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CATEGORICAL DIFFERENTIAL GEOMETRY

by M.V. LOSIK

Résumé. Cet article développe une théorie générale des structures géométriques sur des variétés, basée sur la théorie des catégories. De nombreuses généralisations connues des variétés et des variétés de Riemann rentrent dans le cadre de cette théorie générale. On donne aussi une construction des classes caractéristiques des objets ainsi obtenus, tant classiques que généralisés. Diverses applications sont indiquées, en particulier aux feuilletages.

Introduction

This paper is the result of an attempt to give a precise meaning to some ideas of the "formal differential geometry" of Gel'fand. These ideas of Gel'fand have not been formalized in general but they may be explained by the following example. Let $C^*(W_n;\mathbb{R})$ be the complex of continuous cochains of the Lie algebra W_n of formal vector fields on $\mathbb{R}^n$ with coefficients in the trivial W_n−module $\mathbb{R}$ and let $C^*(W_n, GL(n,\mathbb{R}))$, $C^*(W_n, O(n))$ be its subcomplexes of relative cochains of W_n relative to the groups $GL(n,\mathbb{R})$, $O(n)$, respectively. One can consider cohomologies $H^*(W_n;\mathbb{R})$, $H^*(W_n, GL(n,\mathbb{R}))$, and $H^*(W_n, O(n))$, called the Gel'fand-Fuks cohomologies, as giving the "universal characteristic classes" of foliations of codimension n.

An initial purpose of this paper was to give a general construction of the analogous complex for some class of geometrical structures on manifolds. It appears that one can solve this problem in a very general situation by means of standard methods of category theory, applying the important notion of a Grothendieck topology on a category.

One can define a convenient general notion of a category of local structures on sets or just an LSS-category encompassing many known categories of structures on manifolds; the category Man of finite dimensional manifolds, the category of n-dimensional Riemannian manifolds, the category of $2n$-dimensional symplectic manifolds, and so on are examples of an LSS-category. Any LSS-category A can be included as a full subcategory in the precisely defined LSS-category AS with the terminal object $t(A)$. The category A is called a *model* category and objects of the extended category are called A-*spaces*. A construction of A-spaces is a far generalization of the definition of a manifold by means of charts.

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Nowadays such objects as the leaf space of a foliation or the orbit space of a group of diffeomorphisms of a manifold became standard objects of mathematical study, for example, in Connes' noncommutative differential geometry. There are many generalizations of a manifold giving some structure on such objects: the diffeological spaces of Souriau [16] , the S-manifolds of Van Est [17] , the Satake manifolds [15], and so on. Most of these generalizations are partial cases of A-spaces for a suitable category A.

Moreover, one can show how to transport to A-spaces notions of the model category A having a categorical character. For instance, diffeological spaces are Man-spaces. Therefore one can define such notions as the tangent fiber bundle, the de Rham complex and so on for a diffeological space.

If A is an LSS-category and $F : A \longrightarrow Man$ is a covariant functor one can obtain the de Rham complex $\Omega^*(F(\alpha))$ for any A-space α. Then the cohomology of the de Rham complex $\Omega^*(F(t(A)))$ for the terminal object $t(A)$ give the required "universal characteristic classes" associated to the functor F for the category A and even for the category AS. Indeed, if α is an A-space and $f_\alpha : \alpha \longrightarrow t(A)$ is the corresponding unique morphism of A-spaces, the homomorphism $H^*(\Omega^*(F(t(A))) \longrightarrow H^*(\Omega^*(F(\alpha))$ induced by f_α is the characteristic homomorphism giving the characteristic classes of α associated to F.

For example, let D_n be the subcategory of Man formed by open submanifolds of $\mathbb{R}^n$ and local diffeomorphisms as morphisms and let M_∞ be the category of manifolds with model space $\mathbb{R}^\infty$. Then a slight modification of the above considerations gives, for a covariant functor $F : D_n \longrightarrow M_\infty$, the de Rham complex for $F(t(D_n))$. One can prove that the complexes $C^*(W_n;\mathbb{R})$, $C^*(W_n, GL(n,\mathbb{R}))$, and $C^*(W_n, O(n))$ are isomorphic to the de Rham complexes $\Omega^*(F(t(D_n)))$ for suitable covariant functors $F : D_n \longrightarrow M_\infty$.

Note that the leaf space of a foliation of codimension n and the orbit space of a pseudogroup of local diffeomorphisms of an n-dimensional manifold are typical examples of D_n-spaces. Then the Gel'fand-Fuks cohomologies give the characteristic classes, in particular the Pontrjagin classes, for them. One can prove that these characteristic classes can be nontrivial for the above examples of D_n-spaces even if they are trivial for foliations.

Remark that the main part of the above theory is purely categorical and it may be applied not only in geometry but also in algebra. Unfortunately I do not know interesting algebraic examples. Moreover, there is a dual categorical construction but I know only a few applications.

Preliminary results on this subject were published in [6,7,8].

Note that there are categorical approaches to differential geometry different from our one. We mention only the far developed Frölicher-Kriegl-Michor construction of a cartesian closed category of manifolds containing all finite dimensional ones and some of usual infinite dimensional ones, based on the notion of smooth curves in a manifold [4,9,10].

Throughout the paper all categories are supposed to be small, in particular, we denote by *Set* the small category of sets; by a functor we mean a covariant functor; we call it a contravariant functor otherwise. Manifolds, fiber bundles, maps of manifolds, differential forms, and so on mean the corresponding C^∞ notions. By *Man* we denote the category of finite dimensional manifolds with usual morphisms.

In 1.1 we recall the definition of a Grothendieck topology on a category and give some examples of Grothendieck topologies required in the sequel. In 1.2 we introduce the notion of a category of local structures on sets (an LSS-category), in particular, the notion of a category of local structures on manifolds (an LSM-category) and give some examples. These categories will be used in the sequel as "model categories" for the following constructions.

In 2.1, for an LSS-category A, we define a category AS of A-spaces and show that manifolds and some structures on manifolds are the simplest examples of A-spaces. In 2.2 we prove that an LSS-category A is a full subcategory of the category AS, introduce a natural Grothendieck topology on AS, and construct the terminal object of AS.

In 2.3 we show that each functor from an LSS-category A to an LSS-category B can be naturally extended to a functor from AS to BS. Moreover, a contravariant functor from an LSS-category A to a category B containing projective limits of all diagrams can be naturally extended to a contravariant functor from AS to B.

In 3.1 we show that the category of Man-spaces coincides with the category of diffeological spaces and use the construction of 2.3 to introduce the notions of tangent vectors, tensors of the type $(p, 0)$, differential forms, and so on, for diffeological spaces. We also point out that the category of Fréchet manifolds is a full subcategory of the category of diffeological spaces.

In 3.2 we show that, for the category D_n, there is a large class of D_n-spaces including the leaf space of a foliation of codimension n and the orbit space of a pseudogroup of local diffeomorphisms of an n-dimensional manifold.

