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ORTHOGONALITY AND CLOSURE OPERATORS

by Lurdes SOUSA

Résumé[1]

Étant donné une sous-catégorie pleine et replète $\mathcal{A}$ d'une catégorie $\mathcal{X}$, nous définissons un opérateur de fermeture qui nous permet de caractériser la classe $\mathcal{A}^{\perp}$ (constituée par tous les $\mathcal{X}$-morphismes qui sont orthogonaux à $\mathcal{A}$) et la sous-catégorie pleine $\mathcal{O}(\mathcal{A})$ des $\mathcal{X}$-objets orthogonaux à $\mathcal{A}^{\perp}$, en termes de densité et fermeture, respectivement. En utilisant cette caractérisation, nous obtenons, inter alia, des conditions suffisantes pour que la sous-catégorie soit l'enveloppe réflexive de $\mathcal{A}$ dans $\mathcal{X}$. Nous donnons aussi des relations intéressantes entre l'opérateur de fermeture introduit et l'opérateur de fermeture régulier.

Introduction

Given a category $\mathcal{X}$ and a full subcategory $\mathcal{A}$ of $\mathcal{X}$, let $\mathcal{A}^{\perp}$ be the class of $\mathcal{X}$-morphisms which are orthogonal to $\mathcal{A}$, i. e., of morphisms $f : X \to Y$ such that, for each $A \in \mathcal{A}$, the function $\mathcal{X}(f, A) : \mathcal{X}(Y, A) \to \mathcal{X}(X, A)$ is a bijection. Let $\mathcal{O}(\mathcal{A})$ denote the orthogonal hull of $\mathcal{A}$ in $\mathcal{X}$, i. e., the full subcategory of $\mathcal{X}$ consisting of all $X \in \mathcal{X}$ such that every $f \in \mathcal{A}^{\perp}$ is orthogonal to X. Let $\mathcal{L}(\mathcal{A})$ and $\mathcal{R}(\mathcal{A})$ denote the limit-closure and, assuming existence, the reflective hull of $\mathcal{A}$ in $\mathcal{X}$,

[1] *Keywords*: orthogonal closure, reflective hull, closure operator, orthogonal closure operator, regular closure operator, $\mathcal{A}$-strongly closed object.

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respectively. It is well-known that $\mathcal{A} \subseteq \mathcal{L}(\mathcal{A}) \subseteq \mathcal{O}(\mathcal{A}) \subseteq \mathcal{R}(\mathcal{A})$. Each of these inclusions may be strict but, under suitable conditions, the equalities $\mathcal{R}(\mathcal{A})=\mathcal{O}(\mathcal{A})$ and $\mathcal{O}(\mathcal{A})=\mathcal{L}(\mathcal{A})$ hold. There is an extensive literature on this subject (cf. [6], [12], [16], [17]; for a more complete list see references in [17]). More recently, J. Adámek, J. Rosický and V. Trnková showed that the problem of the existence of a reflective hull of a subcategory may have a negative answer even when we deal with very reasonable categories, such as the category of bitopological spaces (where the subcategory of all spaces in which both topologies are compact Hausdorff does not have a reflective hull, see [2]), and the category of topological spaces (see [18]). In [7], H. Herrlich and M. Hušek present several problems in $\mathcal{T}op$ related to orthogonality and reflectivity.

On the other hand, a categorical notion of closure operator, introduced by D. Dikranjan and E. Giuli in [3], has proved to be a useful tool in investigating a variety of problems in several areas of category theory (see [5] and references there).

In this paper we introduce a new closure operator which allows us to characterize completely the class $\mathcal{A}^{\perp}$ and the orthogonal hull $\mathcal{O}(\mathcal{A})$ (for suitable $\mathcal{A}$) and to give rather "tight" sufficient conditions for the orthogonal hull to be the reflective hull.

The motivating example described next, which was taken from [8], provides a first approach to the problem we are dealing with.

Let $\mathcal{X}$ be the category with objects the *separated quasi-metric spaces*, i. e., sets X equipped with a function $d : X \times X \to [0, +\infty]$ such that, for every $x, y, z \in X$, $d(x,y) = 0$ iff $x = y$, $d(x,y) = d(y,x)$ and $d(x,z) \leq d(x,y) + d(y,z)$; morphisms of $\mathcal{X}$ are *non-expansive maps*, i. e., maps $f : (X,d) \to (Y,e)$ such that $e(f(x), f(y)) \leq d(x,y)$, for every $(x,y) \in X \times X$. Let $\mathcal{A}$ be the full subcategory of $\mathcal{X}$ whose objects are the complete metric spaces. In this case we have that $\mathcal{R}(\mathcal{A})=\mathcal{O}(\mathcal{A})=\mathcal{L}(\mathcal{A})$. The class $\mathcal{A}^{\perp}$ is the class of all dense embeddings, i. e., all dense injective maps $f : (X,d) \to (Y,e)$ such that $d(x,y) = e(f(x), f(y))$ for every $(x,y) \in X \times X$. The orthogonal hull of $\mathcal{A}$, $\mathcal{O}(\mathcal{A})$, is the subcategory of all *complete separated quasi-metric spaces*, i.e., spaces (X,d) in $\mathcal{X}$ in which every Cauchy-sequence converges. Furthermore, we have that the complete separated quasi-metric spaces are just the objects X which are "strongly closed", in the sense that, for every $Y \in \mathcal{X}$, whenever X is a subspace of Y, X is closed in it.

The *orthogonal closure operator* in a category $\mathcal{X}$ with respect to a convenient class of morphisms $\mathcal{M}$ and a subcategory $\mathcal{A}$ of $\mathcal{X}$, which we shall define in section 1, gives us, under suitable assumptions, the means to characterize of $\mathcal{A}^{\perp}$ and $\mathcal{O}(\mathcal{A})$ in terms of denseness and closedness like in the example refered to above.

Throughout this paper, when it is appropriate, we relate the orthogonal closure operator with the corresponding regular one (see [13], [5]). In particular, we show that the orthogonal closure operator induced by a subcategory is, in general, less or equal to the regular one induced by the same category. On the other hand, under mild conditions on $\mathcal{A}$, the subcategory of strongly closed objects relative to the orthogonal closure operator induced by $\mathcal{A}$ is always contained in $\mathcal{O}(\mathcal{A})$, whereas the subcategory of absolutely closed objects relative to the regular closure operator induced by $\mathcal{A}$ has a very irregular behaviour in what concerns its relation to $\mathcal{O}(\mathcal{A})$ or $\mathcal{R}(\mathcal{A})$ (see [14] and remark in 4.5.2).

Some examples are given in 4.5.

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Our references for background on closure operators are [3] and [5]. In order to make the paper selfcontained we recall here the definition of closure operator as well as some basic related notions.

Let $\mathcal{X}$ be an $\mathcal{M}$-complete category where $\mathcal{M}$ is a class of morphisms in $\mathcal{X}$ which contains all isomorphisms and is closed under composition. We recall that $\mathcal{X}$ is said to be $\mathcal{M}$-complete provided that pullbacks of $\mathcal{M}$-morphisms along arbitrary morphisms exist and belong to $\mathcal{M}$, and multiple pullbacks of families of $\mathcal{M}$-morphisms with common codomain exist and belong to $\mathcal{M}$. The pullback of an $\mathcal{M}$-morphism m along a morphism f is called the inverse image of m under f and it is denoted by $f^{-1}(m)$.

Under the above conditions over $\mathcal{X}$ and $\mathcal{M}$, we have that every morphism in $\mathcal{M}$ is a monomorphism and $\mathcal{M}$ is left-cancellable, i. e., if $n \cdot m \in \mathcal{M}$ and $n \in \mathcal{M}$, then $m \in \mathcal{M}$. It is clear that, for each object

$X \in \mathcal{X}$, the preordered class $\mathcal{M}_X$ of all $\mathcal{M}$-morphisms with codomain X is large-complete. Furthermore, there is a (uniquely determined) class of morphisms $\mathcal{E}$ in $\mathcal{X}$ such that $(\mathcal{E}, \mathcal{M})$ is a factorization system of $\mathcal{X}$.

