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MODULES OVER A QUANTALE AND MODELS FOR THE OPERATOR ! IN LINEAR LOGIC

by Kimmo I. ROSENTHAL

Résumé. On démontre que la catégorie *Mod*(**Q**) des modules sur un quantale **Q** (commutatif et unitaire) est un modèle de la logique linéaire pleine au sens de M. Barr. Ainsi, c'est une catégorie *-autonome équippée d'un cotriple ! satisfaisant !(A×B) ~ (!A)⊗(!B) et !1 ~ **Q**, où **Q** en tant que **Q**-module est l'unité pour ⊗ dans *Mod*(**Q**). Pour construire **Q**, on utilise le foncteur libre pour les **Q**-modules ainsi que des formules originellement données par R. Guitart.

INTRODUCTION

*-autonomous categories, originally investigated by Barr [2], have recently become the subject of much interest due to the fact that they provide categorical models for linear logic. Linear logic is a logic of resources developed by J.Y. Girard [6] which has potentially significant applications in theoretical computer science. The precise connection between *-autonomous categories and linear logic was first clarified by Seely [12]. (Also, see Barr [3] and Blute [4].)

One particular aspect of the development of linear logic was the existence of the modal operator 'of course' denoted by !. Seely discussed in [12] some of the categorical properties that ! should possess and ! has been analyzed further by Barr [3] in his recent article. Following Barr, we say that a model of 'full' linear logic is a *-autonomous category $\mathcal{L}$ with finite products together with a cotriple $(!, \epsilon, \delta)$ on $\mathcal{L}$ satisfying that $!(A \times B) \cong (!A) \otimes (!B)$ and $!1 \cong \tau$, where τ is the unit for $\otimes$ in $\mathcal{L}$.

Models for !, i.e. suitable cotriples on *-autonomous categories, have not been easy to find. Girard's original coherent spaces provide a model ([6], [12]) and in [3] Barr discussed modifying the so-called Chu construction to obtain a model for !. Another potentially very interesting model has been investigated by Blute, Panagaden and Seely [5], where ! is modelled by the Fock space construction in functional analysis.

In this article, we provide a new family of models of full linear logic by considering modules over a commutative, unital quantale. Commutative, unital quantales are the commutative monoid objects in the *-autonomous category $\mathcal{S}l$ of sup-lattices. These quantales and their modules were studied by Joyal and Tierney [8]. (For an overview of the theory of quantales, see [9].)

If Q is a commutative, unital quantale, the category $Mod(Q)$, of Q-modules, is a *-autonomous category and we indicate how the free Q-module functor from *Sets* to $Mod(Q)$ extends to a cotriple $! : Mod(Q) \longrightarrow Mod(Q)$ with the requisite structure to make $Mod(Q)$ into a model of full linear logic. Our inspiration and calculations owe a debt to the early work of Guitart [7], where the free Q-module construction is first described and the category $Mod(Q)$ is analyzed in some detail. Guitart's theory of involutive monads deserves further study and may be relevant to developing other examples along these lines.

We begin by briefly describing a simple example, namely the category of sup-lattices. This examples serves to illuminate the more general construction in §2.

§1. An example: the case of sup-lattices

The category Sl of sup-lattices is an example of a *-autonomous category. It was studied in detail by Joyal and Tierney [8] where the *- autonomous structure is described. The covariant power-set functor $\mathcal{P} : Sets \longrightarrow Sl$ is the free sup-lattice functor. It will give rise to a cotriple ! on Sl, which will make Sl into a model of full linear logic, in the sense of Barr [3].

If M is a sup-lattice, define $! : Sl \longrightarrow Sl$ to be the covariant power-set functor. Thus, $!M = \mathcal{P}(M)$, the power set of M.

We have the following two maps:

$\epsilon_M : \mathcal{P}(M) \rightarrow M$ defined by $\epsilon_M(A) = sup A$ for a subset $A \subseteq M$

$\delta_M : \mathcal{P}(M) \rightarrow \mathcal{P}(\mathcal{P}(M))$ given by $\delta_M(A) = \{\{a\} | a \in A\}$.

Proposition: *$(\mathcal{P}, \epsilon, \delta)$ defines a cotriple on Sl. Furthermore, for all sup-lattices A, B, it satisfies that $\mathcal{P}(A \times B) \cong \mathcal{P}(A) \otimes \mathcal{P}(B)$.*

Proof: The fact that $(\mathcal{P}, \epsilon, \delta)$ satisfies the appropriate diagrams for a cotriple is a straightforward exercise. The isomorphism $\mathcal{P}(A \times B) \cong \mathcal{P}(A) \otimes \mathcal{P}(B)$ is discussed in [8] and follows directly from the fact that $\mathcal{P}$ is the free sup-lattice functor.