In 3.3 we define the characteristic classes of an extended category AS, associated to a functor F from A to the category of diffeological spaces, using the generalized de Rham complex for diffeological spaces. The "universal characteristic classes" appear as the cohomology of the de Rham complex of the diffeological space $F(t(A))$, where $t(A)$ is the terminal object of the category AS. Then we apply this construction to the category D_n and show that, for some natural functors from this category to the category of manifolds with model space $\mathbb{R}^\infty$, the "universal characteristic classes" coincide with the Gel'fand-Fuks cohomologies. Some properties of the corresponding characteristic classes for the leaf space of a foliation and the orbit space of a diffeomorphism group are indicated.

In 3.4 we point out that many notions of transverse geometry of foliations such as a transverse Riemannian metric or a projectable connection can be described as structures of suitable A-spaces on the leaf space of a foliation. Moreover, we point out that some results on characteristic classes of Riemannian foliations can be obtained from our considerations.

I would like to express my gratitude to G.I.Zhitomirsky for many useful discussions on category theory.

1. Categories of local structures on sets

1.1. Grothendieck topologies on categories. Let A be a category and $a \in ObA$. A set S of A-morphisms $f : b \longrightarrow a$ for any $b \in ObA$ is an *a-sieve*, if the composite of an arbitrary A-morphism $c \longrightarrow b$ and f belongs to S. The minimal a-sieve containing a given set U of A-morphisms with a target a is called an *a-sieve generated by* U. Suppose that S is an a-sieve and $f : b \longrightarrow a$ is an A-morphism. The set of A-morphisms $c \longrightarrow b$, whose composites with f belong to S, is called a *restriction of* S *to* b (with respect to f) and is denoted by f^*S.

Recall that a *Grothendieck topology* on a category A is constituted by assigning to each $a \in ObA$ a set $Cov(a)$ of a-sieves, called *covers*, such that the following axioms hold:

(1) The a-sieve generated by the identity morphism 1_a is a cover.
(2) For any A-morphism $f : b \longrightarrow a$ and $S \in Cov(a)$, $f^*S \in Cov(b)$.
(3) Let $S \in Cov(a)$ and let R be an a-sieve. Then R is a cover if, for any A-morphism $f : b \longrightarrow a$ from S, $f^*R \in Cov(b)$.

A category A with a Grothendieck topology on A is called a *site*.

In the sequel we shall consider the following standard sites.

1. Let $A = Set$ and, for any $X \in ObSet$, an A-sieve S is called a cover, if images of morphisms of S cover X in the usual sense. Clearly these data define a Grothendieck topology on Set .

2. Let $A = Man$ and, for any manifold M , an M-sieve S is called a cover, if S contains the set of inclusions $M_i \subset M$, where $\{M_i\}$ is an open cover of M . It is evident that these data give a Grothendieck topology on Man .

1.2. Categories of local structures on sets. Now we introduce our main notions.

1.2.1. Definition. *Let A and B be sites. A functor $F : A \to B$ is called cover preserving, if, for every $a \in ObA$, the image of each $S \in Cov(a)$ under F generates a cover from $Cov(F(a))$.*

Remark that, for $B = Set$, the notion of a cover preserving functor is dual to the notion of a sheaf of sets on A .

1.2.2. Definition. *A site A with a cover preserving functor $J = J_A : A \to Set$ is called a category of local structures on sets or just an LSS-category, if the following conditions hold :*

(1) *The functor J is faithful, that is any A-morphism $f : a \to b$ is uniquely determined by a, b and $J(f)$. Moreover, if $J(a) = J(b)$, f is an isomorphism, and $J(f)$ is the identity map, then $a = b$ and $f = 1_a$.*

(2) *Suppose that* $a, b \in ObA$, $l : J(a) \to J(b)$ *is a map, and* $S \in Cov(a)$. *If, for every* $f_1 : c \to a$ *from* S, *there is an* A*-morphism* $f_2 : c \to b$ *such that* $J(f_2) = l \circ J(f_1)$, *then there exists an* A*-morphism* $f : a \to b$ *for which* $J(f) = l$.

An object $a \in ObA$ *is called a structure on the underlying set* $J(a)$.

In applications the functor J is usually a forgetful functor.

Examples of LSS-categories.

1. Suppose that T is a topological space and C_T is the category of open subsets of T whose morphisms are inclusions $U \subset V$ $(U, V \in ObC_T)$. This category has the usual Grothendieck topology whose covers are generated by usual open covers. Clearly the site C_T with the forgetful functor $J : C_T \to Set$ is an LSS-category.

2. The site Man with the forgetful functor $J : Man \to Set$.

3. Let Gr be the category of groups. Given a group G , a G-sieve S is a cover if, for every $g_1, g_2 \in G$, there is a group H and a group homomorphism $h : H \to G$ from S such that $g_1, g_2 \in Im\ h$. Such covers define a Grothendieck topology on Gr and the site Gr with the forgetful functor $J : Gr \to Set$ is an LSS -category.

1.2.3. Definition. *A site* A *with a cover preserving functor* $J_m : A \to Man$ *is called a category of local structures on manifolds or just an* LSM*-category, if* A *with the composite* J *of* J_m *and the forgetful functor* $Man \to Set$ *is an* LSS*-category. An object* $a \in ObA$ *is called a structure on the underlying manifold* $J_m(a)$.

Examples of LSM-categories.

1. The category Man with the identity functor $J_m : Man \to Man$.

2. Let D_n be the subcategory of Man with objects open submanifolds of $\mathbb{R}^n$ and morphisms local diffeomorphisms of one manifold into another one. Then D_n with the Grothendieck topology induced by the Grothendieck topology of Man and the inclusion functor $J_m : D_n \to Man$ is an LSM-category.

3. Let H_{2n} be the subcategory of D_{2n} with the same objects and morphisms which preserve the usual symplectic structure of $\mathbb{R}^{2n}$. Then H_{2n} with the Grothendieck topology induced by the Grothendieck topology of D_{2n} and the forgetful functor $J_m : H_{2n} \to Man$ is an LSM-category.

4. Let R_n be the category of n-dimensional Riemannian manifolds with those local diffeomorphisms of one Riemannian manifold into another one, which compatible with the corresponding Riemannian structures, as morphisms. For any Riemannian manifold M, we define a cover on M as an M-sieve containing the set of inclusions $M_i \subset M$, where $\{M_i\}$ is an open cover of M. These data define a Grothendieck topology on R_n and the forgetful functor $J_m : R_n \to Man$ defines on the site R_n the structure of an LSM-category.

5. Let C_n be the category of manifolds with a linear connection and with those local diffeomorphisms of one such manifold into another one which compatible with the corresponding linear connections, as morphisms. We define a Grothendieck topology on C_n and a structure of an LSM-category on C_n as for the category R_n.

