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ON THE CHARACTERIZATION OF MONADIC CATEGORIES OVER *SET*

by Enrico M. VITALE

Résumé. Dans la première partie de l'article on examine des conditions pour l'exactitude des catégories des algèbres pour une monade. Dans la deuxième partie on démontre une condition nécessaire et suffisante pour l'équivalence entre catégories exactes avec assez de projectifs; on utilise ensuite cette condition pour obtenir une démonstration élémentaire de la caractérisation des catégories algébriques sur les ensembles et des catégories des préfaisceaux.

Introduction

In this work we look for a new proof of the theorem characterizing monadic categories over $\mathcal{SET}$ (see for example [1]); more precisely, we want to stress the role of the exactness condition. Let us recall the theorem (in the following "epi" means regular epimorphism and "projective" means regular projective object):

Let $\mathcal{A}$ be a category; the following conditions are equivalent

1) $\mathcal{A}$ is equivalent to the category of algebras EM($\mathbb{T}$) for a monad $\mathbb{T}$ over $\mathcal{SET}$

2) $\mathcal{A}$ is an exact category and there exists an object $G \in \mathcal{A}$ such that

- G is projective
- $\forall\, I \in \mathcal{SET} \;\; \exists\, I \bullet G$ (the I–indexed copower of G)
- $\forall\, A \in \mathcal{A} \;\; \exists\, I \bullet G \longrightarrow A$ epi

To prove that 1) implies 2) one takes as G the free algebra over the singleton; viceversa the hypothesis over G imply that $\mathcal{A}$ has enough projectives. So this theorem leads us to study exact categories with enough projectives and, on the other hand, to find conditions such that EM($\mathbb{T}$) is exact and the free algebras are projective.

1 Regularity and exactness for a category of algebras

In this section we sketch some elementary facts about EM($\mathbb{T}$) to obtain a topos theoretic example of a free exact category, i.e. of an exact category with enough projectives (cf. [4]).

Proposition 1.1: *let $\mathcal{A}$ be a regular category and $\mathbb{T}$ a monad over $\mathcal{A}$ (with functor part T);*

1) *T preserves epi's if and only if the forgetful functor $U: EM(\mathbb{T}) \longrightarrow \mathcal{A}$ preserves epi's*

2) *if T preserves epi's, then $EM(\mathbb{T})$ is regular and U preserves and reflects the epi-mono factorization.* ■

Proposition 1.2: *let $\mathcal{A}$ be a regular category and $\mathbb{T}$ a monad over $\mathcal{A}$;*

1) *T sends epi's in split epi's (i.e. epi's with a section) if and only if U sends epi's in split epi's*

2) *if T sends epi's in split epi's, then the free algebras are projectives.*

Sketch of the proof: 2) let $f: (D,d) \twoheadrightarrow (TC, \mu_C)$ be an epi in EM($\mathbb{T}$), where (TC, μ_C) is the free algebra over $C \in \mathcal{A}$ ($\mu: T^2 \longrightarrow T$ is the multiplication of $\mathbb{T}$); f is an epi in EM($\mathbb{T}$) and so in $\mathcal{A}$, then Tf is a split epi in $\mathcal{A}$ and using the section of Tf one can construct the section of f in EM($\mathbb{T}$); the proof of 1) is analogous. ■

Lemma 1.3: *let $\mathcal{A}$ be an exact category and $\mathbb{T}$ a monad over $\mathcal{A}$; consider an equivalence relation $e_1, e_2: (E,e) \rightrightarrows (X,x)$ in $EM(\mathbb{T})$ and consider its coequalizer $q: X \longrightarrow Q$ in $\mathcal{A}$; if $TE \overset{Te_1}{\underset{Te_2}{\rightrightarrows}} TX \xrightarrow{Tq} TQ$ is a coequalizer diagram in $\mathcal{A}$, then $e_1, e_2: (E,e) \rightrightarrows (X,x)$ is effective.* ■

Proposition 1.4: *let $\mathcal{A}$ be an exact category and $\mathbb{T}$ a monad over $\mathcal{A}$; if $\mathbb{T}$ preserves the coequalizers in $\mathcal{A}$ of the equivalence relations in $EM(\mathbb{T})$ and the epi's, then $EM(\mathbb{T})$ is exact.* ■

Corollary 1.5: *let $\mathcal{A}$ be an exact category and $\mathbb{T}$ a monad over $\mathcal{A}$;*

