\mathbb{S}^A$ which agree on each and every pair of maps in the very same family* $pres(A)$, giving rise to the natural map $spec(A) \xrightarrow{a} EA$, no doubt the inclusion into $EA$ of it spatial part.

## 2. Coherent spaces constructively

Awaiting introduction is the full subcategory $\mathbf{Lat}_0 \xrightarrow{\subset} \mathbf{Lat}$ of *constructively spatial distributive lattices* which serves as the equalizer of the functor pair,

$$\mathbf{Lat}_0 \xrightarrow{\subset} \mathbf{Lat} \underset{E}{\overset{spec}{\rightrightarrows}}\ \mathbf{Loc}^{op}; \quad (\downarrow a)$$

more perfectly, $\mathbf{Lat}_0$ consists of all that lattice data which is fixed under the natural transformation $spec \xrightarrow{a} E$. The equivalence of categories $\mathbf{Lat} \xrightarrow{\cong} \mathbf{COH}^{op}$ apparently restricts:

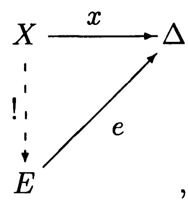
$$\begin{array}{ccc} \mathbf{Lat}_0 & \xrightarrow{\subset} & \mathbf{Lat} \\ {\scriptstyle \cong}\downarrow & & \downarrow{\scriptstyle \cong} \\ \mathbf{Coh}^{op} & \underset{\subset}{\longrightarrow} & \mathbf{COH}^{op} \end{array}$$

to the full subcategories, thereby identifying the category **Coh** of all spatial coherent locales, that is, all sober spaces having coherent topologies, henceforth to be known as *coherent spaces.*

While the category **COH** is complete, and closed under the formation of limits in **Loc**, the non-full inclusion **Coh** $\xrightarrow{\subset}$ **Sp** is (constructively) similiar to the extent that:

**LEMMA: Coh** $\xrightarrow{\subset}$ **Sp** *reflects and preserves limits.*

*Proof:* Given a functor **D** $\xrightarrow{\Delta}$ **Coh** and the diagram:




in which it is assumed that $E \xrightarrow{e} \Delta$ is a cone in **Coh** and, simultaneously, serves as the limit of $\Delta$ in **Sp**, whereby an arbitrary cone $X \xrightarrow{x} \Delta$ in **Coh** factors through $e$ via one and only one continuous map ! as indicated, it follows, since $E$ then comes equipped with the weakest topology compatible with the maps $e$, and an arbitrary compact open in $E$ is a finite union of opens of the form $\cap\{e_i^{-1}(U_i) : i \in F\}$, for some finite set $F \subset |\mathbf{D}|$ and compact opens $U_i \subset \Delta i$, that the map ! is coherent and, therefore, **Coh** $\xrightarrow{\subset}$ **Sp** reflects the limit of $\Delta$.

If, on the other hand, $E \xrightarrow{e} \Delta$ is a limiting cone in **Sp** and $X \xrightarrow{x} \Delta$ is limiting in **Coh** then, in as much as $\mathbf{1} \in |\mathbf{Coh}|$ and any map with singleton domain is coherent, the spaces $X$ and $E$ have the same points. Since $!^{-1} : \Omega E \longrightarrow \Omega X$ is the inclusion of a subframe, and every open $U \in \Omega E$ for which $!^{-1}U$ is compact surely must itself be compact, the maps $e$ are coherent. The spatial limit is then necessarily coherent in the weak topology, hence ! is an isomorphism, and **Coh** $\xrightarrow{\subset}$ **Sp** preserves the limit $X \xrightarrow{x} \Delta$.

*Remark:* **Sp** is closed under formation of limits in the category of all spaces, thus the lemma holds equally well with **Sp** replaced by all spaces throughout, while in no way asserting completeness (the decidability of which comes at issue in section 3) of the category **Coh**.

The results of this section may be summarized as follows:

**THEOREM:** *The following are equivalent for $X \in |\mathbf{Sp}|$:*

1. *$X$ is coherent.*
2. *$X = spec(A)$ for some distributive lattice $A$ for which the inclusion $spec(A) \xrightarrow{a} EA$ is a coherent map.*
3. *$X$ is the equalizer in* **Sp** *of a pair of coherent maps $\mathbb{S}^{\alpha} \rightrightarrows \mathbb{S}^{\beta}$ and the inclusion $X \xrightarrow{\subset} \mathbb{S}^{\alpha}$ is coherent.*

*Proof:* In light of the lemma and the fact that **Loc** $\xrightarrow{pt}$ **Sp** preserves limits, it suffices only to note that, where $E \xrightarrow{\subset} \mathbb{S}^{\alpha}$ is the equalizer in **Loc** of a given (coherent) pair $\mathbb{S}^{\alpha} \rightrightarrows \mathbb{S}^{\beta}$ and $X = ptE$, the inclusion $X \xrightarrow{\subset} E$ is coherent if and only if $X \xrightarrow{\subset} \mathbb{S}^{\alpha}$ is, either condition implying that $X \in |\mathbf{Coh}|$.

*Remark:* As in the case of the lemma, the theorem holds equally well with **Sp** replaced by the category of all spaces throughout; whereas, the question of just which lattices have been implicated must, at present, remain a mystery.