6. Let FB be the category of smooth locally trivial fiber bundles with a fixed fiber F and fiber preserving morphisms. By definition, a cover on $E \in ObFB$ is an E-sieve containing the inclusions $E_i \subset E$, where E_i is the restriction of E to an open subset B_i of some open cover $\{B_i\}$ of the base of E. These data define a Grothendieck topology on FB and the site FB with the functor $J_m : FB \to Man$, for which $J_m(E)$ is the total space of E, is an LSM-category.

2. The extension of an LSS-category

2.1. The definition of the category of A-spaces. Let A be an LSS-category and X a set. A pair (a, l) with a an object of A and $l : J(a) \to X$ a map is called an *A-chart on X*. Given two A-charts (a, l), (b, k) on X, a *morphism from* (a, l) *to* (b, k) is an A-morphism $f : a \to b$ such that $l = k \circ J(f)$. By CX denote the category of A-charts on X. Suppose Φ is a set of A-charts on X and C_Φ is the full subcategory of CX determined by Φ. For $(a, l) \in \Phi$, put $I_\Phi((a, l)) = a$ and, for any morphism f of C_Φ, put $I_\Phi(f) = f$, where on the right f is considered as an A-morphism. Thus we have the functor $I_\Phi : C_\Phi \to A$.

2.1.1. Definition. *A set Φ of A-charts on X is called an A-atlas on X, if the set X with the set of maps $l : J \circ I_\Phi((a, l)) = J(a) \to X$ $((a, l) \in \Phi)$ is an inductive limit* $\operatorname{inj\,lim} J \circ I_\Phi$ *of the functor $J \circ I_\Phi$.*

Let Φ be an A-atlas on X. Evidently the functor $I_\Phi : C_\Phi \to A$ maps C_Φ onto some subcategory A_Φ of A and the pair (X, Φ) can be considered as an inductive limit of the restriction of J to A_Φ. Conversely, for any subcategory B of A, the inductive limit of the restriction of J to B is an A-atlas on some set X. Thus one can define an A-atlas as an inductive limit of the restriction of J to some subcategory of A. The set X of this A-atlas is determined up to an isomorphism of Set by this inductive limit.

Note that, for every set Φ of A-charts on X, by the definition of an inductive limit, $\operatorname{inj\,lim} J \circ I_\Phi$ is a set $\widetilde{X}$ given with a set of maps $\widetilde{l} : J \circ I_\Phi((a, l)) = J(a) \to \widetilde{X}$ such that the set of pairs $(a, \widetilde{l})$ is an A-atlas on $\widetilde{X}$ and there is a unique map $h : \widetilde{X} \to X$ such that, for every $(a, l) \in \Phi$, $l = h \circ \widetilde{l}$.

Let Φ be a set of A-charts on X. By $P(\Phi)$ denote the set of A-charts on X of the type $(b, l \circ J(f))$, where $(a, l) \in \Phi$ and $f : b \to a$ is an A-morphism. Clearly $\Phi \subset P(\Phi)$ and $P^2 = P$. We shall say that an A-chart (a, l) is *obtained by gluing from* Φ, if there is a family $\{(a_i, l_i)\}$ of A-charts of Φ and a set of morphisms $\{f_i : a_i \to a\}$ generating a cover such that $l_i = l \circ J(f_i)$. By $G(\Phi)$ denote the set of A-charts obtained by gluing from Φ. Clearly $\Phi \subset G(\Phi)$ and $G^2 = G$. By definition, put $\overline{\Phi} = G \circ P(\Phi)$.

2.1.2. Proposition. *The assignment $\Phi \to \overline{\Phi}$ is a closure on the set of A-charts on X, that is the following conditions hold: (i) $\Phi \subset \overline{\Phi}$; (ii) If $\Phi_1 \subset \Phi_2$, then $\overline{\Phi_1} \subset \overline{\Phi_2}$; (iii) $\overline{\overline{\Phi}} = \overline{\Phi}$.*

Proof. Properties (i) and (ii) are obvious. To prove (iii) it is sufficient to show that $P \circ G \circ P = G \circ P$. Let $(a, l) \in \overline{\Phi}$ and let $f : b \to a$ be an A-morphism. There is $S \in Cov(a)$ such that (a, l) is obtained by gluing from $P(\Phi)$ by means of S. Then, for each A-morphism $g : c \to b$ from f^*S, $(c, l \circ J(f) \circ J(g)) \in P(\Phi)$. Hence the chart $(b, l \circ J(f))$ is obtained by gluing from $P(\Phi)$ by means of f^*S. □

A set of A-charts on X is called *closed*, if $\overline{\Phi} = \Phi$. By the definition of an inductive limit, the closure $\overline{\Phi}$ is a closed A-atlas on X whenever Φ is an A-atlas on X.

2.1.3. Definition. *A pair (X, Φ), where Φ is a closed A-atlas on X, is called an A-space. For two A-spaces (X_1, Φ_1) and (X_2, Φ_2), a map $h : X_1 \to X_2$ is called a morphism from (X_1, Φ_1) to (X_2, Φ_2) if, for every $(a, l) \in \Phi_1$, $(a, h \circ l) \in \Phi_2$.*

It is clear that, for each A-atlas Φ on X, the pair $(X, \overline{\Phi})$ is an A-space. Thus we obtain the category AS of A-spaces. Note that a D_n-space is an H-manifold of Pradines [13] or an S-manifold of Van Est [17].

Now we indicate some examples of A-spaces.

Let M be an n-dimensional manifold given by an atlas $\{h_i : U_i \to \mathbb{R}^n \mid i \in I\}$, where U_i is an open cover of M, such that, for $i, j \in I$, the restrictions of h_i and h_j on $U_i \cap U_j$ belong to this atlas. Put $k_i = h_i^{-1} : h_i(U_i) \to U_i \subset M$. It is easily checked that the set of D_n-charts $\{(h_i(U_i), k_i) \mid i \in I\}$ is a D_n-atlas on M and a morphism of manifolds as D_n-spaces is a local diffeomorphism of manifolds. It is evident that the Satake manifolds [15] are D_n-spaces. Similarly, 2n-dimensional symplectic manifolds are H_{2n}-spaces. More complicated examples of A-spaces will be pointed out below.

Note that the category of A-spaces has a terminal object. Actually, one can consider $\operatorname{inj\,lim} J$ as an A-atlas Φ_t on some set X_t. Clearly this atlas is closed and the A-space $t(A) = (X_t, \Phi_t)$ is the terminal object of the category AS.

