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CATEGORIES OF CLOSURE SPACES AND CORRESPONDING LATTICES

by **Claude-Alain FAURE**

Résumé: La catégorie des espaces avec un opérateur de fermeture satisfaisant à l'axiome de séparation T_D (et des applications continues) est équivalente à une catégorie de treillis. En restreignant cette équivalence, on obtient, premièrement, une équivalence de catégories entre les espaces topologiques T_D et certains treillis, et, deuxièmement, l'équivalence bien connue entre la catégorie des géométries et celle des treillis géométriques.

For topological spaces the T_D-separation axiom appears naturally in many problems dealing with the algebraic characterization of topological phenomena (cf. for instance [2] and [10]). In [11] W. J. Thron showed that two T_D-spaces having isomorphic lattices of closed subsets must be homeomorphic. Moreover, he remarked that for these lattices the set of completely irreducible elements is order generating (we say that the lattices are *molecular*). It is therefore not surprising that the category of T_D-spaces is actually equivalent to the category of distributive complete molecular lattices (for a natural choice of morphisms).

As a matter of fact, one does not use the distributivity at all, and the basic equivalence lies between the category of T_D-closure spaces and the category of complete molecular lattices. Then it appears that the equivalence of categories between the geometries and the geometric lattices is a special case of this basic equivalence. For several properties of closure operators we determine which are the corresponding lattice properties; in particular for the Steinitz exchange property and the interesting weak exchange property of P. R. Jones [9].

There is a symmetry between the topological properties (of a closure space or a lattice) and the algebraic ones. Many 'algebraic' results are obtained from 'topological' ones by putting the words *directed subset* in place of *finite subset*, and conversely. As an application of this little symmetry principle a direct definition of the morphisms of locales is given at the end of the paper.

All the proofs are elementary. Some are left as exercise, and some others are omitted because of the symmetry principle.

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0. PRELIMINARIES

All through this paper L denotes a *complete* lattice. We write $\sup A$ for the supremum of an arbitrary subset $A \subseteq L$, but $\bigvee F$ for the supremum of a finite subset $F \subseteq L$ and $\bigsqcup D$ for the supremum of a directed subset $D \subseteq L$ (note that a finite subset may be empty, contrary to a directed one).

0.1 __Definition__ We recall that an element $a \in L$ is *completely irreducible* if there exists $p < a$ such that $x < a$ implies $x \leq p$. The element $p \in L$ is clearly unique and denoted by $p(a)$ (for the predecessor of a). An *atom* of L is an element $a \in L$ with $p(a) = 0$. We write $\mathrm{Irr}(L)$ for the set of completely irreducible elements of L and $\mathcal{A}(L)$ for the subset of $\mathrm{Irr}(L)$ formed by the atoms of L.

0.2 __Definition__ An element $p \in L$ is called *coprime* if $p \leq \bigvee F$ for some finite subset $F \subseteq L$ implies that there exists $x \in F$ with $p \leq x$, and an element $c \in L$ is called *compact* if $c \leq \bigsqcup D$ for some directed subset $D \subseteq L$ implies that there exists $x \in D$ with $c \leq x$. The set of coprime elements of L is denoted by $\mathrm{Sp}(L)$ and the set of compact elements of L by $\mathcal{K}(L)$.

0.3 __Definition__ One says that a subset $S \subseteq L$ is *order generating* if for every x in L one has $x = \sup\{s \in S / s \leq x\}$. The lattice L is called *atomistic* if $\mathcal{A}(L)$ is order generating and *molecular* if $\mathrm{Irr}(L)$ is order generating. And it is called *topological* if $\mathrm{Sp}(L)$ is order generating and *algebraic* if $\mathcal{K}(L)$ is so.

0.4 __Remarks__ **1)** If $S \subseteq L$ is order generating, then $\mathrm{Irr}(L)$ is contained in S. In particular, one has $\mathcal{A}(L) = \mathrm{Irr}(L)$ for every atomistic lattice L. **2)** It is well-known that a complete lattice L is topological if and only if it is isomorphic to the lattice of closed subsets of some topological space X.