1) *if T is left exact and preserves epi's, then $EM(\mathbb{T})$ is exact*

2) *if the coequalizer in $\mathcal{A}$ of an equivalence relation in $EM(\mathbb{T})$ is a split epi in $\mathcal{A}$, then $EM(\mathbb{T})$ is exact and free algebras are projectives*

3) *the axiom of choice holds in $\mathcal{A}$ if and only if for every monad $\mathbb{T}$ over $\mathcal{A}$ the category $EM(\mathbb{T})$ is exact and the free algebras are projectives.* ■

As each algebra is a quotient of a free algebra, if free algebras are projective then EM($\mathbb{T}$) has enough projectives; if, moreover, EM($\mathbb{T}$) is exact, one has that EM($\mathbb{T}$) is the free exact category over its full subcategory KL($\mathbb{T}$) of free algebras (cf. [4]). An obvious example of such a situation is when $\mathcal{A}$ is $\mathcal{SET}$, or a power of $\mathcal{SET}$, and we can apply the third point of corollary 1.5. Another example is the following:

Example 1.6: *let $\mathcal{E}$ be an elementary topos; the category of sup-lattices in $\mathcal{E}$ is the free exact category over the category of relations in $\mathcal{E}$.*

Proof: let us consider the covariant monad "power-set" $\mathcal{P}: \mathcal{E} \longrightarrow \mathcal{E}$, for which $\mathrm{EM}(\mathcal{P}) = \mathrm{SL}(\mathcal{E})$ and $\mathrm{KL}(\mathcal{P}) = \mathrm{Rel}(\mathcal{E})$; as the corresponding forgetful functor $\mathrm{SL}(\mathcal{E}) \longrightarrow \mathcal{E}$ sends epi's in split epi's (cf. [5]), $\mathrm{SL}(\mathcal{E})$ is a regular category and the objects of $\mathrm{Rel}(\mathcal{E})$ are projectives in $\mathrm{SL}(\mathcal{E})$. It remains to prove that the second point of corollary 1.5 is satisfied; we sketch the proof using the internal language of $\mathcal{E}$: let $e_1, e_2: E \rightrightarrows X$ be an equivalence relation in $\mathrm{SL}(\mathcal{E})$ and $q: X \longrightarrow Q$ its coequalizer in $\mathcal{E}$; we obtain a section $s: Q \longrightarrow X$ defining $\forall\, y \in Y \quad s(y) = \mathrm{Sup}\{x \in X \mid q(x) = y\}$. ■

For "aesthetic reasons", let us observe that the condition stated in 1.5.2 is also necessary; in fact we have the folowing lemma:

Lemma 1.7: *let $\mathbb{T}$ be a monad over a category $\mathcal{A}$;*

1) *if EM($\mathbb{T}$) is regular and free algebras are projectives, then U sends epi's in split epi's*

2) *if U sends epi's in (split) epi's, then the coequalizer in $\mathcal{A}$ of an exact sequences in EM($\mathbb{T}$) is a (split) epi in $\mathcal{A}$* ■

Now we can summarize the previous discussion as follows:

Proposition 1.8: *let $\mathcal{A}$ be an exact category and $\mathbb{T}$ a monad over $\mathcal{A}$; the following conditions are equivalent:*

1) *EM($\mathbb{T}$) is exact and free algebras are projectives*

2) *the coequalizer in $\mathcal{A}$ of an equivalence relation in EM($\mathbb{T}$) is a split epi in $\mathcal{A}$* ■

2 Exact categories with enough projectives

In this section we obtain a property of exact categories which, in the case of monadic categories over $\mathcal{SET}$, will allow us to give a short proof of the characterizing theorem.

Definition 2.1: *a full subcategory $P_{\mathcal{A}}$ of a category $\mathcal{A}$ is said to be a projective cover of $\mathcal{A}$ if*

- *every object of $P_{\mathcal{A}}$ is projective in $\mathcal{A}$*
- *every object of $\mathcal{A}$ is a quotient of an object of $P_{\mathcal{A}}$*

Lemma 2.2: *let $\mathcal{A}$ be a category with kernel pairs and $P_{\mathcal{A}}$ a projective cover of $\mathcal{A}$; $P_{\mathcal{A}}$ "generates" $\mathcal{A}$ via coequalizers.*

(The assertion means that, given a morphism $f: A \longrightarrow B$ in $\mathcal{A}$, we are able to build up a commutative diagram

$$\begin{array}{ccccc} P' & \overset{a_1}{\underset{a_2}{\rightrightarrows}} & P & \overset{p}{\twoheadrightarrow} & A \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle \overline{f}} & & \downarrow{\scriptstyle f} \\ Q' & \overset{b_1}{\underset{b_2}{\rightrightarrows}} & Q & \overset{q}{\twoheadrightarrow} & B \end{array}$$

such that the left square is in $P_{\mathcal{A}}$ and the two horizontal lines are coequalizers, so that f is the unique extension to the quotient.)