2.2. Properties of the category of A-spaces. For each $a \in ObA$, by $\Phi(a)$ denote an A-atlas on $J(a)$ of A-charts of the type $(b, J(f))$, where $f : b \to a$ is an A-morphism. By axiom (2) of the definition of an LSS-category, the A-atlas $\Phi(a)$ is closed and, by axiom (1) of the same definition, a is uniquely determined by the A-space $(J(a), \Phi(a))$. Moreover, for $a \in ObA$ and an A-space (X, Φ), a map $l : J(a) \to X$ is a morphism from $J(a, \Phi(a))$ to (X, Φ) if and only if $(a, l) \in \Phi$. In particular, for $a, b \in ObA$, a map $l : J(a) \to J(b)$ is a morphism from $(J(a), \Phi(a))$ to $(J(b), \Phi(b))$ if and only if there is an A-morphism $f : a \to b$ such that $J(f) = l$. Thus one can consider A as a full subcategory of the category of A-spaces by means of the identification $a = (J(a), \Phi(a))$.

Suppose that (X, Φ) is an A-space and $(a, l) \in \Phi$. We consider (a, l) as a morphism from a to (X, Φ) and, for an (X, Φ)-sieve S, put $S_A = S \cap \Phi$. We shall say that an (X, Φ)-sieve is a *cover*, if S_A is an A-atlas on X and $\overline{S_A} = \Phi$. Denote by $Cov(X, \Phi)$ the set of such covers. Clearly, for $a \in ObA$, $S \in Cov(J(a), \Phi(a))$ if and only if $S_A \in Cov(a)$.

2.2.1. Theorem. *The assignment $(X, \Phi) \longrightarrow Cov(X, \Phi)$ is a Grothendieck topology on the category of A-spaces.*

Proof. We need to check the axioms of a Grothendieck topology. Obviously axiom (1) holds. Let $h : (Y, \Psi) \to (X, \Phi)$ be a morphism of A-spaces, $S \in Cov((X, \Phi))$, and $(a, l) \in \Psi$. Then the A-chart $(a, h \circ l) \in \Phi$ and it is obtained from S_A by gluing by means of some cover $R \in Cov(a)$ of the site A. Therefore (a, l) is obtained by gluing from $(h^*S)_A$ by means of the same cover R. This means that h^*S is a cover and axiom (2) holds.

Suppose that S is an (X, Φ)-sieve , $R \in Cov(X, \Phi)$, and, for every $h : (Y, \Psi) \to (X, \Phi)$ from R, $h^*S \in Cov(Y, \Psi)$. Let $h = (a, l) \in R_A$. By definition, $(h^*S)_A$ consists of all A-charts $(b, k) \in \Phi(a)$ such that $(b, l \circ k) \in S_A$ and $k = J(f)$ for some A-morphism $f : b \to a$. Then the cover h^*S induces the A-chart (a, l) obtained by gluing from S_A. Therefore $R_A \subset \overline{S_A} \subset \Phi$ and this means that $S \in Cov(X, \Phi)$. Thus axiom (3) holds. □

2.2.2. Corollary. *The site AS of A-spaces with the forgetful functor $J : AS \to Set$ is an LSS-category.*

The proof is obvious.

2.3. Extensions of functors. Let A and B be LSS-categories, $F : A \to B$ a functor, and (X, Φ) an A-space. By the definition of an inductive limit, inj lim $J \circ F \circ I_\Phi$ is a set X' given with the set of maps $l' : J(F(a)) \to X'$ corresponding to A-charts $(a, l) \in \Phi$ and the set of B-charts $(F(a), l')$ is a B-atlas Φ' on X'. By definition, put $\widetilde{F}((X, \Phi)) = (X', \overline{\Phi'})$ and, for every morphism $h : (X, \Phi) \to (Y, \Psi)$ of A-spaces, define $\widetilde{F}(h)$ by the natural way using properties of an inductive limit. Note that the inductive limit inj lim $J \circ F \circ I_\Phi$ determines the set X' up to an isomorphism and precisely saying one must fix some X'.

As it was remarked above one can consider an A-space as the inductive limit of the restriction of the functor $J = J_A$ to the subcategory A_Φ of A determined by Φ. The functor F maps A_Φ onto some subcategory $F(A_\Phi)$ of B and the inductive limit of the restriction of J_B to $F(A_\Phi)$ gives the same B-atlas Φ' on the set X' that was obtained above.

2.3.1. Theorem. *The functor $\widetilde{F}$ is an extension of the functor F and the correspondence $F \to \widetilde{F}$ is the inclusion of the category of functors from A to B into the category of functors from AS to BS.*

Proof. Let $a \in ObA$. Since $(J(a), J(1_a)) \in \Phi(a)$ the set $J(F(a))$ with the set of maps $J \circ F(f) : J(F(b)) \to J(F(a))$, where $f : b \to a$ is an A-morphism, gives inj lim $J \circ F \circ I_{\Phi(a)}$. Hence one can put $\widetilde{F}((J(a), \Phi(a))) = (J(F(a)), \Phi(F(a)))$, that is $\widetilde{F}(a) = F(a)$. The other statements of the theorem follow from the definition of an inductive limit. □

2.3.2. Corollary. *If a functor $F : A \to B$ preserves covers, then the functor $\widetilde{F} : AS \to BS$ also preserves covers.*

Proof. Let S be a (X,Φ)-cover and $(a,l) \in \Phi$. Then (a,l) is obtained by gluing from S_A by means of some cover $R \in Cov(a)$. Therefore the corresponding B-chart $(F(a),l')$ on $\widetilde{F}((X,\Phi))$ is obtained by gluing from the set $F(S_A)$ of B-charts of the B-space $\widetilde{F}((X,\Phi))$ using the $F(a)$-sieve, which is a cover, generated by $F(R)$. Hence an $F((X,\Phi))$-sieve generated by $F(S_A)$ is a cover. □

Now let A be an LSS-category and let B be a category containing projective limits of all diagrams, for example Set, the categories of groups, linear spaces, algebras and so on. Suppose $G : A \to B$ is a contravariant functor. For an A-space (X,Φ), put $\widetilde{G}((X,\Phi)) = \text{proj}\lim G \circ I_\Phi$ and, for every morphism $h : (X,\Phi) \to (Y,\Psi)$ of A-spaces, define $\widetilde{G}(h)$ by the natural way using properties of a projective limit. It is clear that $\widetilde{G}$ is a contravariant functor from AS to B.

2.3.3. Theorem. *The contravariant functor $\widetilde{G}$ is an extension of the contravariant functor G and the correspondence $G \to \widetilde{G}$ is the inclusion of the category of contravariant functors from A to B into the category of contravariant functors from AS to B.*

The proof is analogous to one of Theorem 2.3.1.

2.3.4. Theorem. *Let B be an LSS-category, A a full subcategory of B with the natural structure of a site induced by the Grothendieck topology of B, and $I : A \to B$ the inclusion functor. Then, if $ObB \subset \widetilde{I}(ObAS)$, the extension $\widetilde{I} : AS \to BS$ of the functor I is an isomorphism of categories.*

Proof. Let $b \in ObB$ and $\Phi_A(b)$ be the set of morphisms from objects of A to b. By the conditions of the theorem, one has $\overline{\Phi_A}(b) = \Phi(b)$, where $\Phi_A(b)$ is considered as a set of B-charts on $J(b)$.