Corollary : *The category Sl of sup-lattices, together with the cotriple $(\mathcal{P}, \epsilon, \delta)$, is a model of full linear logic.*

§2. The general case: modules over a commutative unital quantale

A monoid in the category Sl is a sup-lattice Q together with an associative binary operation $\circ$ (with an identity element), which preserves sups in both variables. Such structures have been studied under the name *commutative unital quantale* (see [9] for an overview of quantale theory). Quantales are of interest in a variety of areas, in particular theoretical computer science (e.g. [1]) and linear logic ([9]).

Definition 2.1. *Let Q be a unital, commutative quantale. A Q-module is a sup-lattice M together with a function $\cdot : Q \times M \longrightarrow M$ such that*
1) $e \cdot m = m$ for all $m \in M$, where e is the identity of Q
2) $q \cdot (r \cdot m) = (q \circ r) \cdot m$ for all $q, r \in Q, m \in M$
3) $(sup_\alpha q_\alpha) \cdot m = sup_\alpha(q_\alpha \cdot m)$ for all $\{q_\alpha\} \subseteq Q, m \in M$.
4) $q \cdot (sup_\beta m_\beta) = sup_\beta(q \cdot m_\beta)$ for all $q \in Q, \{m_\beta\} \subseteq M$.

A sup-lattice morphism $\psi : M \to N$ is a *Q-module morphism* iff it satisfies $\psi(q \cdot m) = q \cdot \psi(m)$ for all $q \in Q, m \in M$.

Let $Mod(Q)$ denote the category of Q-modules. This category was also studied in [8] by Joyal and Tierney, and we record the following result.

Theorem 2.1. *$Mod(Q)$ is a *-autonomous category.*

The tensor product $M \otimes_Q N$ is the codomain of the universal bimorphism of modules $M \times N \to M \otimes_Q N$, where a bimorphism is a Q-module map in each variable separately. Q is the unit object for $\otimes_Q$. $Hom_Q(M, N)$ is the module of Q-module morphisms $M \to N$ with the obvious Q-module structure.

We should point out that modules over quantales play a significant role in the categorical treatment of process semantics by Abramsky and Vickers [1].

When $Q = \mathbf{2}$, (with $\mathbf{2}$ the two element Boolean algebra), then it follows that $Mod(\mathbf{2}) \cong Sl$. We would like to generalize the cotriple construction of §1 to this general setting of Q-modules.

We first need to discuss the free Q-module functor defined on *Sets*. The first details of this appear in the work of Guitart [7]. It is also discussed much more briefly in [8].

Let M be a set and let $[M, Q]$ denote the set of all functions (in *Sets*) from M to Q. $[M, Q]$ becomes a Q-module under the action $(q \cdot f)(m) = q \cdot f(m)$ for all $m \in M$. Define $! : Sets \longrightarrow Mod(Q)$ by $!(M) = [M, Q]$. ! becomes a covariant functor as follows. If $F : M \to N$ is a function, then define $(!F) : [M, Q] \to [N, Q]$ by $(!F)(f)(n) = sup\{f(m) | F(m) = n\}$.

That !(F) is a Q-module morphism follows directly from the fact that in a quantale $q \cdot (\,)$ preserves suprema. ! lifts to a functor $Mod(Q) \longrightarrow Mod(Q)$.

We record the following result from Guitart [7].

Theorem 2.2 $! : Sets \longrightarrow Mod(Q)$ *is the free Q-module functor.*

We now wish to endow !, viewed as a functor from $Mod(Q)$ to $Mod(Q)$, with the structure of a cotriple by generalizing the construction for sup-lattices (the case $Q = \mathbf{2}$). We shall need to utilize the following functions in $[M, Q]$.

If $m \in M$ and $e \in Q$ is the identity, define $\eta_m : M \to Q$ by $\eta_m(x) = e$ if $x = m$ and $\eta_m(x) = 0$ if $x \neq m$.

To obtain a cotriple structure on !, we need to define appropriate ϵ and δ.

Define $\epsilon_M : [M, Q] \to M$ by $\epsilon_M(f) = sup\{f(m) \cdot m | m \in M\}$.
Define $\delta_M : [M, Q] \to [[M, Q], Q]$ by $\delta_M(f)(g) = f(m)$ if $g = \eta_m$ for some $m \in M$ and $\delta_M(f)(g) = 0$ otherwise.

Both ϵ_M and δ_M are easily seen to be Q- module morphisms.

We record the following simple lemma, which we shall need.