Proof: given A in $\mathcal{A}$, there exists P in $P_{\mathcal{A}}$ and an epi $p\colon P \longrightarrow A$; now consider the kernel pair $N(p) \overset{p_1}{\underset{p_2}{\rightrightarrows}} P \overset{p}{\longrightarrow} A$ and again there exists an epi $p'\colon P' \longrightarrow N(p)$ with P' in $P_{\mathcal{A}}$, so that p is the coequalizer of p_1 and p_2 and then of $p'p_1 = a_1$ and $p'p_2 = a_2$; analogously one can work over B and now the three dotted arrows making the following diagram commutative arise respectively from the fact that P is projective and q is an epi, from the universality of $q_1, q_2\colon N(q) \rightrightarrows Q$ and from the fact that P' is projective and q' is an epi

$$\begin{array}{ccccccc} P' & \overset{p'}{\twoheadrightarrow} & N(p) & \overset{p_1}{\underset{p_2}{\rightrightarrows}} & P & \overset{p}{\twoheadrightarrow} & A \\ {\scriptstyle f'}\vdots & & \vdots{\scriptstyle \tilde{f}} & & \vdots{\scriptstyle \overline{f}} & & {\scriptstyle f}\downarrow \\ Q' & \overset{q'}{\twoheadrightarrow} & N(q) & \overset{q_1}{\underset{q_2}{\rightrightarrows}} & Q & \overset{q}{\twoheadrightarrow} & B \end{array}$$

■

Proposition 2.3: *let $\mathcal{A}$ and $\mathcal{B}$ be two exact categories with enough projectives, $P_{\mathcal{A}}$ and $P_{\mathcal{B}}$ two projective covers and $P(\mathcal{A})$ and $P(\mathcal{B})$ the full subcategories of projective objects;*

1) $\mathcal{A}$ is equivalent to $\mathcal{B}$ if and only if $P(\mathcal{A})$ is equivalent to $P(\mathcal{B})$

2) if $P_{\mathcal{A}}$ is equivalent to $P_{\mathcal{B}}$, then $P(\mathcal{A})$ is equivalent to $P(\mathcal{B})$

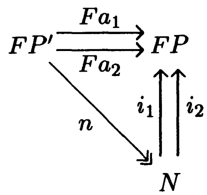
Proof: 1) the non-trivial implication is the "if": let $F\colon P(\mathcal{A}) \longrightarrow P(\mathcal{B})$ be an equivalence; define $F'\colon \mathcal{A} \longrightarrow \mathcal{B}$ as follows: if $f\colon A \longrightarrow B$ is in $\mathcal{A}$, consider its presentation as in the previous lemma

$$\begin{array}{ccccc} P' & \overset{a_1}{\underset{a_2}{\rightrightarrows}} & P & \overset{p}{\twoheadrightarrow} & A \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle \overline{f}} & & \downarrow{\scriptstyle f} \\ Q' & \overset{b_1}{\underset{b_2}{\rightrightarrows}} & Q & \overset{q}{\twoheadrightarrow} & B \end{array}$$

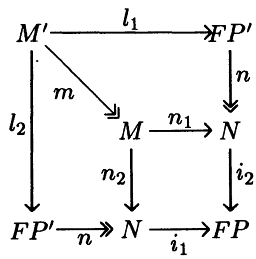
and put $F'f$ as the unique extension to the quotient of

$$\begin{array}{ccccc} FP' & \overset{Fa_1}{\underset{Fa_2}{\rightrightarrows}} & FP & \overset{F'p}{\twoheadrightarrow} & F'A \\ {\scriptstyle Ff'}\downarrow & & \downarrow{\scriptstyle F\overline{f}} & & \downarrow{\scriptstyle F'f} \\ FQ' & \overset{Fb_1}{\underset{Fb_2}{\rightrightarrows}} & FQ & \overset{F'q}{\twoheadrightarrow} & F'B \end{array}$$