Suppose that (X,Φ) is a B-space, S is the cover generated by Φ, S_A is the set of morphisms from objects of A to (X,Φ), and $\overline{S_A}$ is the (X,Φ)-sieve generated by S_A. If $h : b \to (X,\Phi)$ is a morphism of B-spaces, $h^*\widetilde{S_A} \supset \Phi_A(b)$ and, hence, $h^*\overline{S_A} \in Cov(b)$. Thus, by axioms (2) and (3) of a Grothendieck topology, $\widetilde{S_A} \in Cov(X,\Phi)$ and S_A is a closed A-atlas on X. Put $I^*((X,\Phi)) = (X,S_A)$ and define values of I^* on morphisms of BS by the obvious way. Clearly I^* is an inverse to the functor $\widetilde{I}$. □

For example, let D be the full subcategory of Man whose objects are the open submanifolds of $\mathbb{R}^n$ $(n = 1,2,...)$. It is easily seen that the category of D-spaces is the category of diffeological spaces of Souriau [16]. Then, by Theorem 2.3.4, the category of Man-spaces is isomorphic to the category of D-spaces. Analogously, the category of R_n-spaces is isomorphic to the category of R_n^0-spaces, where R_n^0 is the full subcategory of R_n of Riemannian structures on open submanifolds of $\mathbb{R}^n$.

2.3.5. Corollary. *The LSS-category AS of A-spaces is isomorphic to the category of AS-spaces.*

The proof immediately follows from Theorem 2.3.4 applied to the inclusion of A into AS.

3. Some applications

3.1. Diffeological spaces. Consider the category of Man-spaces, that is diffeological spaces or just D-spaces. Let us take, for example, the following natural functors from Man to Man: T, T^p and $\Lambda^p T$ such that, for a manifold M, $T(M)$ is the tangent fiber bundle of M, $T^p(M)$ and $\Lambda^p T(M)$ are its p-th tensor and p-th exterior degree, respectively. One also has the natural projections $T(M) \to M$, $T^p(M) \to M$, and $\Lambda^p T(M) \to M$ which are functor morphisms from the above functors to the identity functor of Man. The extensions of the above functors give the generalized fiber bundles $\widetilde{T}((X,\Phi)) \to (X,\Phi)$, $\widetilde{T^p}(M) \to (X,\Phi)$, and $\widetilde{\Lambda^p T}((X,\Phi)) \to (X,\Phi)$. Since these morphisms of D-spaces are maps of the underlying sets one can define fibers at points of X and sections of the above fiber bundles in the usual manner. Note that these fibers have no algebraic structure in general.

Points of $\widetilde{T}((X,\Phi))$, $\widetilde{T^p}((X,\Phi))$, and $\widetilde{\Lambda^p T}((X,\Phi))$ we can call *tangent vectors, tensors of the type* $(p,0)$, and $p-$*vectors* at corresponding points of X, respectively. Morphisms of D-spaces, which are sections of the above fiber bundles, we can call *vector fields, tensor fields of the type* $(p,0)$, and *fields of p-vectors* .

Consider the contravariant functors Ω^p $(p = 0, 1, ...)$ from Man to the category of real vector spaces such that, for a manifold M, $\Omega^p(M)$ is the space of differential p-forms on M and the functors morphisms $d^p : \Omega^p(M) \to \Omega^{p+1}(M)$, where d^p is the exterior derivative. Then one has the extensions $\widetilde{\Omega^p}$ of Ω^p and the maps

$$\widetilde{d^p} : \widetilde{\Omega^p}((X,\Phi)) \to \widetilde{\Omega^{p+1}}(X,\Phi),$$

where $\widetilde{\Omega^p}((X,\Phi))$ is a real vector space and $\widetilde{d^p}$ is a linear map satisfying the equation $\widetilde{d^{p+1}} \circ \widetilde{d^p} = 0$. One can define the exterior composition

$$\widetilde{\Omega^p}((X,\Phi)) \otimes \widetilde{\Omega^q}((X,\Phi)) \to \widetilde{\Omega^{p+q}}((X,\Phi))$$

in the usual manner and it has the usual properties. Thus one defines the de Rham complex $\widetilde{\Omega^*}((X,\Phi))$ for a D-space (X,Φ).

Analogously, since a construction of the complex of singular smooth chains or cochains with some coefficients has a functorial character, one can define these notions for the category of D-spaces. Moreover, the same reasons give the definitions of an integral of a differential p-form on (X,Φ) over a singular smooth p-cochain with integral or real coefficients and the Stokes theorem for D-spaces.

Let A be an LSM-category with the forgetful functor $J_m : A \to Man$ and let $\widetilde{J_m}$ be the extension of J_m. Then the composites of $\widetilde{J_m}$ and the constructed above functors for the category of D-spaces extend the corresponding notions to the category of A-spaces.

Suppose that M is a C^∞-Fréchet manifold with model space E. Consider M as a D-space with a diffeology defined by the set of C^∞-maps from open subsets of $\mathbb{R}^n$ $(n = 1, 2, ...)$ to M.

3.1.1. Theorem. [8] *The diffeological structure of M uniquely determines the initial structure of a Fréchet manifold. Any morphism of Fréchet manifolds, as D-spaces, is a morphism of Fréchet manifolds. The tangent fiber bundle $\widetilde{T}(M)$ of M, as a D-space, have a natural structure of a Fréchet manifold with model space E^2. A fiber of $\widetilde{T}(M)$ has a natural structure of a Fréchet space isomorphic to E. The fiber bundles $\widetilde{T^p}(M)$ and $\widetilde{\Lambda^p T}(M)$ of M, as D-spaces, are the algebraic p-th tensor degree and the p-th exterior degree of $\widetilde{T}(M)$ with structures of D-spaces, respectively.*

Note that the main results of Theorem 3.1.1 are valid, if one uses, instead of diffeology, the weaker Frölicher-Kriegl structure of an infinite dimensional manifold on a Fréchet manifold [4,10].

3.2. D_n-spaces. At first we indicate some examples of D_n-spaces. Clearly any D_n-space is a D-space.

Suppose that F is a foliation of codimension n on an $(m+n)$-dimensional manifold M and Γ is a pseudogroup of local diffeomorphisms of M preserving F. Then one has two equivalence relations ε_1 and ε_2 on M: classes of ε_1 are leaves of F and classes of ϵ_2 are orbits of Γ. Let ε be the minimal equivalence relation on M containing ε_1 and ε_2.