0.5 __Definition__ A *closure space* X is a set (also denoted by X) with a closure operator, i.e. a map $\mathcal{C} : \mathcal{P}(X) \to \mathcal{P}(X)$ which satisfies **1)** $A \subseteq \mathcal{C}(A)$, **2)** $A \subseteq B$ implies $\mathcal{C}(A) \subseteq \mathcal{C}(B)$, **3)** $\mathcal{C}(\mathcal{C}(A)) = \mathcal{C}(A)$. A subset $E \subseteq X$ is called *closed* if $\mathcal{C}(E) = E$. The complete lattice of closed subsets of X is denoted by $\Gamma(X)$.

0.6 __Definition__ Let X, Y be two closure spaces. A map $f : X \to Y$ is called *continuous* if $f(\mathcal{C}(A)) \subseteq \mathcal{C}(f(A))$ for every subset $A \subseteq X$. One thus gets a category (the category of closure spaces).

0.7 __Definition__ Let X be a closure space. One says that X is a

a) T_0-*closure space* if $\mathcal{C}(\phi) = \phi$, and $\mathcal{C}(x) = \mathcal{C}(y)$ implies $x = y$;

b) T_D-*closure space* if $\mathcal{C}(x) \setminus x$ is closed for every $x \in X$;

c) T_1-*closure space* if $\mathcal{C}(\phi) = \phi$, and $\mathcal{C}(x) = \{x\}$ for every $x \in X$.

Clearly, any $\mathrm{T_1}$-closure space is a $\mathrm{T_D}$-closure space, and any $\mathrm{T_D}$-closure space is a $\mathrm{T_0}$-closure space (note that $\mathcal{C}(\phi) = \phi$).

1. THE EQUIVALENCE OF CATEGORIES

1.1 Lemma *One has $\mathcal{A}(\Gamma(X)) \subseteq \mathrm{Irr}(\Gamma(X)) \subseteq \{\mathcal{C}(x) / x \notin \mathcal{C}(\phi)\}$ for any closure space X. Moreover, the latter subset is order generating in $\Gamma(X)$.*

Proof. Trivial verification. ■

1.2 Proposition *Let X be a closure space and denote by $[x]$ the equivalence class of $x \in X$ for the equivalence relation $x \sim y$ iff $\mathcal{C}(x) = \mathcal{C}(y)$. Then for any point $x \notin \mathcal{C}(\phi)$ the two following hold:*

1) *$\mathcal{C}(x)$ is completely irreducible in $\Gamma(X)$ iff $\mathcal{C}(x) \setminus [x]$ is closed;*
2) *$\mathcal{C}(x)$ is an atom of $\Gamma(X)$ iff $\mathcal{C}(\phi) \cup [x]$ is closed.*

Proof. 1) Suppose that $\mathcal{C}(x)$ is completely irreducible in $\Gamma(X)$. Denote by P the predecessor of $\mathcal{C}(x)$. If $y \in \mathcal{C}(x) \setminus P$, then $\mathcal{C}(y) \subseteq \mathcal{C}(x)$ and $\mathcal{C}(y) \nsubseteq P$ imply $\mathcal{C}(y) = \mathcal{C}(x)$. Hence $y \in [x]$ and one gets $\mathcal{C}(x) \setminus P \subseteq [x]$. Since $[x] \cap P = \phi$ one has $\mathcal{C}(x) \setminus [x] = P$. In particular, $\mathcal{C}(x) \setminus [x]$ is closed. Conversely, if $\mathcal{C}(x) \setminus [x]$ is closed, then it is obviously the predecessor of $\mathcal{C}(x)$.