## 3. The prime ideal theorem for spatial frames

A class of lattices $\mathcal{A}$ is said to satisfy the *prime ideal theorem* provided $\mathcal{A} \subset |\mathbf{Lat}_0|$, that is, $EA$ is a spatial locale, for each $A \in \mathcal{A}$, or equivalently, for each pair of ideals $I, J \in Idl(A)$, if $I \not\subset J$, then there exists a prime ideal $P$ such that $I \not\subset P$ and $J \subset P$. While it is well-known that the prime ideal for *all* lattices holds (that is, $spec \xrightarrow{a} E$ is an isomorphism) if and only if, for every pair of distinct elements of a distributive lattice there is a prime ideal containing one and not the other, or, equivalently, every distributive lattice *has* a prime ideal, or merely that every Boolean algebra has a prime ideal; such reductions will not be anticipated for a general class $\mathcal{A}$, in particular, when $\mathcal{A}$ is the class of *spatial frames* $\{A \in |\mathbf{Lat}| \ / \ \exists X \in |\mathbf{Sp}| : A = \Omega X\}$ appearing in the main result.

**THEOREM:** *The following are logically equivalent:*

1. *The natural maps* $spec \xrightarrow{a} E$ *are coherent.*

2. **Coh** *has equalizers of all pairs of (coherent) maps* $\mathbb{S}^\alpha \rightrightarrows \mathbb{S}^\beta$.

3. *The functor* $\mathbf{Lat} \xrightarrow{spec} \mathbf{Loc}^{op}$ *takes coherent values.*

4. $\mathbf{Coh}^{op} \xrightarrow{\kappa} \mathbf{Lat}$ *has a left adjoint.*

5. **Coh** *is complete.*

6. *For each spatial frame* $\Omega$*, the map* $spec(\Omega) \xrightarrow{a\Omega} E\Omega$ *is coherent.*

7. *The prime ideal theorem holds for the class of spatial frames.*

8. **Coh** *is a (non-full) reflective subcategory of* **Sp**.

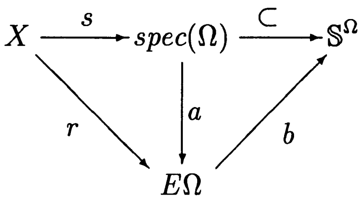
*Proof:* The equivalence of statements $1-3$ is implicit in the proof of Theorem 2, when taken in conjunction with the fact that, for each lattice homomorphism $A \xrightarrow{\theta} B$, the map $spec(\theta)$ appearing in the diagram:

$$\begin{array}{ccc} spec(B) & \xrightarrow{\subset} & \mathbb{S}^B \\ {\scriptstyle spec(\theta)} \downarrow & & \downarrow {\scriptstyle \mathbb{S}^\theta} \\ spec(A) & \xrightarrow[\subset]{} & \mathbb{S}^A \end{array}$$

of **Sp**-morphisms is the unique factorization of $\mathbb{S}^\theta \circ \subset$ through the equalizer $spec(A) \xrightarrow{\subset} \mathbb{S}^A$, and therefore $spec(\theta)$ is coherent provided the inclusions of the spectra are.

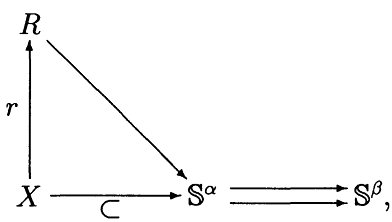
The contravariant hom-functor $\mathbf{Coh}(-, \mathbb{S}) : \mathbf{Coh}^{op} \longrightarrow Sets$, in any case, admits a left adjoint in powers of the object $\mathbb{S} \in |\mathbf{Coh}|$, giving rise to the distributive lattice monad in $Sets$, with Eilenberg-Moore comparison functor $\kappa$, whereby the equivalence $2 \iff 4$ follows from Beck's theorem [6]. While if, on the other hand, $\kappa$ has a left adjoint ($\mathbf{Lat}_0 \overset{\subset}{\longrightarrow} \mathbf{Lat}$ is reflective, and) **Coh** is complete (and co-complete!) thus establishing the equivalence of the statements $1 - 5$.

$1 \Longrightarrow 6$ is trivial. To establish $6 \Longrightarrow 7$, consider $X \in |\mathbf{Sp}|$ with topology $\Omega$, and the diagram:




where all paths $X \longrightarrow \mathbb{S}^{\Omega}$ compose to the canonical *evaluation map.* The map $X \overset{r}{\longrightarrow} E\Omega$ serves as the reflection of the *locale* $X$ into **COH** and, under the hypothesis, the map $s$ must factor *coherently* through it, from which it follows that the inclusion $spec(\Omega) \overset{a}{\longrightarrow} E\Omega$ is an isomorphism. If, on the other hand, $a$ is an isomorphism, then clearly $X \overset{s}{\longrightarrow} spec(\Omega)$ is the reflection of the *space* $X$ into **Coh**, establishing $7 \Longrightarrow 8$.

Finally, if $X$ is the equalizer in **Sp** of the coherent pair illustrated:




and $X \overset{r}{\longrightarrow} R$ the reflection of $X$ into **Coh**, the coherent map $R \longrightarrow \mathbb{S}^{\alpha}$ must factor, in turn, through the equalizer $X$ via a map $R \longrightarrow\!\!\!\!\rightarrow X$ with respect to which $X$ has the quotient topology, whereby the inclusion $X \overset{\subset}{\longrightarrow} \mathbb{S}^{\alpha}$ is coherent, establishing $8 \Longrightarrow 2$, to complete the proof.

*Remark:* Weaker (perhaps) than the equivalent conditions of the theorem is that *the spatial part of every coherent locale is itself coherent,* which would allow the sharpening of Theorem 2 to the statement that *an arbitrary topological space is coherent if and only if it is the spatial equalizer of a pair of coherent maps between powers of* $\mathbb{S}$.

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P.B.JOHNSON
*Masarykova Univerzita, Janáčkovo náměstí 2A, 662 95 Brno*
*(email: johnson@math.muni.cz)*