3.2.1. Theorem. *(cf.* [3,7]*) The quotient set M/ε has a canonical structure of D_n-space. The projection $M \to M/\varepsilon$ is a morphism of D-spaces.*

Proof. Let K be the atlas of charts $h : W \to \mathbb{R}^{m+n}$ on M, where W is an open subset of M, such that the composite of h and the projection $p : \mathbb{R}^{m+n} \to \mathbb{R}^n$ is constant on leaves of the restriction of F to W. Suppose that $h \in K$, then $U = p \circ h(W)$ is an open subset of $\mathbb{R}^n$ and, for each $u \in U$, there is a unique leaf of F containing $(p \circ h)^{-1}(u)$. Consider a D_n-chart $k : U \to M/\varepsilon$ on M/ε such that $(p \circ h)^{-1}(u) \in k(u)$ and the set Φ of such D_n-charts on M/ε associated to all charts of K.

Suppose that $h, h' \in K$, $k : U \to M/\varepsilon$, and $k' : U' \to M/\varepsilon$ are the corresponding D_n-charts on M/ε, and $k(u) = k'(u')$, where $u \in U$ and $u' \in U'$. To prove that Φ is a D_n-atlas on M/ε one must show that there is a local diffeomorphism f of $\mathbb{R}^n$ defined in some neighborhood of u such that $f(u) = u'$ and k coincides with $k' \circ f$ in this neighborhood.

corresponding leaf of F. Therefore there is a finite sequence of charts $h_i : W_i \to \mathbb{R}^{m+n}$ $(i = 1, ..., r)$ of K such that $\{W_i\}$ cover this curve, $h_1 = h$, $h_r = h'$, and $W_i \cap W_{i+1} \neq \varnothing$ $(i = 1, ..., r-1)$. Moreover, one can assume that there are points x_i $(i = 1, ..., r-1)$ of our curve such that $x_i \in W_i \cap W_{i+1}$ and the intervals of this curve between x and x_1, x_i and x_{i+1}, x_{r-1} and x' are contained in W_1, W_i and W_r, respectively. It is obvious that our problem reduces to the case $r = 1$. Then the local diffeomorphism $h' \circ h^{-1}$ of $\mathbb{R}^{m+n}$ induces the required diffeomorphism f. In the second case, the equations of γ with respect to the charts h and h' give a local diffeomorphism of $\mathbb{R}^{m+n}$ which induces the required local diffeomorphism f of $\mathbb{R}^n$. The last statement of the theorem is obvious. □

Let F be a foliation of codimension n on a manifold M, M/F the leaf space of F, and $(M/F, \Phi_F)$ the D_n-space defined by Theorem 3.1.1. It is easily seen that the generalized de Rham complex $\widetilde{\Omega}^*((M/F, \Phi_F))$ is the complex of basic forms of F and, then, its cohomology coincides with the Reinhart basic cohomology of F [14].

Now we define a holonomy of a D_n-space (X, Φ). Let $x \in X$, $(U, l), (V, k) \in \Phi$, and $u \in U$, $v \in V$ such that $l(u) = k(v) = x$. By the definition of an inductive limit there exists an open neighborhood W of u contained in U and a morphism of D_n-charts on X $f : W \to V$ from $(V, l|_W)$ to (V, k) such that $f(u) = v$. By $\Gamma_{x,u,v}$ denote the set of germs at u of such morphisms f. For $u = v$, $\Gamma_{x,u} = \Gamma_{x,u,u}$ is a group called the *holonomy group* of x at u. If we take another D_n-chart $(U', l') \in \Phi$ and $u' \in U'$ such that $l'(u') = x$, it is easily proved that the groups $\Gamma_{x,u}$ and $\Gamma_{x,u'}$ are isomorphic. For D_n-space $(M/F, \Phi_F)$ corresponding to a foliation F on a manifold M, our definition of the holonomy group is equivalent to the usual definition of leaf holonomy [11].

Remark that the underlying set of the terminal object $t(D_n)$ of the category D_n is a one-point set.

3.3. Characteristic classes. Let A be an LSS-category and let F be a functor from A to the category of D-spaces. By Corollary 2.3.5 the extension $\widetilde{F}$ of F is a functor from the category AS to the category of D-spaces. Consider the composite $\widetilde{\Omega^*} \circ \widetilde{F}$ of the extended functors and, for an A-space (X, Φ), the cohomology $H_F(X, \Phi)$ of the complex $\widetilde{\Omega^*} \circ \widetilde{F}((X, \Phi))$, in particular, the cohomology $H_F(t(A))$, where $t(A)$ is the terminal object of the category AS. Then one has the canonical homomorphism $\chi : H_F(t(A)) \to H_F(X, \Phi)$ which is induced by the unique morphism $(X, \Phi) \to t(A)$ of A-spaces. We shall say that χ is the *characteristic homomorphism associated to* F and that classes of ImF are *the characteristic classes of A-space* (X, Φ) *associated to* F. One can consider $H_F(t(A))$ as the "universal characteristic classes" of the category AS associated to the functor F.

Consider some examples of the above characteristic classes.

Let M be an n-dimensional manifold. By $S(M)$ denote the space of invertible ∞-jets at $0 \in \mathbb{R}^n$ of smooth maps from $\mathbb{R}^n$ to M. It is known [1] that $S(M)$

is a manifold with model space $\mathbb{R}^\infty$. The natural actions of the groups $GL(n,\mathbb{R})$ and $O(n)$ on $\mathbb{R}^n$ induce the actions of these groups on $S(M)$. The quotient spaces $S'(M) = S(M)/GL(n,\mathbb{R})$ and $S''(M) = S(M)/O(n)$ are also manifolds with model space $\mathbb{R}^\infty$. It is known [1] that the natural projections $S(M) \to P(M)$, where $P(M)$ is the bundle of frames of M, $S'(M) \to M$, and $S''(M) \to M$ are smooth homotopy equivalences, and that the cohomologies of the de Rham complexes of $S(M)$, $S'(M)$, and $S''(M)$ as manifolds with model space $\mathbb{R}^\infty$ coincide with the real cohomologies of $P(M)$, M, and M, respectively. By Theorem 3.1.1 the category M_∞ of manifolds with model space $\mathbb{R}^\infty$ is a full subcategory of D and one can prove that the de Rham complex of the manifold N with model space $\mathbb{R}^\infty$ and the de Rham complex of N as a D-space coincide. It is easily checked that S, S', and S'' are functors from the category D_n to the category of manifolds with model space $\mathbb{R}^\infty$ (or to the category of D-spaces).

Let W_n be the topological Lie algebra of ∞-jets at the point $0 \in \mathbb{R}^n$ of vector fields on $\mathbb{R}^n$ with the bracket induced by the Lie bracket of vector fields and the projective limit topology of the corresponding spaces of finite order jets of vector fields. Suppose that $C^*(W_n;\mathbb{R})$ is the complex of standard continuous cochains of W_n with coefficients in the trivial W_n-module $\mathbb{R}$ and $C^*(W_n, GL(n,\mathbb{R}))$, $C^*(W_n, O(n))$ are its subcomplexes of relative cochains relative to the groups $GL(n;\mathbb{R})$ and $O(n)$, acting naturally on $C^*(W_n;\mathbb{R})$, respectively.