Now, let $\mathcal{M}$ be a class of monomorphisms in $\mathcal{X}$ which contains all isomorphisms, is closed under composition and is left-cancellable. Let us consider $\mathcal{M}$ as a full subcategory of $\mathcal{X}^2$ and let $U : \mathcal{M} \to \mathcal{X}$ be the codomain functor. A *closure operator* on $\mathcal{X}$ with respect to $\mathcal{M}$ consists of a functor $C : \mathcal{M} \to \mathcal{M}$, such that $UC = U$, endowded with a natural transformation $\delta : Id_{\mathcal{M}} \to C$ such that $U\delta = Id_U$.

If $\mathcal{X}$ is $\mathcal{M}$-complete, the closure operator $C : \mathcal{M} \to \mathcal{M}$ may be equivalently described by a family of operators $(C_X : \mathcal{M}_X \to \mathcal{M}_X)_{X \in \mathcal{X}}$, where $C_X(m) = C(m)$ for each m, satisfying the conditions:

1) $m \leq C_X(m)$, $m \in \mathcal{M}_X$;

2) if $m \leq n$, then $C_X(m) \leq C_X(n)$, $m, n \in \mathcal{M}_X$;

3) $C_X(f^{-1}(m)) \leq f^{-1}(C_Y(m))$, for $(f : X \to Y) \in Mor(\mathcal{X})$ and $m \in \mathcal{M}_Y$.

For each $(m : X \to Y) \in \mathcal{M}$ we put $C(m) = ([m] : [X] \to Y)$ and we denote by $\delta(m)$ the morphism such that $\delta_m = (\delta(m), 1_Y)$, i. e., $[m] \cdot \delta(m) = m$.

A morphism $(m : X \to Y) \in \mathcal{M}$ is called C-*dense* (respectively, C-*closed*) if $C(m) \cong 1_Y$ (respectively, $C(m) \cong m$).

We say that C is *weakly hereditary* (respectively, *idempotent*) if, for each $m \in \mathcal{M}$, $\delta(m)$ is C-dense (respectively, $C(m)$ is C-closed).

1 The orthogonal closure operator

Throughout this paper $\mathcal{X}$ is an $\mathcal{M}$-complete category with pushouts, where $\mathcal{M}$ contains all isomorphisms and is closed under composition, and $(\mathcal{E}, \mathcal{M})$ is the existing factorization system of $\mathcal{X}$.

All subcategories of $\mathcal{X}$ are assumed to be full and isomorphism-closed.

Let $\mathcal{A}$ be a subcategory of $\mathcal{X}$. In order to define the orthogonal closure operator induced by $\mathcal{A}$, we consider, for each $m : X \to Y$ in $\mathcal{M}$, a morphism $[m] : [X] \to Y$ in $\mathcal{M}$ obtained as follows:

(C) For each $(g : X \to A)$ with A in $\mathcal{A}$, we form the pushout $(\overline{m}, g')$ of (m, g) in $\mathcal{X}$. Let $m' \cdot e$ be the $(\mathcal{E}, \mathcal{M})$-factorization of $\overline{m}$ and (m_g, g^*) the pullback of (m', g').

$$
\begin{tikzcd}
X \arrow[rr, "m"] \arrow[dr] \arrow[dd, "g"'] & & Y \arrow[dd, "g'"] \\
& X_g \arrow[ur, "m_g"'] \arrow[dd, "g^*"] & \\
A \arrow[rr, "\overline{m}"'] \arrow[dr, "e"'] & & \bullet \\
& \bullet \arrow[ur, "m'"'] &
\end{tikzcd}
$$

Let $P(m)$ be the class of all $m_g : X_g \to Y$ obtained that way. The morphism $[m] : [X] \to Y$ is the intersection of $P(m)$. It is clear that it is in $\mathcal{M}$.

Proposition 1.1 *With $C_{\mathcal{A}}(m) = [m]$, $m \in \mathcal{M}$, one obtains a closure operator $C_{\mathcal{A}}$ with respect to $\mathcal{M}$.*

Proof. Let $(p, f) : (m : X \to Y) \to (n : Z \to W)$ be a morphism in the category $\mathcal{M}$. We are going to define $C_{\mathcal{A}}(p, f)$. For $(h : Z \to A) \in \mathcal{X}(Z, \mathcal{A})$, let $(\overline{n}, h')$ be the pushout of (n, h), let $n' \cdot q$ be the $(\mathcal{E}, \mathcal{M})$-factorization of $\overline{n}$ and let (n_h, h^*) be the pullback of (n', h'). For $g = h \cdot p$, let the morphisms $\overline{m}$, g', m', e, m_g and g^* be as in (C). Since

$$(h' \cdot f) \cdot m = h' \cdot n \cdot p = \overline{n} \cdot h \cdot p = \overline{n} \cdot g$$

and $(\overline{m}, g')$ is the pushout of (m, g), there is a unique morphism d such that $h' \cdot f = d \cdot g'$ and $\overline{n} = d \cdot \overline{m}$. From the last equality, we get $n' \cdot q = d \cdot m' \cdot e$ and, by the diagonal property, there is a unique morphism k such that $k \cdot e = q$ and $n' \cdot k = d \cdot m'$.

$$\begin{tikzcd}
X \arrow[rrrr, "m"] \arrow[ddr, "p"] \arrow[ddd, "g"'] & & & & Y \arrow[ddr, "f"] \arrow[ddd, "g'"] & \\
 & & X_g \arrow[urr, "m_g"] \arrow[dd, "g^*"] & & & \\
 & Z \arrow[rrrr, "n"] \arrow[dd, "h"] & & & & W \arrow[dd, "h'"] \\
A \arrow[rr, "e"] \arrow[dr, "1_A"'] & & \bullet \arrow[rr, "m'"] \arrow[dr, "k"] & & \bullet \arrow[dr, "d"] & \\
 & A \arrow[rr, "q"] & & \bullet \arrow[rr, "n'"] & & \bullet
\end{tikzcd}$$

Then we have

$$n' \cdot (k \cdot g^*) = d \cdot m' \cdot g^* = d \cdot g' \cdot m_g = h' \cdot (f \cdot m_g).$$

Since (n_h, h^*) is the pullback of (n', h'), there exists a morphism r_h such that

$$f \cdot m_{h \cdot p} = f \cdot m_g = n_h \cdot r_h.$$

Now, for each $h \in \mathcal{X}(Z, \mathcal{A})$ let t_h be the unique morphism that fulfils $m_{h \cdot p} \cdot t_h = [m]$. Then, since $[n] : [Z] \to W$ is the intersection of $P(n)$ and the equalities

$$n_h \cdot (r_h \cdot t_h) = f \cdot m_{h \cdot p} \cdot t_h = f \cdot [m], h \in \mathcal{X}(Z, \mathcal{A}),$$

hold, there is a unique morphism $u : [X] \to [Z]$ such that $f \cdot [m] = [n] \cdot u$. Taking

$$C_{\mathcal{A}}(p, f) = (u, f) : C_{\mathcal{A}}(m) \to C_{\mathcal{A}}(n),$$

it is easy to see that $C_{\mathcal{A}} : \mathcal{M} \to \mathcal{M}$ is a functor for which $UC = U$, where U is the codomain functor from $\mathcal{M}$ to $\mathcal{X}$.