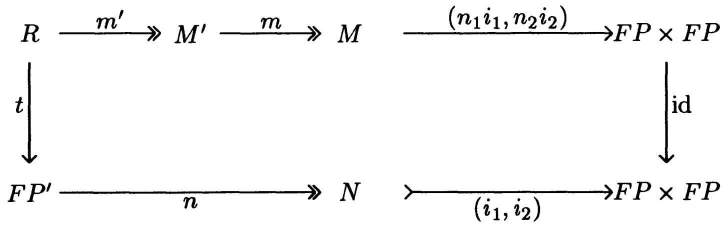
The existence of F' depends on the fact that the (jointly) monic part (i_1, i_2) of the epi-(jointly) mono factorization



is an equivalence relation in $\mathcal{B}$; this follows from the fact that the pair (a_1, a_2) is a pseudo-equivalence relation in $P(\mathcal{A})$ (i.e. as an equivalence relation but we do not require that a_1 and a_2 are jointly monic) and so the same holds for (Fa_1, Fa_2) in $P(\mathcal{B})$. See for instance the transitivity condition: consider the following diagram



where M is the pullback of i_1 and i_2 and M' the pullback of Fa_1 and Fa_2, so that the unique factorization $m: M' \longrightarrow M$ is an epi; consider again a projective cover $m': R \twoheadrightarrow M'$; the transitivity of (Fa_1, Fa_2) in $P(\mathcal{B})$ means exactly that there exists a morphism $t: R \longrightarrow FP'$ making commutative the following diagram



The fact that $m'm$ is an epi and (i_1, i_2) is a mono implies the existence of a morphism $\tau: M \longrightarrow N$ which exhibits the transitivity of $i_1, i_2: N \rightrightarrows FP$. To show that F' is a full and essentially surjective functor is quite obvious (for this recall that F is an equivalence); the faithfulness of F' essentially depends on the fact that the image of (Fb_1, Fb_2), being an equivalence relation in $\mathcal{B}$, is the kernel pair of its coequalizer $F'q$.

2) is trivial under the only condition that $\mathcal{A}$ and $\mathcal{B}$ have enough projectives. ■

The previous proposition explains the name "free" given to an exact category with enough projectives: it is completely determined by the full subcategory of projective objects. In [4] we have discussed the universal property satisfied by this kind of categories.

3 Characterization theorem

Proposition 3.1: *let $\mathcal{C}$ be a category; the following conditions are equivalent:*

1) $\mathcal{C}$ is equivalent to the category KL($\mathbb{T}$) for a monad $\mathbb{T}$ over $\mathcal{SET}$

2) there exists an object $G \in \mathcal{C}$ such that

- $\forall\, I \in \mathcal{SET} \quad \exists\, I \bullet G$
- $\forall\, X \in \mathcal{C} \ \exists\, I \in \mathcal{SET}$ *such that* $X \cong I \bullet G$

Proof: 2) $\Rightarrow$ 1) consider the pair of functors

$$\mathcal{SET} \underset{- \bullet G}{\overset{\mathcal{C}(G,-)}{\rightleftarrows}} \mathcal{C}$$

The first condition says that $- \bullet G$ is left adjoint to $\mathcal{C}(G,-)$; the second condition says that the comparison functor $\mathrm{KL}(\mathbb{T}) \longrightarrow \mathcal{C}$ is essentially surjective and so it is an equivalence (here $\mathbb{T}$ is the monad induced by $- \bullet G \dashv \mathcal{C}(G,-)$). ■

Proposition 3.2: *let $\mathcal{A}$ be a category; the following conditions are equivalent:*

1) $\mathcal{A}$ is equivalent to the category EM($\mathbb{T}$) for a monad $\mathbb{T}$ over $\mathcal{SET}$

2) $\mathcal{A}$ is an exact category and there exists an object $G \in \mathcal{A}$ such that

- *G is projective*
- $\forall\, I \in \mathcal{SET} \quad \exists\, I \bullet G$
- $\forall\, A \in \mathcal{A} \ \exists\, I \bullet G \longrightarrow A$ *epi*

Proof: 2) $\Rightarrow$ 1) let $\mathcal{C}$ be the full subcategory of $\mathcal{A}$ spanned by $I \bullet G$ for $I \in \mathcal{SET}$; by proposition 3.1, $\mathcal{C} \simeq \mathrm{KL}(\mathbb{T})$ for a monad $\mathbb{T}$ over $\mathcal{SET}$; so, by proposition 2.3, $\mathcal{A} \simeq \mathrm{EM}(\mathbb{T})$ because $\mathcal{C}$ is a projective cover of $\mathcal{A}$ and $\mathrm{KL}(\mathbb{T})$ is a projective cover of $\mathrm{EM}(\mathbb{T})$. ■

4 Presheaf categories

The two previous propositions can be generalized to characterize KL($\mathbb{T}$) and EM($\mathbb{T}$) when $\mathbb{T}$ is a monad over $\mathcal{SET}^X$ for $X \in \mathcal{SET}$ (to get examples as presheaf categories); the short proof suggested for proposition 3.2 remains, of course, unchanged. It is not surprising (cf. [6]) that proposition 2.3 allows us also to give a short proof for the characterization of presheaf categories (cf. [2], [3]). In the next lemma, Fam$\mathbb{C}$ is the sum completion of a small category $\mathbb{C}$.