2) Left as exercise; one shows that $\mathcal{C}(x) = \mathcal{C}(\phi) \cup [x]$. ■

1.3 Corollary 1) *A closure space X is a $\mathrm{T_D}$-closure space if and only if it is a $\mathrm{T_0}$-closure space and $\mathrm{Irr}(\Gamma(X)) = \{\mathcal{C}(x) / x \in X\}$ (this condition implies that $\Gamma(X)$ is a molecular lattice by 1.1).* 2) *A closure space X is a $\mathrm{T_1}$-closure space if and only if it is a $\mathrm{T_0}$-closure space and $\mathcal{A}(\Gamma(X)) = \{\mathcal{C}(x) / x \in X\}$ (this condition implies that $\Gamma(X)$ is an atomistic lattice).*

Proof. For a $\mathrm{T_0}$-closure space one has $\mathcal{C}(\phi) = \phi$ and $[x] = \{x\}$. ■

Therefore, if X is a $\mathrm{T_D}$-closure space, then the lattice $\Gamma(X)$ is complete and molecular. Conversely, we shall associate to any complete (molecular) lattice L a $\mathrm{T_D}$-closure space $\mathrm{Irr}(L)$ (formed by the completely irreducible elements of L).

1.4 Proposition *Let L be a complete lattice. On the set $\mathrm{Irr}(L)$ we consider the operator $\mathcal{C} = \mathcal{C}_L$ defined by $\mathcal{C}_L(A) := \{a \in \mathrm{Irr}(L) / a \leq \sup A\}$. Then $\mathrm{Irr}(L)$ with this operator is a $\mathrm{T_D}$-closure space. Moreover, if L is an atomistic lattice, then $\mathrm{Irr}(L) = \mathcal{A}(L)$ is a $\mathrm{T_1}$-closure space.*

Proof. Clearly, $\mathcal{C}$ is a closure operator on $\mathrm{Irr}(L)$. And for any $a \in \mathrm{Irr}(L)$ the set $\mathcal{C}(a) \setminus a = \{b \in \mathrm{Irr}(L) / b < a\} = \mathcal{C}(p(a))$ is closed. Finally, if L is atomistic, then one has $\mathcal{C}(a) = \{b \in \mathcal{A}(L) / b \leq a\} = \{a\}$ for every $a \in \mathcal{A}(L)$. ■

1.5 Proposition *For any* T_D*-closure space* X *the map* $\eta_X : X \to \mathrm{Irr}(\Gamma(X))$ *defined by* $\eta_X(x) = \mathcal{C}(x)$ *is a homeomorphism.*

Proof. This map is bijective by Corollary 1.3. It remains to show that for every $A \subseteq X$ one has the equality $\eta_X(\mathcal{C}(A)) = \mathcal{C}'(\eta_X(A))$, where $\mathcal{C}'$ denotes the operator on $\mathrm{Irr}(\Gamma(X))$. This follows from the formula $\sup \eta_X(A) = \mathcal{C}(A)$. ■

1.6 Proposition *Let* L *be a complete molecular lattice. Then for any* $x \in L$ *the subset* $\eta_L(x) = \{a \in \mathrm{Irr}(L) \,/\, a \leq x\}$ *is closed in* $\mathrm{Irr}(L)$*, and one thus gets an isomorphism (of lattices)* $\eta_L : L \to \Gamma(\mathrm{Irr}(L))$.

Proof. Since $\mathrm{Irr}(L)$ is order generating one has $\sup \eta_L(x) = x$. Hence $\eta_L(x)$ is closed in $\mathrm{Irr}(L)$. One easily verifies that the map $\varepsilon_L : \Gamma(\mathrm{Irr}(L)) \to L$ defined by $\varepsilon_L(E) = \sup E$ is the inverse of η_L. ■

1.7 Definition Let L and M be complete molecular lattices. One says that a map $g : L \to M$ is a *morphism* if it satisfies

1) $g(\sup A) = \sup g(A)$ for every subset $A \subseteq L$,
2) $g(\mathrm{Irr}(L)) \subseteq \mathrm{Irr}(M)$.

One thus gets a category (the category of complete molecular lattices).

1.8 Lemma *Let* $f : X \to Y$ *be a continuous map between* T_D*-closure spaces. Then the map* $g = \Gamma f : \Gamma(X) \to \Gamma(Y)$ *defined by* $g(E) = \mathcal{C}(f(E))$ *is a morphism of complete molecular lattices. Moreover, if* $f = f_2 \circ f_1$*, then* $\Gamma f = \Gamma f_2 \circ \Gamma f_1$.