3.3.1. Theorem. *There are the following natural isomorphisms:*

$$H_S(t(D_n)) \cong H^*(W_n;\mathbb{R}),$$

$$H_{S'}(t(D_n)) \cong H^*(W_n, GL(n,\mathbb{R})),$$

$$H_{S''}(t(D_n)) \cong H^*(W_n, O(n)).$$

Proof. Consider the canonical Gel'fand-Kazhdan 1-form ω on $S(M)$ with values in W_n [1] satisfying the Maurer-Cartan equation: $d\omega = -1/2[\omega,\omega]$. It is known that ω maps the tangent space at each point of $S(M)$ isomorphically onto W_n and it is compatible with D_n-morphisms. Then one has the homomorphisms of complexes $C^*(W_n;\mathbb{R}) \to \Omega^*(S(M))$, $C^*(W_n, GL(n,\mathbb{R})) \to \Omega^*(S'(M))$, and $C^*(W_n, O(n)) \to \Omega^*(S''(M))$ which are compatible with D_n-morphisms and, then, they can be considered as functor morphisms of contravariant functors from D_n to the category of graded differential algebras. Their extensions give the following homomorphisms of complexes:

$$C^*(W_n;\mathbb{R}) \to \widetilde{\Omega}^* \circ \widetilde{S}(t(D_n)),$$

$$C^*(W_n, GL(n,\mathbb{R})) \to \widetilde{\Omega}^* \circ \widetilde{S'}(t(D_n)),$$

$$C^*(W_n, O(n)) \to \widetilde{\Omega}^* \circ \widetilde{S''}(t(D_n)),$$

respectively. It is easily proved that $\widetilde{S}(t(D_n))$ is an one-point set and the tangent space at this point is canonically isomorphic to W_n. Then these homomorphisms of complexes are isomorphisms. □

Thus, for D_n-space (X,Φ) and the functors S, S', and S'', the characteristic homomorphisms are the following homomorphisms $H^*(W_n;\mathbb{R}) \to H_S((X,\Phi))$, $H^*(W_n, GL(n,\mathbb{R})) \to H_{S'}((X,\Phi))$, and $H^*(W_n, O(n)) \to H_{S''}((X,\Phi))$.

Note that one can extend functors S, S', and S'' to the category of foliations of codimension n with the natural preserving leaves morphisms. Indeed, let F be a foliation of codimension n on a manifold M. Then $S(F)$ is the space of ∞-jets at points $x \in M$ of local submersions $f : W \to \mathbb{R}^n$, where W is an open subset of M, such that f is constant on leaves of the restriction of F on W and $f(x) = 0$. The natural actions of the groups $GL(n,\mathbb{R})$ and $O(n)$ on $\mathbb{R}^n$ induce the actions of these groups on $S(F)$ and one obtains the quotient spaces $S'(F) = S(F)/GL(n,\mathbb{R})$ and $S''(F) = S(F)/O(n)$. It is known [1] that $S(F)$, $S'(F)$, and $S''(F)$ are manifolds with model space $\mathbb{R}^\infty$ and the projections $S(F) \to P(F)$, $S'(F) \to M$, and $S''(F) \to M$, where $P(F)$ is the bundle of frames of the normal bundle of F, are smooth homotopy equivalences which induce the isomorphisms of the corresponding de Rham complexes.

Let M/F be the leaf space of F and let $(M/F,\Phi)$ be the D_n-space defined in the proof of Theorem 3.2.1. We have the natural maps $S(F) \to \widetilde{S}((M/F,\Phi_F))$, $S'(F) \to \widetilde{S}'((M/F,\Phi_F))$, and $S''(F) \to \widetilde{S}''((M/F,\Phi_F))$ which are diffeological morphisms. Consider the homomorphisms of the de Rham complexes induced by these maps and the corresponding cohomology homomorphisms: $H_S((M/F,\Phi_F)) \to H^*(P(F);\mathbb{R})$, $H_{S'}((M/F,\Phi_F)) \to H^*(M;\mathbb{R})$, and $H_{S''}((M/F,\Phi_F)) \to H^*(M;\mathbb{R})$. Then the composites

$$H^*(W_n,\mathbb{R}) \to H_S((M/F,\Phi_F)) \to H^*(P(F);\mathbb{R}),$$

$$H^*(W_n, GL(n,\mathbb{R})) \to H_{S'}((M/F,\Phi_F)) \to H^*(M;\mathbb{R}),$$

$$H^*(W_n, O(n)) \to H_{S''}((M/F,\Phi_F)) \to H^*(P(F);\mathbb{R})$$

give the known characteristic homomorphisms of foliations of Bott-Haefliger-Bernstein-Rosenfeld [1,2]. Thus these characteristic homomorphisms factor through our ones.

When (X,Φ) is a manifold one has the dimensional restrictions of nontriviality of the corresponding characteristic classes but, for general D_n-spaces, one has no such restrictions. For example, it is shown in [5] that the characteristic class corresponding to the two-dimensional generator of $H^*(W_1, GL(1,\mathbb{R}))$ is nontrivial for some D_1-space obtained as the leaf space of a certain foliation of codimension 1 or as the orbit space of a certain discrete group of diffeomorphisms of a circle. Moreover, one can prove that values of the Pontrjagin classes on smooth singular cocycles of D_n-spaces may be arbitrary real numbers.

Note that the Gel'fand-Feǐgin-Fuks deformation theory of characteristic classes of foliations [5] is valid also for the characteristic classes of D_n-spaces corresponding to foliations.

Let $R(M)$ be the space of ∞-jets of Riemannian metrics on M at points of M. It is clear that R is a functor from D_n to the category M_∞. Now we define a functor morphism $q : R \to S''$ in the following way. Let $z = j_x^\infty \in R(M)$, where $x \in M$ and g is a Riemannian metric on M, and let h be a normal chart of g with origin at x. One can consider $k = h^{-1}$ as a D_n-chart on M, then $j_0^\infty k \in S(M)$. Clearly the image $q(z)$ of $j_0^\infty k$ under the projection $S(M) \to S''(M)$ depends only on z and the assignment $z \to q(z)$ defines the smooth map $q(M) : R(M) \to S''(M)$. It is evident that q is a functor morphism from R to S''.

3.3.2. Theorem. *The functor morphism $q : R \to S''$ induces the isomorphism $H_R(t(D_n)) \cong H_{S''}(t(D_n))$ of the "universal characteristic classes" associated to the functors R and S'', hence, $H_R(t(D_n) \cong H^*(W_n, O(n))$.*

Proof. Suppose that g is a Riemannian metric on M and $x \in M$. The ∞-jet $j_x^\infty g \in R(M)$ is uniquely determined by coefficients of formal Taylor series at $0 \in \mathbb{R}^n$ of components g_{ij} of g relative to some normal chart h of g with origin at x. It is known [18] that nontrivial coefficients of these series are universal polynomials of components of the curvature tensor and its covariant derivatives of finite orders at x with respect to h.