Let $(m : X \to Y) \in \mathcal{M}$. For each $g \in \mathcal{X}(X, \mathcal{A})$, there is a unique morphism $d_g : X \to X_g$ such that $m_g \cdot d_g = m$ and $g^* \cdot d_g = e \cdot g$. Then, since $[m] : [X] \to Y$ is the intersection of $P(m)$, from the first equality, there is a unique morphism $\delta(m) : X \to [X]$ such that $[m] \cdot \delta(m) = m$. The family of morphisms

$$\delta_m = ((\delta(m), 1_Y) : m \to C(m)), \ m \in \mathcal{M},$$

defines a natural transformation $\delta : Id_{\mathcal{M}} \to C$ such that $U\delta = Id_U$. $\square$

Definition 1.2 We shall call the closure operator $C_{\mathcal{A}} : \mathcal{M} \to \mathcal{M}$ the *orthogonal closure operator* of $\mathcal{X}$ with respect to $\mathcal{M}$ induced by $\mathcal{A}$.

As in the above proof, throughout this paper $\delta(m)$ always denotes the unique morphism such that $m = [m] \cdot \delta(m)$. We use $\delta_{\mathcal{A}}$ instead of δ when it is necessary to specify what subcategory $\mathcal{A}$ is involved.

Next we state some properties of the orthogonal closure operator.

Proposition 1.3 *For subcategories $\mathcal{A}$ and $\mathcal{B}$ of $\mathcal{X}$, we have that:*

1) The orthogonal closure operator induced by a subcategory $\mathcal{A}$ is discrete in the subclass of morphisms with domain in $\mathcal{A}$.

2) If $\mathcal{A} \subseteq \mathcal{B}$ then $C_{\mathcal{B}} \leq C_{\mathcal{A}}$.

3) Under the assumption $SplitMono(\mathcal{X}) \subseteq \mathcal{M}$, for each pair of morphisms $a, b : Y \to A$, with $A \in \mathcal{A}$, if $a \cdot m = b \cdot m$, with $(m : X \to Y) \in \mathcal{M}$, then $a \cdot C_{\mathcal{A}}(m) = b \cdot C_{\mathcal{A}}(m)$.

Proof. 1) and 2) are immediate.

3) Let $g = a \cdot m = b \cdot m$ and let $(\overline{m}, g')$, $m' \cdot e$ and (m_g, g^*) be as in (C). We are going to show that $a \cdot m_g = b \cdot m_g$. The equality $1_A \cdot g = a \cdot m$ implies the existence of a unique morphism t such that $t \cdot \overline{m} = 1_A$ and $t \cdot g' = a$; hence $t \cdot m' \cdot e = t \cdot \overline{m} = 1_A$ and so, since $e \in \mathcal{E}$, e is an isomorphism. Analogously, there is a unique morphism t' such that $t' \cdot \overline{m} = 1_A$ and $t' \cdot g' = b$. Then

$$\begin{aligned} a \cdot m_g = {} & t \cdot g' \cdot m_g = t \cdot m' \cdot g^* = t \cdot \overline{m} \cdot e^{-1} \cdot g^* = e^{-1} \cdot g^* = \\ = {} & t' \cdot \overline{m} \cdot e^{-1} \cdot g^* = t' \cdot m' \cdot g^* = t' \cdot g' \cdot m_g = b \cdot m_g. \end{aligned}$$

Let t_g be the morphism that fulfils the equality $m_g \cdot t_g = C_{\mathcal{A}}(m)$. Hence

$$a \cdot C_{\mathcal{A}}(m) = a \cdot m_g \cdot t_g = b \cdot m_g \cdot t_g = b \cdot C_{\mathcal{A}}(m).$$

□

Using 1.3.3), one may relate this closure operator to the regular closure operator induced by the same category.

We recall that, for a subcategory $\mathcal{A}$ of $\mathcal{X}$ and $RegMono(\mathcal{X}) \subseteq \mathcal{M}$, we define a closure operator $R_{\mathcal{A}} : \mathcal{M} \to \mathcal{M}$ assigning, to each $m \in \mathcal{M}$, the intersection of all n such that $m \leq n$ and n is the equalizer of a pair of morphisms with codomain in $\mathcal{A}$. Such closure operators are called *regular closure operators*. They were introduced by Salbany [13] for $\mathcal{X}$= $\mathcal{T}op$ and $\mathcal{M}$ the class of embeddings, and have been widely investigated (see [3], [5] and references there).

Proposition 1.4 *If $RegMono(\mathcal{X}) \subseteq \mathcal{M}$, then for each subcategory $\mathcal{A}$ of $\mathcal{X}$ we have that $C_{\mathcal{A}} \leq R_{\mathcal{A}}$.*

Proof. It is a consequence of 1.3.3). □

Given a closure operator $C : \mathcal{M} \to \mathcal{M}$ on $\mathcal{X}$, a morphism is called *C-dense* provided that the morphism of $\mathcal{M}$ which is part of its $(\mathcal{E}, \mathcal{M})$-factorization is C-dense. If $R_{\mathcal{A}} : \mathcal{M} \to \mathcal{M}$ is a regular closure operator on $\mathcal{X}$ and $\mathcal{X}$ has equalizers, then the $R_{\mathcal{A}}$-dense morphisms of $\mathcal{X}$ are just the *$\mathcal{A}$-cancellable* morphisms, i. e., morphisms f such that if $a \cdot f = b \cdot f$ with the codomain of a and b in $\mathcal{A}$, then $a = b$ (cf. [3] and [5]). The same is not true, in general, for orthogonal closure operators: if $\mathcal{A} = \mathcal{X}$, then the class of $C_{\mathcal{A}}$-dense morphisms coincides with the class of $\mathcal{E}$-morphisms and this one can obviously be different from that of epimorphisms. One example with $\mathcal{A} \neq \mathcal{X}$ and such that $\mathcal{A}$-cancellable morphisms are not necessarily $C_{\mathcal{A}}$-dense is given in 4.5.2 below.

Now we are going to see that $C_{\mathcal{A}}$-dense morphisms play an important role in characterizing $\mathcal{A}^{\perp}$-morphisms, for suitable subcategories $\mathcal{A}$.

Proposition 1.5 *Under the assumption $SplitMono(\mathcal{X}) \subseteq \mathcal{M}$, every $C_{\mathcal{A}}$-dense morphism in $\mathcal{M}$ is $\mathcal{A}$-cancellable.*

Proof. Let $a \cdot m = b \cdot m$, where a and b are morphisms with codomain in $\mathcal{A}$ and $m : X \to Y$ is a dense morphism in $\mathcal{M}$. Then $[m] \cong 1_Y$ and, from 1.3.3), it follows that $a = b$. □

Corollary 1.6 *Assuming that $\mathcal{E}$ is a class of epimorphisms, every $C_{\mathcal{A}}$-dense morphism is $\mathcal{A}$-cancellable.*

2 Dense morphisms and $\mathcal{A}^{\perp}$-morphisms

From now on we assume further that $\mathcal{X}$ is an $(\mathcal{E}, \mathbb{M})$-category, with $\mathbb{M}$ a conglomerate of monosources. It follows that $\mathcal{M} = \mathbb{M} \cap Mor(\mathcal{X})$ and that $\mathcal{E}$ is a class of epimorphisms. (cf. [1], [16]).

Let $\mathcal{A}$ be a subcategory of $\mathcal{X}$. By $\mathbb{M}(\mathcal{A})$ we denote the full subcategory of $\mathcal{X}$ consisting of all $X \in \mathcal{X}$ such that the source $\mathcal{X}(X, \mathcal{A})$

belongs to $\mathbb{M}$ which, as it is well known, is the $\mathcal{E}$-reflective hull of $\mathcal{A}$ in $\mathcal{X}$ (cf. [1]).

We remark that an $\mathcal{X}$-morphism f is orthogonal to $\mathcal{A}$ if and only if the image of f by the reflector in $\mathbb{M}(\mathcal{A})$ is orthogonal to $\mathcal{A}$, and that the orthogonal hull of $\mathcal{A}$ in $\mathcal{X}$ coincides with the orthogonal hull of $\mathcal{A}$ in $\mathbb{M}(\mathcal{A})$ ([15]). So, in order to look for a characterization of the orthogonal hull as well as for conditions under which $\mathcal{O}(\mathcal{A})$ is the reflective hull, we can assume without loss of generality that $\mathbb{M}(\mathcal{A}) = \mathcal{X}$. This is often assumed for the rest of the paper.