Proof. 1) If $\mathcal{A} \subseteq \Gamma(X)$ is an arbitrary subset, then $g(\sup \mathcal{A}) = \mathcal{C}(f(\sup \mathcal{A})) = \mathcal{C}(f(\bigcup \mathcal{A})) = \mathcal{C}(\bigcup f(\mathcal{A})) = \mathcal{C}(\bigcup g(\mathcal{A})) = \sup g(\mathcal{A})$.

2) If $\mathcal{C}(x) \in \mathrm{Irr}(\Gamma(X))$, then $g(\mathcal{C}(x)) = \mathcal{C}(f(x)) \in \mathrm{Irr}(\Gamma(Y))$ because Y is a T_D-closure space, cf. 1.3. The functoriality is clear. ■

1.9 Lemma *Let* $g : L \to M$ *be a morphism between complete molecular lattices. Then the map* $f = \mathrm{Irr}\, g : \mathrm{Irr}(L) \to \mathrm{Irr}(M)$ *obtained by restriction of* g *is continuous. Moreover, if* $g = g_2 \circ g_1$*, then* $\mathrm{Irr}\, g = \mathrm{Irr}\, g_2 \circ \mathrm{Irr}\, g_1$.

Proof. Easy verification. ■

1.10 Theorem *The category of* T_D*-closure spaces (and continuous maps) is equivalent to the category of complete molecular lattices (and morphisms).*

Proof. It is enough to verify that the homeomorphisms η_X of Proposition 1.5 and the isomorphisms η_L of Proposition 1.6 yield natural transformations. ■

1.11 Corollary *The category of* T_1*-closure spaces (and continuous maps) is equivalent to the category of complete atomistic lattices (and morphisms).*

2. RESTRICTIONS OF THE EQUIVALENCE (I)

In this section we shall prove that the equivalence of Theorem 1.10 restricts to an equivalence between the category of T_D-spaces (= T_D-topological spaces) and the category of distributive complete molecular lattices. The two following lemmas are classical results.

2.1 Lemma *For any closure space X the following are equivalent:*

1) *$\mathcal{C}$ is topological, i.e. $\mathcal{C}(\bigcup \mathcal{F}) = \bigcup \mathcal{C}(\mathcal{F})$ for every finite subset $\mathcal{F} \subseteq \mathcal{P}(X)$,*
2) *$\Gamma(X)$ is closed under finite unions,*
3) *$\{\mathcal{C}(x)/x \in X\} \subseteq \mathrm{Sp}(\Gamma(X))$.*

2.2 Lemma *For any complete molecular lattice L the following are equivalent:*

1) *L is topological, i.e. $\mathrm{Sp}(L)$ is order generating,*
2) *L is distributive,*
3) *$\mathrm{Irr}(L) \subseteq \mathrm{Sp}(L)$.*

2.3 Proposition *If a closure space X is topological, then $\Gamma(X)$ is a distributive lattice. And if a complete molecular lattice L is distributive, then $\mathrm{Irr}(L)$ is a topological space. Hence the category of T_D-spaces is equivalent to the category of distributive (or topological) complete molecular lattices.*

Proof. The first assertion is trivially verified. Let $\mathcal{F} \subseteq \mathcal{P}(\mathrm{Irr}(L))$ be any finite subset. If $a \in \mathcal{C}(\bigcup \mathcal{F})$, then $a \leq \sup(\bigcup \mathcal{F}) = \bigvee \sup \mathcal{F}$, and since $a \in \mathrm{Irr}(L)$ is a coprime element there exists a set $A \in \mathcal{F}$ with $a \leq \sup A$. Therefore $a \in \mathcal{C}(A) \subseteq \bigcup \mathcal{C}(\mathcal{F})$ and this proves that $\mathrm{Irr}(L)$ is topological. ■

2.4 Corollary *Let X and Y be T_D-spaces. If the lattices $\Gamma(X)$ and $\Gamma(Y)$ are isomorphic, then X and Y are homeomorphic (Theorem 2.1 in [11]).*