Lemma 2.1 *(1) If M is a Q-module and $m \in M$, then we have $\delta_M(\eta_m) = \eta_{\eta_m}$. (2) Given $g \in [M, Q]$, we have $g = sup_m\{g(m) \cdot \eta_m\}$.*

Theorem 2.3. *$(!, \epsilon, \delta)$ is a cotriple on $Mod(Q)$. Furthermore, we have for all Q-modules M and N that $(!M) \otimes (!N) \cong !(M \times N)$, and $!1 \cong Q$.*

Proof: First, to obtain a cotriple structure, we must verify that several equations hold. Given a Q-module M, we first of all need to obtain the identity function on $[M, Q]$ from the following two maps.
$\epsilon_{[M,Q]} \circ \delta_M : [M, Q] \to [[M, Q], Q] \to [M, Q]$
$(!\epsilon_M) \circ \delta_M : [M, Q] \to [[M, Q], Q] \to [M, Q]$.

To see the first of these, $\epsilon_{[M,Q]}(\delta_M)(g) = sup_f\{(\delta_M(g)(f) \cdot f\}$. But, if $f \neq \eta_m$ for some $m \in M$, then $(\delta_M(g)(f) = 0$. Therefore, our supremum now becomes $sup_m\{(\delta_M(g)(\eta_m) \cdot \eta_m\} = sup_m\{g(m) \cdot \eta_m\}$, by Lemma 2.1.

For the second equality, note that upon applying the functoriality of !, we obtain that $(!\epsilon_M)(\delta_M)(g)(m) = sup_f\{(\delta_M)(g)(f) | \epsilon_M(f) = m\}$. But, $(\delta_M)(g)(f)$ takes on the value 0 unless $f = \eta_m$, in which case we get $g(m)$. Since $\epsilon_M(\eta_m) = m$, it follows that $(!\epsilon_M)(\delta_M)(g)(m) = g(m)$, as desired.

The remaining conditions that need to be verified in checking that $(!, \epsilon, \delta)$ defines a cotriple is that the two composites

$!(\delta_M) \circ \delta_M : [M, Q] \to [[M, Q], Q] \to [[[M, Q], Q], Q]$ and

$\delta_{[M,Q]} \circ \delta_M : [M, Q] \to [[M, Q], Q] \to [[[M, Q], Q], Q]$ are, in fact, equal.
The latter map is most easily analyzed. For a function $k \in [[M, Q], Q]$, we have that $(\delta_{[M,Q]})(\delta_M)(g)(k) = (\delta_M)(g)(f)$ provided $k = \eta_f$ and is 0 otherwise. But, $(\delta_M)(g)(f) = g(m)$ if $f = \eta_m$ and is 0 otherwise. Piecing these facts together, $(\delta_M)(\delta_M)(g)(k) = g(m)$ provided that $k = \eta_{\eta_m}$ and is 0 otherwise.

We must now obtain this calculation for $!(\delta_M) \circ \delta_M$. By definition, we have that $!(\delta_M)(\delta_M)(g)(k) = sup_f\{(\delta_M)(g)(f) | (\delta_M)(f) = k\}$. Since $(\delta_M)(f) = 0$ unless $f = \eta_m$, this equals $sup_m\{g(m) | (\delta_m)(\eta_m) = k\}$. But, by Lemma 2.1., we have that $\delta_M(\eta_m) = \eta_{\eta_m}$ and if $k = \delta_M(\eta_m) = \eta_{\eta_m}$, it must be for a unique m. Thus, we have shown that $!(\delta_M)(\delta_M)(g)(k) = g(m)$ if $k = \eta_{\eta_m}$ and is 0 otherwise, proving that $!(\delta_M)(\delta_M)(g) = (\delta_{[M,Q]})(\delta_M)(g)$ for all g, as desired. This finishes the verification that $(!, \epsilon, \delta)$ forms a cotriple on $Mod(Q)$.

The assertion $(!M) \otimes (!N) \cong !(M \times N)$ follows from the fact that ! is the free Q-module functor and that $\otimes_Q$ is left adjoint to Hom_Q. For any Q-module L, we have the following isomorphisms : $Hom_Q(!M \otimes_Q !N, L) \cong Hom_Q(!M, Hom_Q(!N, L)) \cong$

$Sets(M, Hom_Q(!N, L))$. This , in turn, is isomorphic to $Sets(M, Sets(N, L)) \cong Sets(M \times N, L)) \cong Hom_Q(1(M \times N), L)$. This proves that $(!M) \otimes (!N) \cong !(M \times N)$ and we are done

It may be possible to generalize this construction further as follows. By a *quantaloid* we mean a category Q enriched in Sl. These are a natural generalization of unital quantales, which are quantaloids with one object. Much of the theory of quantales generalizes to quantaloids (see [11]), and it was recently shown in [10] that the notion of Q-bimodule leads to a cyclic (non-symmetric) *-autonomous category. A natural question to consider next is whether one can obtain a suitable model for ! on the category of Q-bimodules, where Q is a quantaloid.

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