In the sequel, we will often make use of the following

Lemma 2.1 *If $\mathcal{A}$ is a subcategory of $\mathcal{X}$ such that $\mathbb{M}(\mathcal{A}) = \mathcal{X}$, then a morphism of $\mathcal{X}$ is $\mathcal{A}$-cancellable if and only if it is an epimorphism.*

Proof. If $f : X \to Y$ is an $\mathcal{A}$-cancellable morphism and $a, b : Y \to Z$ are a pair of morphisms such that $a \cdot f = b \cdot f$, then, for each $g \in \mathcal{X}(Z, \mathcal{A})$, $g \cdot a \cdot f = g \cdot b \cdot f$ and so $g \cdot a = g \cdot b$. As $\mathcal{X}(Z, \mathcal{A})$ is a monosource, it follows that $a = b$. $\square$

We denote by $PS(\mathcal{M})$ the subclass of $\mathcal{M}$ consisting of morphisms for which the pushout along any morphism belongs to $\mathcal{M}$. It is clear that the class $PS(\mathcal{M})$ is pushout stable and that, since $\mathcal{M}$ is closed under composition and left cancellable, the same holds for $PS(\mathcal{M})$. The class $PS(\mathcal{M})$ plays a crucial role in almost all the results presented in the rest of the paper. This is due to the fact that the class $\mathcal{A}^{\perp}$ is pushout stable and that, under the assumption that $\mathbb{M}(\mathcal{A}) = \mathcal{X}$, $\mathcal{A}^{\perp} \subseteq \mathcal{M}$.

For a subcategory $\mathcal{A}$ of $\mathcal{X}$, let $Inj(\mathcal{A})$ be the class of all morphisms $f : X \to Y$ such that, for every $A \in \mathcal{A}$, the map $\mathcal{X}(f, A) : \mathcal{X}(Y, A) \to \mathcal{X}(X, A)$ is surjective.

Lemma 2.2 *If $\mathbb{M}(\mathcal{A}) = \mathcal{X}$, we have that:*

1) $Inj(\mathcal{A})$ consists of all $m \in \mathcal{M}$ such that every pushout of m along a morphism with codomain in $\mathcal{A}$ is a split monomorphism.

2) $\mathcal{A}^{\perp}$ consists of all $m \in PS(\mathcal{M})$ such that every pushout of m along a morphism with codomain in $\mathcal{A}$ is an isomorphism.

Proof. 1) It is clear that an $\mathcal{X}$-morphism f belongs to $Inj(\mathcal{A})$ if and only if the pushout of f along a morphism with codomain in $\mathcal{A}$ is a split monomorphism. It remains to show that $Inj(\mathcal{A}) \subseteq \mathcal{M}$. Let

$(f : X \to Y) \in Inj(\mathcal{A})$ and $m \cdot e$ be the $(\mathcal{E}, \mathcal{M})$-factorization of f. Let $(f_i)_I$ be the source of all morphisms from X to $\mathcal{A}$. For each $i \in I$, there is some f_i' such that $f_i' \cdot f = f_i$. Then we have $(f_i' \cdot m) \cdot e = f_i \cdot 1_X, i \in I$. Since $(f_i)_I \in \mathbb{M}$ and $e \in \mathcal{E}$, there is a morphism d such that $d \cdot e = 1_X$. Hence, since $e \in \mathcal{E}$, it is an isomorphism and so $f \in \mathcal{M}$.

2) If $f \in \mathcal{A}^{\perp}$, then f is $\mathcal{A}$-cancellable and so, by 2.1, it is an epimorphism. Hence, using the fact that epimorphisms are pushout stable, it is easily seen that a morphism f belongs to $\mathcal{A}^{\perp}$ if and only if the pushout of m along a morphism with codomain in $\mathcal{A}$ is an isomorphism. On the other hand, from 1), $\mathcal{A}^{\perp} \subseteq \mathcal{M}$ and, since $\mathcal{A}^{\perp}$ is stable under pushouts, it follows that $\mathcal{A}^{\perp} \subseteq PS(\mathcal{M})$. □

Theorem 2.3 *For a subcategory $\mathcal{A}$ of $\mathcal{X}$ such that $\mathbb{M}(\mathcal{A}) = \mathcal{X}$, $\mathcal{A}^{\perp}$ consists of all $C_{\mathcal{A}}$-dense morphisms in $PS(\mathcal{M})$.*

Proof. Let $m \in \mathcal{A}^{\perp}$. Then, by 2.2.2), $m \in PS(\mathcal{M})$ and every $m_g \in P(m)$ is an isomorphism, which implies $[m] \cong 1_Y$, i. e., m is dense. Conversely, let $m : X \to Y$ be in $PS(\mathcal{M})$ such that $[m] \cong 1_Y$. Hence every $m_g \in P(m)$ must be an isomorphism. Now, let us recall that every pullback of a pushout is a pushout, i. e., if (m', g') is the pushout of (m, g) and (m^*, g^*) is the pullback of (m', g') then (m', g') is the pushout of (m^*, g^*). Then, for each $g \in \mathcal{X}(X, \mathcal{A})$, the pushout of (m, g) is the pushout of m_g along a certain morphism, thus it is an isomorphism. Therefore, by 2.2.2), $m \in \mathcal{A}^{\perp}$. □

Corollary 2.4 *Let $\mathcal{D}$ be the class of all morphisms n such that $n \cong \delta_{\mathcal{A}}(m)$ for some $m \in PS(\mathcal{M})$. Then $\mathcal{D} \subseteq Inj(\mathcal{A})$ and, whenever $\mathbb{M}(\mathcal{A}) = \mathcal{X}$, a $\mathcal{D}$-morphism is $C_{\mathcal{A}}$-dense if and only if it is an epimorphism.*

Proof. If $(\delta(m) : X \to [X]) \in \mathcal{D}$ and $g : X \to A$ is a morphism with codomain in $\mathcal{A}$, let (m_g, g^*) be the pullback of the pushout of (m, g) and let d_g be the morphism such that $m_g \cdot d_g = [m]$; then we have $g = (g^* \cdot d_g) \cdot \delta(m)$. Thus $\delta(m) \in Inj(\mathcal{A})$. Now, since $\mathbb{M}(\mathcal{A}) = \mathcal{X}$, $\mathcal{A}^{\perp} = Inj(\mathcal{A}) \cap Epi(\mathcal{X})$ (using 2.1) and $\mathcal{A}^{\perp} \subseteq \mathcal{D}$(by 2.3); so we have that $\mathcal{A}^{\perp} = \mathcal{D} \cap Epi(\mathcal{X})$, from what follows that every epimorphism belonging to $\mathcal{D}$ is dense. □

3 Strongly closed objects and the orthogonal hull

For a closure operator $C : \mathcal{M} \to \mathcal{M}$ of $\mathcal{X}$, an object $X \in \mathcal{X}$ is said to be *C-absolutely closed* if every morphism in $\mathcal{M}$ with domain X is C-closed. For the case in which C is a regular closure operator, the C-absolutely closed objects were studied in [4] and in [14].

In order to characterize the orthogonal hull of a subcategory of $\mathcal{X}$ by means of the orthogonal closure operator, we consider the following

Definition 3.1 An object $X \in \mathcal{X}$ is said to be *$\mathcal{A}$-strongly closed* provided that each morphism in $PS(\mathcal{M})$ with domain X is $C_{\mathcal{A}}$-closed.

We denote by $\mathcal{SCl}(\mathcal{A})$ the subcategory of all $\mathcal{A}$-strongly closed objects .

We shall prove that, under convenient assumptions, the subcategory $\mathcal{SCl}(\mathcal{A})$ of $\mathcal{A}$-strongly closed objects and the orthogonal hull $\mathcal{O}(\mathcal{A})$ coincide.