2.5 Corollary *Let X be a T_0-space and Y a sober T_D-space. If the lattices $\Gamma(X)$ and $\Gamma(Y)$ are isomorphic, then X and Y are homeomorphic (compare to Theorem 2.3 in [11]).*

Proof. We recall that a topological space Y is sober if and only if it is a T_0-space and $\{\mathcal{C}(y)/y \in Y\} = \mathrm{Sp}(\Gamma(Y))$. By 1.3 one gets $\mathrm{Irr}(\Gamma(Y)) = \mathrm{Sp}(\Gamma(Y))$. But then $\mathrm{Irr}(\Gamma(X)) = \mathrm{Sp}(\Gamma(X))$ implies $\mathrm{Irr}(\Gamma(X)) = \{\mathcal{C}(x)/x \in X\}$, and X is also a (sober) T_D-space. So one can use the preceding corollary. ■

2.6 Example On the set $\overline{\mathbb{N}} = \mathbb{N} \cup \{\infty\}$ one takes as closed subsets $\overline{\mathbb{N}}$ and all finite subsets of $\mathbb{N}$. Then $\overline{\mathbb{N}}$ is a sober space but not a T_D-space, and it follows that $\overline{\mathbb{N}}$ cannot be homeomorphic to its T_1-subspace $\mathbb{N}$. However, the inclusion $f : \mathbb{N} \hookrightarrow \overline{\mathbb{N}}$ yields an isomorphism $\Gamma f : \Gamma(\mathbb{N}) \to \Gamma(\overline{\mathbb{N}})$ (cf. Example 2.1 in [11]).

We also deduce that the category of T_1-spaces is equivalent to the category of distributive complete atomistic lattices. The following remark gives a characterization of those lattices which correspond to T_2-spaces.

2.7 Definition Let L be a lattice with 0. An element $w \in L$ is called *weakly coprime* if $\bigvee F = 1$ for some finite subset $F \subseteq L$ implies that there exists $x \in F$ with $w \leq x$. The set of weakly coprime elements of L is denoted by $\mathcal{W}(L)$.

2.8 Remark Let X be any topological space. One easily shows that a closed subset $A \subseteq X$ is a weakly coprime element of $\Gamma(X)$ if and only if no two points of A can be separated (by disjoint open neighbourhoods). Therefore X is a T_2-space if and only if it is a T_0-space and $\mathcal{A}(\Gamma(X)) = \mathcal{W}(\Gamma(X))$.

Finally, we give the 'algebraic version' of Proposition 2.3.

2.9 Lemma *For any closure space X the following are equivalent:*

1) $\mathcal{C}$ *is algebraic, i.e.* $\mathcal{C}(A) = \bigcup\{\mathcal{C}(B) \,/\, B \subseteq A \text{ finite}\}$,
2) $\mathcal{C}(\bigcup \mathcal{D}) = \bigcup \mathcal{C}(\mathcal{D})$ *for every directed subset* $\mathcal{D} \subseteq \mathcal{P}(X)$,
3) $\Gamma(X)$ *is closed under directed unions,*
4) $\{\mathcal{C}(x) \,/\, x \in X\} \subseteq \mathcal{K}(\Gamma(X))$.

2.10 Lemma *For any complete molecular lattice L the following are equivalent:*

1) L *is algebraic, i.e.* $\mathcal{K}(L)$ *is order generating,*
2) L *is* $\wedge$*-continuous, i.e.* $\wedge$ *distributes over directed suprema,*
3) $\mathrm{Irr}(L) \subseteq \mathcal{K}(L)$.

2.11 Proposition *If a closure space X is algebraic, then $\Gamma(X)$ is a $\wedge$-continuous lattice. And if a complete molecular lattice L is $\wedge$-continuous, then $\mathrm{Irr}(L)$ is an algebraic closure space. Hence the category of algebraic T_D-closure spaces is equivalent to the category of $\wedge$-continuous (or algebraic) complete molecular lattices (the proposition is proved as for Proposition 2.3).*

3. RESTRICTIONS OF THE EQUIVALENCE (II)

We shall determine the lattices which correspond to T_D-closure spaces with the weak exchange property (WEP) or the well-known exchange property (EP). The next proposition generalizes Proposition 1.2 (and is proved similarly).