Lemma 4.1: *let $\mathbb{C}$ be a small category and $\mathcal{B}$ the full subcategory of $\mathcal{SET}^{\mathbb{C}^{op}}$ spanned by sums of representable functors; $\mathcal{B}$ is equivalent to Fam$\mathbb{C}$.*

Proof: consider the unique extension $Y'\colon \text{Fam}\mathbb{C} \longrightarrow \mathcal{B}$ of the Yoneda embedding $Y\colon \mathbb{C} \longrightarrow \mathcal{B}$; obviously Y' is essentially surjective; its fullness and faithfulness easily follow from Yoneda's lemma. ■

Lemma 4.2: *let $\mathcal{B}$ be a category with disjoint sums and strict initial object; the following conditions are equivalent*

(1) $\mathcal{B}$ is equivalent to the category Fam$\mathbb{C}$ for a small category $\mathbb{C}$

(2) there exists a small subcategory $\mathbb{C}$ of $\mathcal{B}$ such that

- $\forall\, B \in \mathcal{B}\ \exists\ \{C_i\}_I$ *with* $C_i \in \mathbb{C}$ *such that* $B \cong \coprod_I C_i$
- $\forall\, f\colon C \longrightarrow \coprod_I C_i$ *with* $C, C_i \in \mathbb{C}\ \ \exists\, i_0 \in I$ *such that f can be factorized through the injection* $C_{i_0} \longrightarrow \coprod_I C_i$
- *the initial object* $0 \notin \mathbb{C}$

Proof: 2) $\Rightarrow$ 1) consider the unique extension $F\colon \text{Fam}\mathbb{C} \longrightarrow \mathcal{B}$ of the full inclusion of $\mathbb{C}$ in $\mathcal{B}$; the first condition implies that F is essentially surjective; the second conditions implies that F is full; the third condition (together with the disjointness and the fact that the initial object is strict) implies that F is faithful. ■

Proposition 4.3: *let $\mathcal{A}$ be an exact category with disjoint sums and strict initial objects; the following conditions are equivalent*

(1) $\mathcal{A}$ is equivalent to the category of presheaves on a small category

(2) $\mathcal{A}$ has a set $\{G_j\}_J$ of regular generators such that

- $\forall\, j \in J\ \ G_j$ *is projective*
- $\forall\ f\colon G \longrightarrow \coprod_I G_i$ *with* $G, G_i \in \{G_j\}_J\ \ \exists\ i_0 \in I$ *such that f can be factorized through the injection* $G_{i_0} \longrightarrow \coprod_I G_i$

(3) $\mathcal{A}$ has a family of absolutely presentable generators

Proof: 1) $\Rightarrow$ 3) and 3) $\Rightarrow$ 2) are obvious (recall that an object $G \in \mathcal{A}$ is absolutely presentable if $\mathcal{A}(G,-)\colon \mathcal{A} \longrightarrow \mathcal{SET}$ preserves colimits).
2) $\Rightarrow$ 1): two cases: first, if the initial object $0 \in \{G_j\}_J$ but $\{G_j\}_J \backslash 0$ is not a family of generators, then $\{G_j\}_J = \{0\}$ and so $\mathcal{A} \simeq 1 \simeq \mathcal{SET}^{\emptyset}$; second, if $0 \notin \{G_j\}_J$ let $\mathbb{C}$ be the full subcategory of generators and $\mathcal{B}$ the full subcategory spanned by sums of generators; by lemma 4.2, $\mathcal{B} \simeq$ Fam$\mathbb{C}$ and, by lemma 4.1, Fam$\mathbb{C}$ is a projective cover of $\mathcal{SET}^{\mathbb{C}^{op}}$; but $\mathcal{B}$ is a projective cover of $\mathcal{A}$, so, by proposition 2.3, $\mathcal{A} \simeq \mathcal{SET}^{\mathbb{C}^{op}}$.
■

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