It is obvious that every object in $\mathcal{A}$ is $C_{\mathcal{A}}$-absolutely closed and that $\mathcal{SCl}(\mathcal{A})$ contains all $C_{\mathcal{A}}$-absolutely closed objects. Next we show that, when $\mathcal{X}$ is the $\mathcal{E}$-reflective hull of $\mathcal{A}$, we also have that $\mathcal{SCl}(\mathcal{A}) \subseteq \mathcal{O}(\mathcal{A})$.

Proposition 3.2 *For $\mathcal{A}$ such that $\mathbb{M}(\mathcal{A}) = \mathcal{X}$, we have that $\mathcal{SCl}(\mathcal{A}) \subseteq \mathcal{O}(\mathcal{A})$.*

Proof. Assume that $\mathbb{M}(\mathcal{A}) = \mathcal{X}$ and let X be an $\mathcal{A}$-strongly closed object of $\mathcal{X}$; in order to show that $X \in \mathcal{O}(\mathcal{A})$, consider $(m : Y \to Z) \in \mathcal{A}^{\perp}$ and $f : Y \to X$; let $(n : X \to W,\ f' : Z \to W)$ be the pushout of (m, f). Then $(n : X \to W) \in \mathcal{A}^{\perp}$ and so, by 2.3, it is $C_{\mathcal{A}}$-dense, i.e., $[n] \cong 1_W$. Since X is strongly closed, $[n] \cong n$. Hence we have $n \cong 1_W$ and $(n^{-1} \cdot f') \cdot m = f$. Therefore X is m-injective and, since, by 2.1, $m \in Epi(\mathcal{X}), X \in \mathcal{O}(\mathcal{A})$. $\square$

Remark 3.3 The inclusion of $\mathcal{A}$ in the subcategory of all $C_{\mathcal{A}}$-absolutely closed objects and the inclusion of this one in $\mathcal{SCl}(\mathcal{A})$ may be strict (see 4.5 below). But we do not know any example with $\mathcal{O}(\mathcal{A}) \neq \mathcal{SCl}(\mathcal{A})$.

We recall that the class $PS(\mathcal{M})$ is said to be $C_{\mathcal{A}}$*-stable* provided that $C_{\mathcal{A}}(m) \in PS(\mathcal{M})$ for every $m \in PS(\mathcal{M})$. In what follows we show that the assumption that $PS(\mathcal{M})$ is $C_{\mathcal{A}}$-stable implies some relevant properties of $\mathcal{O}(\mathcal{A})$.

Theorem 3.4 *For $\mathcal{A}$ such that $\mathbb{M}(\mathcal{A}) = \mathcal{X}$, the orthogonal closure operator $C_{\mathcal{A}}$ is weakly hereditary in $PS(\mathcal{M})$ if and only if $PS(\mathcal{M})$ is $C_{\mathcal{A}}$-stable. In this case, $C_{\mathcal{A}} : PS(\mathcal{M}) \to PS(\mathcal{M})$ is an idempotent weakly hereditary closure operator and $\mathcal{O}(\mathcal{A}) = \mathcal{SC}l(\mathcal{A})$.*

Proof. If $C_{\mathcal{A}}$ is weakly hereditary in $PS(\mathcal{M})$, consider morphisms $(m : X \to Y) \in PS(\mathcal{M})$ and $f : [X] \to Z$, and let $(m^\sharp, f^\sharp)$ be the pushout of $([m], f)$. The morphism $\delta(m) : X \to [X]$ is an epimorphism, from 2.4 and the fact that it is dense. Hence, since $(m^\sharp, f^\sharp)$ is the pushout of $([m], f)$, it is easy to see that $(m^\sharp, f^\sharp)$ is also the pushout of $(m, f \cdot \delta(m))$. Then $m^\sharp \in \mathcal{M}$.

Conversely, let $(m : X \to Y) \in PS(\mathcal{M})$ be such that $C_{\mathcal{A}}(m) \in PS(\mathcal{M})$. We want to show that $[\delta(m)] \cong 1_{[X]}$. For each $g \in \mathcal{X}(X, \mathcal{A})$, let $(n, \hat{g})$ be the pushout of $(\delta(m), g)$. Since $PS(\mathcal{M})$ is left-cancellable, $\delta(m) \in PS(\mathcal{M})$ and, then, $n \in \mathcal{M}$. Let $(\hat{m}_g, g^\sharp)$ be the pullback of $(n, \hat{g})$, (r, g') be the pushout of $([m], \hat{g})$, (s, h) be the pullback of (r, g') and (d, g^*) be the pulback of (n, h) as illustrated in the following diagram.

$$\begin{tikzcd}
X \arrow[rr, "\delta(m)"] \arrow[ddd, "g"'] & & {[X]} \arrow[rr, "{[m]}"] \arrow[ddd, "\hat{g}"] & & Y \arrow[ddd, "g'"] \\
& \hat{X}_g \arrow[ur, "\hat{m}_g"] \arrow[ddl, "g^\sharp"'] & & \bullet \arrow[ur, "s"] \arrow[ddl, "h"] & \\
& X_g \arrow[urr, "d"] \arrow[dl, "g^*"] & & & \\
\bullet \arrow[rr, "n"] & & \bullet \arrow[rr, "r"] & & \bullet
\end{tikzcd}$$

Then $(s \cdot d, g^*)$ is the pullback of the pushout of (m, g). So $s \cdot d \in P(m)$ and then, there exists some morphism $t_g : [X] \to X_g$ such that $s \cdot d \cdot t_g = [m]$. Thus, we have

$$r \cdot \hat{g} = g' \cdot [m] = g' \cdot s \cdot d \cdot t_g = r \cdot h \cdot d \cdot t_g = r \cdot n \cdot g^* \cdot t_g.$$

It follows that $\hat{g} \cdot 1_{[X]} = n \cdot g^* \cdot t_g$, because $r \in \mathcal{M}$. Then, since $(\hat{m}_g, g^\sharp)$ is the pullback of $(n, \hat{g})$, there is a morphism $w : [X] \to \hat{X}_g$ such that $\hat{m}_g \cdot w = 1_{[X]}$. Thus, for each $g \in \mathcal{X}(X, \mathcal{A})$, we have that $\hat{m}_g \cong 1_{[X]}$, so $[\delta(m)] \cong 1_{[X]}$.

Now, let $C_{\mathcal{A}} : PS(\mathcal{M}) \to PS(\mathcal{M})$ be a weakly hereditary closure operator. By 2.3 and taking into account that $\mathcal{A}^\perp$ is closed under composition, we have that the $C_{\mathcal{A}}$-dense $PS(\mathcal{M})$-morphisms are closed under composition. With the fact that $C_{\mathcal{A}} : PS(\mathcal{M}) \to PS(\mathcal{M})$ is weakly hereditary, this implies that $C_{\mathcal{A}} : PS(\mathcal{M}) \to PS(\mathcal{M})$ is idempotent (see [5]).