3.1 Notations Let E be a closed subset of a closure space X. We denote by $[E, 1]$ the segment $\{F \in \mathcal{L}(X) \,/\, E \subseteq F \subseteq X\}$ and by $[x]_E$ the equivalence class of $x \in X$ for the equivalence relation $x \sim_E y$ iff $\mathcal{C}(E \cup x) = \mathcal{C}(E \cup y)$.

3.2 __**Proposition**__ *Let X be a closure space and let $E \subseteq X$ be a closed subset. Then for any point $x \notin E$ the two following hold:*

1) *$\mathcal{C}(E \cup x)$ is completely irreducible in $[E, 1]$ iff $\mathcal{C}(E \cup x) \setminus [x]_E$ is closed;*
2) *$\mathcal{C}(E \cup x)$ is an atom of $[E, 1]$ iff $E \cup [x]_E$ is closed.*

This proposition is used to prove the following characterization of the WEP.

3.3 __**Proposition**__ *For any closure space X the following are equivalent:*

1) *WEP: if $\mathcal{C}(A \cup x) = \mathcal{C}(B)$, then there exists $y \in B$ with $x \in \mathcal{C}(A \cup y)$,*
2) *$\mathrm{Irr}([E, 1]) = \{\mathcal{C}(E \cup x) / x \notin E\}$ for every closed subset $E \subseteq X$.*

Proof. $(1 \Rightarrow 2)$ We show that $\mathcal{C}(E \cup x)$ is completely irreducible in $[E, 1]$ for every $x \notin E$. Suppose that $\mathcal{C}(\mathcal{C}(E \cup x) \setminus [x]_E) = \mathcal{C}(E \cup x)$. Then by hypothesis there would exist $y \in \mathcal{C}(E \cup x) \setminus [x]_E$ with $x \in \mathcal{C}(E \cup y)$. Thus $y \in [x]_E$ and one gets a contradiction. This proves that $\mathcal{C}(\mathcal{C}(E \cup x) \setminus [x]_E) \subseteq \mathcal{C}(E \cup x) \setminus [x]_E$.

$(2 \Rightarrow 1)$ One may assume that $x \notin E := \mathcal{C}(A)$. Then $\mathcal{C}(A \cup x)$ is completely irreducible in $[E, 1]$, and since $\mathcal{C}(A \cup x) \setminus [x]_E$ is closed one concludes that there exists $y \in B \cap [x]_E$. So the assertion follows. ■

3.4 __**Remark**__ In [9] P. R. Jones introduced this property, and he showed that every algebraic closure space with the WEP satisfies the *basis property*, i.e. any two independent subsets with the same closure have the same cardinality.

3.5 __**Definition**__ A lattice L is called an *AC-lattice* if it is atomistic and satisfies the covering law (for atoms):

a) $a \in \mathcal{A}(L)$ and $a \not\leq x$ imply $a \vee x \in \mathcal{A}([x, 1])$.

Similarly, L is called an *MC-lattice* if it is molecular and satisfies the following covering law (for completely irreducible elements):

b) $a \in \mathrm{Irr}(L)$ and $a \not\leq x$ imply $a \vee x \in \mathrm{Irr}([x, 1])$.

Note that $p(a) \vee x$ is not necessarily the predecessor $p_x(a \vee x)$ of $a \vee x$ in $[x, 1]$ (but it is the case when L is a modular lattice).

3.6 __**Proposition**__ *If a closure space X has the weak exchange property, then $\Gamma(X)$ is an MC-lattice. And if L is a complete MC-lattice, then $\mathrm{Irr}(L)$ has the weak exchange property.*

Proof. 1) By considering $E = \mathcal{C}(\phi)$ in Proposition 3.3 one obtains that $\Gamma(X)$ is a molecular lattice. And it clearly satisfies the covering law.