The above coefficients are arbitrary solutions of an infinite system of homogeneous linear equations defining a closed infinite dimensional subspace V in the space $\mathbb{R}^\infty = \prod_{p>3} \mathbb{R}^{np}$ of components of tensors of the types $(0,p)$ $(p = 4, 5, ...)$ in $\mathbb{R}^n$ [18]. The natural action of the group $O(n)$ on normal charts of g with origin at x induces the usual action of $O(n)$ on the space V as the space of components of tensors of the types $(0,p)$ $(p = 4, 5, ...)$. Thus $R(M) = S(M) \times_{O(n)} V$ and relative to this representation of $R(M)$ the map $S(M) \times_{O(n)} V \to S''(M)$, induced by the projection $S(M) \times V \to S(M)$, coincides with $q(M)$.

Now we prove that $q(M)$ is a smooth homotopy equivalence of $R(M)$ and $S''(M)$ which is compatible with morphisms of D_n. It is sufficient to prove that the composite of $q(M)$ and the map $S''(M) \to R(M)$, induced by the zero section $S(M) \to S(M) \times V$, is homotopic to the identity map of $R(M)$ and this homotopy is compatible with morphisms of D_n. But the required homotopy is induced by the map $H : S(M) \times V \times [0,1] \to S(M) \times V$ defined by the following way: $H(s, v, t) = (s, tv)$ $(s \in S(M), v \in V, t \in [0,1])$. Then the cohomology homomorphism $q^*(M) : H_R(M) \to H_{S''}(M)$ is an isomorphism which is compatible with morphisms of D_n, that is we obtain the functor isomorphism $q^* : H_R \to H_{S''}$. Finally, the extended functor $\tilde{q}^*$ gives the required isomorphism $q^*(t(D_n)) : H_R(t(D_n)) \to H_{S''}(t(D_n))$. □

Now let $C(M)$ be the space of ∞-jets of linear connections on an n-dimensional manifold M. Clearly C is a functor from D_n to the category of manifolds with model space $\mathbb{R}^\infty$. Using normal coordinates, as for Riemannian manifolds, we obtain the functor morphism $p : C \to S'$. One can prove that p induces the isomorphism $H_C(t(D_n)) \cong H_{S'}(t(D_n))$ in the same way as in Theorem 3.3.1.

3.4. Transverse A-foliations. Let A be an LSM-category such that the functor $J_m : A \to Man$ factors through the forgetful functor $D_n \to Man$ and, therefore, one can consider J_m as a functor from A to D_n. Then, for every A-space (X, Φ), one has a structure of D_n-space $\widetilde{J_m}((X, \Phi))$ on the same set X.

Suppose F is a foliation of codimension n on a manifold M, M/F is the leaf space of F, and $(M/F, \Phi_F)$ is the D_n-space defined in the proof of Theorem 3.2.1.

3.4.1. Definition. *A foliation F with the structure $(M/F, \Psi)$ of an A-space on the set M/F such that $\widetilde{J_m}((M/F, \Psi)) = (M/F, \Phi_F)$ is called a transverse A-foliation.*

It can be easily proved that many notions of foliation theory can be expressed in terms of transverse A-foliations. For example, Riemannian foliations (see, for example, [11]) are transverse R_n-foliations, a transverse C_n-foliation is a foliation with a projectable infinitesimal connection and so on. General theory of A-spaces allows to simplify proofs of some theorems of foliation theory. For example, let $Q : R_n \to C_n$ be the functor, which assigns to each Riemannian manifold the underlying manifold with the Levi-Civita connection. Then the extension $\widetilde{Q} : R_nS \to C_nS$ of Q gives, for every transverse R_n-foliation $(F, (M/F, \Psi))$, the transverse C_n-foliation $(F, \widetilde{Q}((M/F, \Psi)))$. This proves the existence of the transverse Levi-Civita connection for Riemannian foliations.

By $PO(M)$ we denote the fiber bundle of orthogonal frames of a Riemannian manifold M. There is the natural smooth map $N(M) : PO(M) \to S(M)$ which assigns to each orthogonal frame of M at $x \in M$ the jet $j_0^\infty k \in S(M)$, where $h = k^{-1}$ is the normal chart with origin at x determined by this frame. One can consider N as a functor morphism from PO to S. Then the extension $\widetilde{N}$ of N gives, for every R-space (X, Φ), in particular, for every Riemannian foliation $(F, (M/F, \Psi))$, the morphism of D-spaces $\widetilde{N}((X, \Phi)) : \widetilde{PO}((X, \Phi)) \to \widetilde{S}((X, \Phi))$. The inverse image of the generalized Gel'fand-Kazhdan 1-form ω on $\widetilde{S}((X, \Phi))$ under $\widetilde{N}((X, \Phi))$ is a 1-form on $\widetilde{PO}((X, \Phi))$ with values in W_n and this form defines many notions of Riemannian geometry: the canonical 1-form ϑ with values in $\mathbb{R}^n$, the Levi-Civita connection form, the curvature form, and the horizontal 1-forms which have as coefficients covariant derivatives of the curvature tensor of all orders. The Maurer-Cartan condition for ω gives all classical relations between these objects, for example, the structure equation for the Levi-Civita connection, the Bianchi identity, and so on.

Note that the functor morphisms $PO \to P$ and $P \to PO$, induced by the inclusion $PO(M) \subset P(M)$ and the map $P(M) \to PO(M)$ obtained by the usual orthogonalization, respectively, induce the functor morphism $H_{PO} \to H_P$ for the category of R_n-spaces. Then the two functor morphisms $N : PO \to S$ and $S \to P$, induced by the projection $S(M) \to P(M)$, induce the functor isomorphism $H_S \cong H_{PO}$. Analogously, one can obtain functor isomorphisms $H_{S'} \cong H_{S''} \cong H_{J_m}$. Thus, in this case the characteristic classes associated to the functors S' and S'' take their values in the cohomology of the complex $\widetilde{\Omega}^* \circ \widetilde{J_m}((X, \Phi))$ for any R_n-space (X, Φ). Moreover, one has, for the characteristic classes of R_n-spaces associated to the functors S, S', and S'', the same dimensional restrictions as for n-dimensional manifolds. This implies, for example, the Pasternack vanishing theorem for the Pontrjagin classes of the normal bundle of Riemannian foliations [12].

For the category of C-spaces and the functors S, S', P, and J_m the same arguments give functor isomorphisms: $H_S \cong H_P$ and $H_{S'} \cong H_{J_m}$ with the same conclusions.

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