In order to show that $\mathcal{O}(\mathcal{A}) \subseteq \mathcal{SC}l(\mathcal{A})$, let $X \in \mathcal{O}(\mathcal{A})$. If $(m : X \to Y) \in PS(\mathcal{M})$, $\delta(m)$ is a dense $PS(\mathcal{M})$-morphism and, then, by 2.3, it belongs to $\mathcal{A}^\perp$. The fact that $X \in \mathcal{O}(\mathcal{A})$ and $(\delta(m) : X \to [X]) \in \mathcal{A}^\perp$ implies that $\delta(m)$ is an isomorphism and, then, $[m] \cong m$. Thus X is $\mathcal{A}$-strongly closed. Therefore, from 3.2, we have that $\mathcal{O}(\mathcal{A}) = \mathcal{SC}l(\mathcal{A})$. $\square$

Corollary 3.5 *If $\mathcal{M} = PS(\mathcal{M})$ and $\mathbb{M}(\mathcal{A}) = \mathcal{X}$, then $C_{\mathcal{A}} : \mathcal{M} \to \mathcal{M}$ is an idempotent weakly hereditary closure operator and $\mathcal{O}(\mathcal{A}) = \mathcal{SC}l(\mathcal{A})$.*

We note that, for instance, in $\mathcal{T}op_0$ and in the category of separated quasi-metric spaces described in the Introduction, we have $\mathcal{M} = PS(\mathcal{M})$ for $\mathcal{M}$ the class of all embeddings. However, this relation does not hold for any epireflective subcategory of $\mathcal{T}$ contained in $\mathcal{T}op_1$ and having a space with more than one point[2]: To prove this assertion, we first recall that, under these conditions, the subcategory $\mathcal{X}$ contains necessarily the 0-dimensional Hausdorff spaces. Now, let $X = [0,1] \cap \mathbb{Q}$ (with the euclidean topology), and consider the embedding $m : X \backslash \{\frac{1}{2}\} \to X$. Let $D = \{0,1\}$ be discrete, and let $f : X \backslash \{\frac{1}{2}\} \to D$ be defined by $f(x) = 0$ for all $x < \frac{1}{2}$ and $f(x) = 1$ for all $x > \frac{1}{2}$. Then the pushout of m along f in $\mathcal{X}$ is $D \to \{*\}$, so $m \notin PS(\mathcal{M})$.

Remark 3.6 Let $\mathcal{X}$ have equalizers, $RegMono(\mathcal{X}) \subseteq \mathcal{M}$, $\mathcal{M} = PS(\mathcal{M})$ and $\mathbb{M}(\mathcal{A}) = \mathcal{X}$. If the regular closure operator $R_{\mathcal{A}}$ is weakly hereditary

[2]M. M. Clementino, private communication

and all $\mathcal{X}$-epimorphisms are $C_{\mathcal{A}}$-dense, then $R_{\mathcal{A}} \cong C_{\mathcal{A}}$. Indeed, under these conditions, the $R_{\mathcal{A}}$-dense morphisms are just the $\mathcal{X}$-epimorphisms and, since, by 3.5, $C_{\mathcal{A}}$ is an idempotent weakly hereditary closure operator, $\mathcal{X}$ has an orthogonal ($C_{\mathcal{A}}$-dense, $C_{\mathcal{A}}$-closed)-factorization system with respect to $\mathcal{M}$ (cf. [5]). Let $m \in \mathcal{M}$; then $C_{\mathcal{A}}(m) \cdot \delta_{\mathcal{A}}(m)$ is a ($C_{\mathcal{A}}$-dense, $C_{\mathcal{A}}$-closed)-factorization of m. By 1.4, there is a morphism d such that $m = R_{\mathcal{A}}(m) \cdot d \cdot \delta_{\mathcal{A}}(m)$. Since $R_{\mathcal{A}}$ is weakly hereditary, the morphism $d \cdot \delta_{\mathcal{A}}(m)$ is an epimorphism, so it is $C_{\mathcal{A}}$-dense. Thus, from the diagonal property, there is a morphism t such that $t \cdot d \cdot \delta_{\mathcal{A}}(m) = \delta_{\mathcal{A}}(m)$, from what follows that d is an isomorphism and, consequently, $R_{\mathcal{A}}(m) \cong C_{\mathcal{A}}(m)$.

This is what holds in example 4.5.1 below. On the other hand, in example 4.5.2 we have that $R_{\mathcal{A}} \ncong C_{\mathcal{A}}$ as a consequence of the fact that the class of $\mathcal{X}$-epimorphisms is different from the class of $C_{\mathcal{A}}$-dense morphisms.

In the following proposition we state some relations between the orthogonal closure operator and the orthogonal hull relative to different subcategories of $\mathcal{X}$.

Proposition 3.7 *Let $\mathcal{A}$ and $\mathcal{B}$ be subcategories of $\mathcal{X}$ with $\mathbb{M}(\mathcal{A}) = \mathcal{X}$. Then:*

1) If $C_{\mathcal{A}} \leq C_{\mathcal{B}}$ when restricted to $PS(\mathcal{M})$ then $\mathcal{O}(\mathcal{B}) \subseteq \mathcal{O}(\mathcal{A})$.

2) If $PS(\mathcal{M})$ is $C_{\mathcal{A}}$-stable and $\mathcal{B} \subseteq \mathcal{O}(\mathcal{A})$ then $C_{\mathcal{A}} \leq C_{\mathcal{B}}$ when restricted to $PS(\mathcal{M})$.

Proof. 1) If $C_{\mathcal{A}} \leq C_{\mathcal{B}}$ in $PS(\mathcal{M})$, then every $C_{\mathcal{A}}$-dense $PS(\mathcal{M})$-morphism is $C_{\mathcal{B}}$-dense. Thus, using 2.3, 2.4 and 1.5, we have that

$$\begin{aligned} \mathcal{A}^{\perp} &= \{m \in PS(\mathcal{M}) \,|\, m \text{ is } C_{\mathcal{A}}\text{-dense}\} \\ &\subseteq \{m \in PS(\mathcal{M}) \,|\, m \text{ is } C_{\mathcal{B}}\text{-dense}\} \mathrel{\dot{\subseteq}} B^{\perp} \end{aligned}$$

and so $\mathcal{O}(\mathcal{B}) \subseteq \mathcal{O}(\mathcal{A})$.

2) Let $(m : X \to Y) \in PS(\mathcal{M})$; $C_{\mathcal{B}}(m)$ is the intersection of all $m_h : X_h \to Y$ such that, for some h^*, (m_h, h^*) is the pullback of the pushout of (m, h), with $h \in \mathcal{X}(X, \mathcal{B})$. We are going to show that, for each such m_h, $C_{\mathcal{A}}(m) \leq m_h$, from what follows that $C_{\mathcal{A}}(m) \leq C_{\mathcal{B}}(m)$. Let $(h : X \to B) \in \mathcal{X}(X, \mathcal{B})$, let (m', g') be the pushout of (m, g) and let

(m_h, h^*) be the pullback of (m', g'). Since $PS(\mathcal{M})$ is $C_{\mathcal{A}}$-stable and left-cancellable, $\delta_{\mathcal{A}}(m) \in PS(\mathcal{M})$ and, by 3.4, and 2.3, $\delta_{\mathcal{A}}(m) \in \mathcal{A}^{\perp}$. Then, since $B \in \mathcal{O}(\mathcal{A})$, there is a morphism $h^\sharp$ such that $h^\sharp \cdot \delta_{\mathcal{A}}(m) = h$. Thus, we have that $h' \cdot C_{\mathcal{A}}(m) \cdot \delta_{\mathcal{A}}(m) = m' \cdot h^\sharp \cdot \delta_{\mathcal{A}}(m)$ and, from the fact that $\delta_{\mathcal{A}}(m)$ is an epimorphism (by 2.1), it follows that $h' \cdot C_{\mathcal{A}}(m) = m' \cdot h^\sharp$. Hence, as (m_h, h^*) is the pullback of (m', g'), there exists a morphism t such that $m_h \cdot t = C_{\mathcal{A}}(m)$, that is, $C_{\mathcal{A}}(m) \leq m_h$. □

Corollary 3.8 *If $\mathcal{M} = PS(\mathcal{M})$ and $\mathcal{A}$ and $\mathcal{B}$ are subcategories of $\mathcal{X}$ such that $\mathbb{M}(\mathcal{A}) = \mathbb{M}(\mathcal{B}) = \mathcal{X}$, then $C_{\mathcal{A}} = C_{\mathcal{B}}$ if and only if $\mathcal{O}(\mathcal{A}) = \mathcal{O}(\mathcal{B})$.*

4 The orthogonal closure operator versus reflectivity

Let $\mathbb{M}(\mathcal{A}) = \mathcal{X}$. It is clear that if $\mathcal{O}(\mathcal{A})$ is reflective in $\mathcal{X}$ then, for each $X \in \mathcal{X}$, the reflection of X in $\mathcal{O}(\mathcal{A})$ is a morphism of $PS(\mathcal{M})$ with codomain in $\mathcal{O}(\mathcal{A})$. The next theorem, which is the main result of this section, states that, when $PS(\mathcal{M})$ is $C_{\mathcal{A}}$-stable, the existence, for each $X \in \mathcal{X}$, of a morphism $(m : X \to Y) \in PS(\mathcal{M})$ with $Y \in \mathcal{O}(\mathcal{A})$ is also a sufficient condition for $\mathcal{O}(\mathcal{A})$ to be the reflective hull of $\mathcal{A}$.