2) Let $A, B \subseteq \mathrm{Irr}(L)$ and $a \in \mathrm{Irr}(L)$ with $\mathcal{C}(A \cup a) = \mathcal{C}(B)$. We may assume that $a \notin E := \mathcal{C}(A)$. Then $a \not\leq x := \sup A$ and by hypothesis $a \vee x$ is completely irreducible in $[x, 1]$. Since $\sup B = a \vee x$ there exists $b \in B$ with $b \not\leq p_x(a \vee x)$. Hence $b \vee x = a \vee x$ and one obtains $a \in \mathcal{C}(A \cup b)$. ■

3.7 Remark The preceding proposition provides a large and interesting class of complete molecular lattices. For instance, the lattice of all subgroups of any finite p-group, or more generally of any primary $\widetilde{N}$-group, is an MC-lattice (use the main theorem in [9] and Theorem 4.6 in [8]). But many other examples can be deduced from the work of P. R. Jones.

We have similar results for the exchange property (left as exercises).

3.8 Proposition *For any closure space X the following are equivalent:*
1) *EP: if $x \in \mathcal{C}(A \cup y)$ and $x \notin \mathcal{C}(A)$, then $y \in \mathcal{C}(A \cup x)$;*
2) *$\mathcal{A}([E,1]) = \{\mathcal{C}(E \cup x) / x \notin E\}$ for every closed subset $E \subseteq X$.*

3.9 Proposition *If a closure space X has the exchange property, then $\Gamma(X)$ is an AC-lattice. And if L is a complete AC-lattice, then $\mathrm{Irr}(L) = \mathcal{A}(L)$ has the exchange property.*

Hence the category of geometries (i.e. algebraic T_1-closure spaces with EP) is equivalent to the category of geometric lattices (i.e. algebraic AC-lattices).

3.10 Remark Let E be a closed subset of a closure space X, and denote by X/E the quotient of $X \setminus E$ by the equivalence relation $\sim_E$ of 3.1. On X/E we consider the operator $\mathcal{C}_E(A) = \pi(\mathcal{C}(\pi^{-1}(A) \cup E) \setminus E)$, where π is the canonical projection. Then X/E is a T_0-closure space and one easily shows that $\Gamma(X/E)$ is isomorphic to the segment $[E,1]$.

From Propositions 3.3 and 3.8 we also obtain the following result:

3.11 Proposition *A closure space X has the weak exchange property if and only if X/E is a T_D-closure space for every closed subset E of X. And X has the exchange property if and only if X/E is a T_1-closure space for every closed subset E of X (left as exercise).*

4. CONCLUDING REMARKS

(1) Morphisms of projective geometries

For two projective geometries X and Y the continuous maps $f : X \to Y$ can be described by natural geometric conditions:

a) if x, y, z are collinear in X, then fx, fy, fz are collinear in Y.

b) if moreover $y \neq z$ and $fy = fz$, then $fx = fy$.

More generally, A. Frölicher and the author introduced morphisms of projective geometries as partially defined maps $f : X \setminus E \longrightarrow Y$ satisfying similar geometric conditions, cf. 3.1.1 in [4]. The following example shows that it is very natural to consider morphisms which are not everywhere defined:

4.1 **Example** Every semilinear map $h : V \longrightarrow W$ between vector spaces determines a map $\mathbf{P}h : \mathbf{P}(V) \setminus \mathbf{P}(\ker h) \longrightarrow \mathbf{P}(W)$, $[v] \mapsto [hv]$, which is a morphism of projective geometries (cf. Proposition 3.2.2 in [4] for details).

4.2 **Remark** The corresponding morphisms of projective lattices $g : L \longrightarrow M$ are obtained by considering in place of the condition $g(\mathcal{A}(L)) \subseteq \mathcal{A}(M)$ (cf. 1.7) the weaker $g(\mathcal{A}(L)) \subseteq \mathcal{A}(M) \cup \{0\}$.