Theorem 4.1 *If $\mathcal{A}$ is a subcategory of $\mathcal{X}$ such that $\mathbb{M}(\mathcal{A}) = \mathcal{X}$, $PS(\mathcal{M})$ is $C_{\mathcal{A}}$-stable and for every $X \in \mathcal{X}$ there is a morphism in $PS(\mathcal{M})$ with domain X and codomain in $\mathcal{O}(\mathcal{A})$, then $\mathcal{O}(\mathcal{A})$ is the reflective hull of $\mathcal{A}$ in $\mathcal{X}$.*

Proof. Firstly, we are going to prove that if Y is an $\mathcal{A}$-strongly closed object and $(m : X \to Y) \in PS(\mathcal{M})$ then $[X]$ (i. e., the domain of $[m]$) is an $\mathcal{A}$-strongly closed object. Consider such Y and m, and let $n : [X] \to Z$ be a morphism in $PS(\mathcal{M})$. We want to show that $[n] \cong n$. Let the diagram

$$\begin{tikzcd} {[X]} \arrow[r, "n"] \arrow[d, "{[m]}"'] & Z \arrow[d, "u"] \\ Y \arrow[r, "n'"'] & \bullet \end{tikzcd}$$

represent the pushout of $(n, [m])$. Then $n' \in PS(\mathcal{M})$ and $[n'] \cong n'$. The $\mathcal{X}^2$-morphism $([m], u) : n \to n'$ is a morphism in the category $PS(\mathcal{M})$. Let $C_{\mathcal{A}}(([m], u)) = (t, u) : [n] \to [n']$. Since $[n'] \cong n'$, we have that, for a suitable t', the following diagram is commutative

$$\begin{tikzcd} \bullet \arrow[r, "{[n]}"] \arrow[d, "t'"'] & Z \arrow[d, "u"] \\ Y \arrow[r, "n'"'] & \bullet \end{tikzcd}$$

Hence

$$n' \cdot [m] \cdot \delta(m) = u \cdot n \cdot \delta(m) = u \cdot [n] \cdot \delta(n) \cdot \delta(m) = n' \cdot t' \cdot \delta(n) \cdot \delta(m).$$

Since n' is a monomorphism, then $[m] \cdot \delta(m) = t' \cdot \delta(n) \cdot \delta(m)$. The morphisms $\delta(n)$ and $\delta(m)$ are $C_{\mathcal{A}}$-dense $PS(\mathcal{M})$-morphisms, hence its composition is a $C_{\mathcal{A}}$-dense $PS(\mathcal{M})$-morphism. Since, by 3.4, $C_{\mathcal{A}} : PS(\mathcal{M}) \to PS(\mathcal{M})$ is an idempotent weakly hereditary closure operator, $\mathcal{X}$ has an orthogonal ($C_{\mathcal{A}}$-dense, $C_{\mathcal{A}}$-closed)-factorization system with respect to $PS(\mathcal{M})$ (cf. [5]). Thus there exists a morphism s such that $s \cdot (\delta(n) \cdot \delta(m)) = \delta(m)$; hence $s \cdot \delta(n) = 1_{[X]}$, from what follows that $\delta(n)$ is an isomorphism and, consequently, $[n] \cong n$.

Now, let $X \in \mathcal{X}$ and $(m : X \to Y) \in PS(\mathcal{M})$ with $Y \in \mathcal{O}(\mathcal{A})$. Then, from 2.3 and 3.4, it follows that $\delta(m) : X \to [X]$ is a reflection from $\mathcal{X}$ to $\mathcal{O}(\mathcal{A})$. □

It is clear that a subcategory $\mathcal{A}$ of $\mathcal{X}$ is $\mathcal{M}$-reflective in $\mathcal{X}$ if and only if $\mathcal{A}$ is reflective in $\mathcal{X}$ and $\mathbb{M}(\mathcal{A}) = \mathcal{X}$. It is easy to see that, if $\mathcal{A}$ is $\mathcal{M}$-reflective in $\mathcal{X}$, the corresponding orthogonal closure operator $C_{\mathcal{A}}$ with respect to $PS(\mathcal{M})$ is defined as follows:

For each $(m : X \to Y) \in PS(\mathcal{M})$, let $r_X : X \to RX$ be the reflection of X. Form the pushout (m', r') of (m, r_X). Then $C_{\mathcal{A}}(m)$ is just the pullback of m' along r'.

Proposition 4.2 *If $\mathcal{A}$ is $\mathcal{M}$-reflective in $\mathcal{X}$ and $PS(\mathcal{M})$ is $C_{\mathcal{A}}$-stable, then the corresponding reflector preserves morphisms of $PS(\mathcal{M})$.*

Proof. Let $(m : X \to A) \in PS(\mathcal{M})$ with $A \in \mathcal{A}$; let $m^\sharp$ be the unique morphism such that $m^\sharp \cdot r_X = m$, where $r_X : X \to RX$ is the reflection of X in $\mathcal{A}$. Since $\mathcal{A}$ is reflective, the equality $\mathcal{A} = \mathcal{O}(\mathcal{A})$ holds and then, following proof of 4.1, we have that $r_X \cong \delta(m)$ and, thus,($[m] : [X] \to A) \cong (m^\sharp : RX \to A)$. Now, let $(m : X \to Y) \in PS(\mathcal{M})$ and let $r_Y : Y \to RY$ be the reflection of Y in $\mathcal{A}$. Since $r_Y \in \mathcal{A}^\perp$, $r_Y \in PS(\mathcal{M})$ and so $r_Y \cdot m \in PS(\mathcal{M})$. Then, we have that $Rm \cong [r_Y \cdot m]$ and, as $PS(\mathcal{M})$ is $C_\mathcal{A}$-stable, $Rm \in PS(\mathcal{M})$.

□

Corollary 4.3 *If $\mathcal{M} = PS(\mathcal{M})$ and $\mathcal{A}$ is $\mathcal{M}$-reflective in $\mathcal{X}$, then the corresponding reflector preserves morphisms of $\mathcal{M}$.*

Remark 4.4 Let $\mathcal{M}$ satisfy the following condition:

> If $m, n, d \in \mathcal{M}$ and $m = n \cdot d$ with $m \in Epi(\mathcal{X})$ then $d \in Epi(\mathcal{X})$.

Then, for every $\mathcal{M}$-reflective subcategory $\mathcal{A}$ of $\mathcal{X}$, $PS(\mathcal{M})$ is $C_\mathcal{A}$-stable. Indeed, let $(m : X \to Y) \in PS(\mathcal{M})$ and consider the pullback of the pushout of (m, r_X) as illustrated by the diagram

$$
\begin{tikzcd}
X \arrow[rr, "m"] \arrow[dr, "\delta(m)"] \arrow[dd, "r_X"'] & & Y \arrow[dd, "r'"] \\
& \bullet \arrow[ur, "{[m]}"'] \arrow[dl, "r^*"] & \\
RX \arrow[rr, "m'"'] & & \bullet
\end{tikzcd}
$$

Since $r_X \in PS(\mathcal{M})$, $r' \in \mathcal{M}$ and so $r^* \in \mathcal{M}$. Hence, using the above property of $\mathcal{M}$ and the fact that $r_X \in Epi(\mathcal{X})$ (by 2.1), it follows that $\delta(m) \in Epi(\mathcal{X})$ and, thus, by 3.4 and 2.4, $[m] \in PS(\mathcal{M})$.

Examples 4.5

1. Let $\mathcal{X}=\mathcal{T}op_0$, i. e., $\mathcal{X}$ is the category of all T_0 spaces and continuous maps, and let $\mathbb{M}$ be the conglomerate of all initial monosources. Then $\mathcal{M}$ is the class of all embeddings and it coincides with $PS(\mathcal{M})$. We can consider every $(m : X \to Y) \in \mathcal{M}$ as an inclusion of a subspace X in Y and identify m with X. Thus, if $C : \mathcal{M} \to \mathcal{M}$ is a closure operator, we identify $C(m)$ with the corresponding subspace of Y which we denote by $C(X)$.

Let $\mathcal{S}$ be the full subcategory having the Sierpiński spaces as its only objects; it is well known that $\mathbb{M}(\mathcal{S}) = \mathcal{X}$. It was shown in [13] that the corresponding regular closure operator $R_{\mathcal{S}} : \mathcal{M} \to \mathcal{M}$ is the b-closure, i. e., given $Y \in \mathcal{T}op_0$, for every subspace X of Y, $R_{\mathcal{S}}(X) = \{y \in Y \mid \overline{\{y\}} \cap H \cap X \neq \emptyset$ for every open neighborhood H of y in $Y\}$. As we proved in 1.4, $C_{\mathcal{S}}(X) \subseteq R_{\mathcal{S}}(X)$ for every subspace X of Y. We are going to show that $R_{\mathcal{S}}(X) \subseteq C_{\mathcal{S}}(X)$, so that $C_{\mathcal{S}}(X) = R_{\mathcal{S}}(X)$.

Let $y \in R_{\mathcal{S}}(X)$. Let S be the Sierpiński space $\{0, 1\}$ with $\{1\}$ the only non-trivial open set. Then $\mathcal{X}(X, S) = \{\chi_G : X \to S \mid G$ is an open set in $X\}$, where

$$\chi_G(x) = \begin{cases} 0 & \text{if } x \notin G \\ 1 & \text{if } x \in G. \end{cases}$$

For a given open set H of Y, let $g = \chi_{H \cap X}$. If W is the pushout object of the inclusion of X into Y along g, then it is easily checked that: if $y \in H$, y is identifyed with 1 in W; if $y \notin H$, y is identifyed with 0 in W. Thus $y \in X_g$ and, since this holds for every $g \in \mathcal{X}(X, S)$, $y \in C_{\mathcal{S}}(X)$.

By 3.5 we have that $\mathcal{SCl}(\mathcal{S}) = \mathcal{O}(\mathcal{S})$, and that $\mathcal{SCl}(\mathcal{S})$ is the reflective hull of $\mathcal{S}$ in $\mathcal{X}$ which, as it is well-known, is the subcategory of all sober spaces, i. e., spaces in which every non-empty irreducible closed set is the closure of a unique point.

The example mentioned in the Introduction provides a situation similar to the last one, that is, for $\mathcal{X}$ the category of separated quasi-metric spaces, $\mathbb{M}$ the conglomerate of initial monosources of $\mathcal{X}$ and $\mathcal{A}$ the subcategory of complete metric spaces, it holds that $\mathcal{M} = PS(\mathcal{M})$, $\mathbb{M}(\mathcal{A}) = \mathcal{X}$, $C_{\mathcal{A}} = R_{\mathcal{A}}$ and $\mathcal{SCl}(\mathcal{A})$ is the reflective hull of $\mathcal{A}$ in $\mathcal{X}$. In this case, we have that, for a subspace X of Y, $C_{\mathcal{A}}(X)$ coincides with the usual closure in $\mathcal{T}op$ of X in Y.

2. Let $\mathcal{X}$ and $\mathbb{M}$ be as in 4.5.1 and let $\mathcal{N}$ be the full subcategory of $\mathcal{T}op_0$ having as objects those spaces which are isomorphic to N, where N is the set $\mathbb{N} = \{1, 2, ...\}$ with the upper topology with respect to the natural order.

   Since $\mathbb{M}(\mathcal{N}) = \mathcal{X}$, we have that $R_{\mathcal{N}} = R_{\mathcal{X}}$ (cf. [5]), i. e., $R_{\mathcal{N}}$ is the b-closure. But the inequality $C_{\mathcal{N}} \leq R_{\mathcal{N}}$ is strict; in fact, let Y

be the set $\mathbb{N} \cup \{\infty\}$ endowded with the topology whose non-empty open sets are all $n \uparrow \cup \{\infty\}$, $n \in \mathbb{N}$. Thus N is a subspace of Y; it is clear that $R_{\mathcal{N}}(N) = Y$ and, on the other hand, $C_{\mathcal{N}}(N) = N$, since $N \in \mathcal{N}$ (see 1.3.1)).

As in 4.5.1, the subcategory $\mathcal{SCl}(\mathcal{N})$ is the reflective hull of $\mathcal{N}$ in $\mathcal{T}op_0$.

Remark. Obviously, in this and the above examples, the notions of $\mathcal{A}$-strongly closed and $C_{\mathcal{A}}$-absolutely closed object coincide. We observe that, in some sense, this concept of closedness for objects has a better behaviour when we deal with orthogonal closure operators than when we deal with regular ones. Indeed, as we have seen, under mild conditions, $\mathcal{A} \subseteq \mathcal{SCl}(\mathcal{A}) \subseteq \mathcal{O}(\mathcal{A})$ and, adding the $C_{\mathcal{A}}$-stability of $PS(\mathcal{M})$, $\mathcal{O}(\mathcal{A}) = \mathcal{SCl}(\mathcal{A})$; whereas, with respect to the regular closure operator, we have that, for instance, N is not $R_{\mathcal{N}}$-absolutely closed although it belongs to $\mathcal{N}$ (cf. [14]).

3. The following categories $\mathcal{X}$ and $\mathcal{A}$ were given in [11] (see also [12]). Let $\mathcal{A}$ have as objects all pairs (X, x) where X is a non-empty set and $x = (x_i)_{i \in Ord}$ is a collection of elements of X, indexed by the class of all ordinals, such that, if $x_i = x_k$ for some pair (i, k) with $i < k$, then, for all $j \geq i$, $x_j = x_i$; the morphisms $f : (X, x) \to (Y, y)$ are the maps $f : X \to Y$ for which $f(x_i) = y_i$, $i \in Ord$. Let $\mathcal{X}$ be obtained from the coproduct of $\mathcal{A}$ with $\mathcal{S}et$ by adding the morphisms $f : X \to (Y, y)$ where $(Y, y) \in \mathcal{A}$ and $f : X \to Y$ is a map. The category $\mathcal{X}$ is concrete over $\mathcal{S}et$.

   Let $\mathbb{M}$ be the conglomerate of all initial monosources of $\mathcal{X}$; clearly, $\mathbb{M}(\mathcal{A}) = \mathcal{X}$. In this case we have that $\mathcal{M} \neq PS(\mathcal{M})$; in fact, let X be a set, let $Y = X \dot{\cup} \{a\}$ and let $y = (y_i)_{i \in Ord}$ with $y_i = a$ for every i. Then $m : X \to (Y, y)$, where m is the inclusion of X in Y, is a morphism of $\mathcal{M} \backslash PS(\mathcal{M})$. It is easy to see that the closure operators $C_{\mathcal{A}}$ and $R_{\mathcal{A}}$ coincide in $\mathcal{M}$, that the class $PS(\mathcal{M})$ is $C_{\mathcal{A}}$-stable and the $C_{\mathcal{A}}$-dense $PS(\mathcal{M})$-morphisms are just the isomorphisms, hence $\mathcal{SCl}(\mathcal{A}) = \mathcal{X}$.

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