4.3 **Theorem** *Let V, W be vector spaces and let $f : \mathbf{P}(V) \setminus E \longrightarrow \mathbf{P}(W)$ be a morphism of projective geometries with a non-degenerate image (i.e. the image contains at least three non-collinear points). Then there exists a semilinear map $h : V \longrightarrow W$ such that $f = \mathbf{P}h$.*

Proof. Theorem 5.4.1 in [5]. ∎

(2) The symmetry principle

There is an evident symmetry between the 'topological' lemmas 2.1 and 2.2 and their 'algebraic' version 2.9 and 2.10. They are obtained from each other by exchanging finiteness and directedness. Therefore coprime elements correspond to compact elements, distributivity corresponds to $\wedge$-continuity, etc. Of course, this symmetry principle has obvious limits. For instance, there is no equivalent to the formula $\sup A = \bigsqcup \{\bigvee B \,/\, B \subseteq A \text{ finite}\}$ or to Zorn's lemma. However, it can be used to obtain certain elementary results; in the following we shall use it to characterize the morphisms of locales.

4.4 **Definition** Let L, M be complete lattices. A *Galois connection* between L and M is a pair of monotone maps $g : L \longrightarrow M$ and $h : M \longrightarrow L$ (called *adjoint maps*) satisfying $gx \leq y$ iff $x \leq hy$ (for all $x \in L$ and $y \in M$).

4.5 **Proposition** *Let $g : L \longrightarrow M$ and $h : M \longrightarrow L$ be two adjoint maps. Then the following conditions are equivalent:*

1) *$h(\bigsqcup D) = \bigsqcup h(D)$ for every directed subset $D \subseteq M$,*
2) *$gx \leq \bigsqcup D$ implies $x \leq \bigsqcup h(D)$,*
3) *if $U \subseteq L$ is Scott-open, then $\uparrow g(U)$ is also Scott-open.*

Moreover, if L is an algebraic lattice, all these conditions are equivalent to

4) *$g(\mathcal{K}(L)) \subseteq \mathcal{K}(M)$.*

Proof. Theorem IV.1.4 and Proposition IV.1.11 in [6]. ∎

The 'dual' notion of a Scott-closed subset is an ideal. So we introduce:

4.6 Definition Let L be a complete lattice. A subset $A \subseteq L$ is called an *anti-ideal* if it satisfies the two following conditions:

1) $x \in A$ and $x \leq y$ imply $y \in A$,
2) if $\bigvee F \in A$ for some finite subset $F \subseteq L$, then there exists $x \in A \cap F$.

Therefore A is an anti-ideal of L if and only if $L \setminus A$ is an ideal.

4.7 Proposition *Let $g : L \to M$ and $h : M \to L$ be two adjoint maps. Then the following conditions are equivalent:*

1) $h(\bigvee F) = \bigvee h(F)$ *for every finite subset* $F \subseteq M$,
2) $gx \leq \bigvee F$ *implies* $x \leq \bigvee h(F)$,
3) *if* $A \subseteq L$ *is an anti-ideal, then* $\uparrow g(A)$ *is also an anti-ideal.*

Moreover, if L is a topological lattice, all these conditions are equivalent to

4) $g(\mathrm{Sp}(L)) \subseteq \mathrm{Sp}(M)$.

Proof. (3 $\Rightarrow$ 1) Consider the anti-ideal $A = \{x \in L \,/\, x \nleq \bigvee h(F)\}$. ■

4.8 Definition We recall that a *locale* L is a complete lattice which satisfies the distributive law $x \wedge \sup A = \sup (x \wedge A)$ (for all $x \in L$ and $A \subseteq L$), and that a map $g : L \to M$ between locales is a *morphism* if it has an adjoint $h : M \to L$ which preserves finite infima.

4.9 Corollary *A map $g : L \to M$ between locales is a morphism of locales if and only if it preserves arbitrary infima and antifilters:*

1) $g(\bigwedge B) = \bigwedge g(B)$ *for every subset* $B \subseteq L$,
2) *if* $A \subseteq L$ *is an antifilter, then* $\downarrow g(A)$ *is also an antifilter.*

Proof. The adjoint $h : M \to L$ preserves suprema and finite infima. ■

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