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REPRESENTATIONS OF MODULES AND CAUCHY COMPLETENESS

by SHU HAO SUN and R.F.C. WALTERS

Soit R un anneau, non nécessairement commutatif, et soit M un R-module à droite. Soit $\mathcal{V}$ la catégorie monoidale des idéaux faibles à droite de R. Nous décrivons une $\mathcal{V}$-catégorie Cauchy-complète construite à partir de M, dont les objets sont les éléments de M.

1. Introduction

Recently, Borceux and Van den Bossche presented in [1] an interesting 'generic' sheaf representation of commutative rings from a quantale theoretic point of view. But it does not seems to be very clear how this is connected with classical sheaves.

On the other hand, F.W. Lawvere introduced in [3] the notion of *cauchy complete* $\mathcal{V}$-category, a notion which generalizes that of cauchy complete metric space. In [4], [3], R.F.C. Walters showed that sheaves on a space could be regarded as cauchy complete categories on the distributive bicategory of relations of the space.

The authors believe that this language of cauchy complete categories based on a distributive bicategory provides a flexible and precise language for analysing sheaf-like representations of algebraic structures. The general methods of enriched category theory provide an excellent guide to this particular situation.

We intend a series of investigations, of which this is the first, into different bicategories suitable for analysing different aspects of the representation of $R - mod$, R a non-commutative ring.

Firstly, if we choose the base bicategory to be the monoidal category Id (R), whose objects are ideals, whose arrows are inclusions, and whose tensor product is the product of ideals (i.e., the bicategory is a quantale), then it is not too hard to show that each R-module M occurs as a symmetric skeletal cauchy complete $\mathcal{V}$-category $L(M)$, whose objects are the elements of M (more details will be given in Section 2).

However, in the light of papers [3] and [4], to obtain representations which are more sheaf-like, we prefer to choose the bicategory to be a locale or a category having the form Rel(C) (the definition see [3]) with a small category C.

For this reason we introduce the notion of *weak right ideals* of a *non-commutative* ring R: weak right ideals are subsets of R closed under right-multiplication by elements of the ring and *"weak addition"*. The poset of weak right ideals is then a locale. We consider in this paper the monoidal category $\mathcal{V}$ whose objects are *weak right ideals* of a *non-commutative* ring R, whose arrows are inclusions, and whose tensor products is intersection. Given a right-R module (or left-R module) M we construct a symmetric skeletal cauchy complete $\mathcal{V}$-category $L(M)$, whose objects are the elements of M.

In our future paper, we will consider such representations whose base bicategory has the form Rel(C) with C a small category.
The notation of this paper is that of [3] and [4], with the simplification that the base category is a monoidal category $\mathcal{V}$ rather than a bicategory $\mathcal{B}$.
After this paper was written we became aware of further work of Borceux, with Cruciani, extending [1] to the non-commutative case. Their main contribution is to present a notion of sheaf over a quantale, but again without our enriched-categorical setting.

2. The base bicategory is a quantale

In this short section, we sketch how the results in [1] are connected with the theory of enriched category, in particular, with the theory of cauchy completion. Our main results will be given in next two sections.
Let R be a commutative ring with an identity and $\mathcal{V} = Id(R)$ the distributive monoidal category of ideals of R.

Definition 2.1. *If M is a right-R-module, then $\mathcal{V}$-category $L(M)$ is defined by*
(i) the set of objects of $L(M)$ is M;
(ii) if $m, m' \in M$ then the hom $d(m, m')$ is $Ann(m - m')$,
where $Ann(m) = \{r \in R | 0 = mr\}$.

Lemma 2.1. *Each $L(M)$ is a symmetric skeletal $\mathcal{V}$-category.*

Proof. See section 4. □

Lemma 2.2. *For any adjoint pair of $\mathcal{V}$-modules*

$$\phi : \{*\} \longrightarrow\!\!\!\!\!\!\mapsto M \qquad and \qquad \psi : M \longrightarrow\!\!\!\!\!\!\mapsto \{*\},$$

it is the case that $\phi(*, m) = \psi(m, *)$ for all $m \in M$.

Proof. Note that R is commutative. □

Lemma 2.3. *Each $L(M)$ is cauchy complete.*

Proof. For the details see section 4. □

Now let $f : M_1 \to M_2$ be a right R-module morphism and let $Lf = f$. Then we have,

Lemma 2.4. *Lf is a $\mathcal{V}$-functor from $\mathcal{V}$-category $\mathcal{V}_1$ to the $\mathcal{V}$-category $\mathcal{V}_2$.*

Proof. It suffices to show that $d_1(m, m') \leq d_2(fm, fm')$ but this is obvious. □

We have thus proved the following

Theorem 2.1. *L is a functor from the category MOD-R to the category $\mathcal{V} - Cat_{cc}$ of symmetric skeletal cauchy complete $\mathcal{V}$-categories and $\mathcal{V}$-functors.*

After this paper was written we became aware of further work of Borceux, with Cruciani, extending [1] to the non-commutative case. Their proof and ours in the next sections suggest that the above statements remain true if a commutative ring R is replaced by a non-commutative one.
However, our main contributions are contained in the next two sections.

3. The locale of weak right ideals

Let R be a not-necessarily commutative ring with identity.

Definition 3.1. A subset I of R is called a weak right ideal of R if

(i) $IR \subseteq I$;

(ii) for all $a \in R$ and central elements $u_1, u_2, \ldots, u_n \in R$ with $\sum_{i=1}^{n} u_i = 1$, if $au_i \in I$ for $i = 1, \cdots, n$, then $a \in I$.

For convenience, if I satisfies (ii), we will say that I is closed under *weak addition*. It is clear that each right ideal is a weak right ideal.

Example 3.1. Let R be a (not necessarily commutative) ring without central elements except 0 and invertible elements. Then any union of right ideals of R is a weak right ideal. The free ring R with more than one generator, over a skew-field, is such an example. Moreover, any simple ring (i.e., a ring with no non-trivial two sided ideals) is such a ring. A particular example is the ring of $n \times n$ matrices over a skew-field.

Example 3.2. On the other hand, however, in the Euclidean domain, $\mathbf{Z}$, the fact that I is closed under weak addition implies that I is closed under addition; that is that I is an ideal. The proof is as follows. If I is a weak ideal and m, n lie in I and $d = HCF(m,n)$ then write $m = dm'$, $n = dn'$ and solve $u_1 + u_2 = m'x + n'y = 1$. Since $du_1 = mx \in I$ and $du_2 = ny \in I$ then closure under weak addition implies that $d \in I$. It is then trivial that $m + n = d(m' + n') \in I$.

Lemma 3.1. *The set $WRId(R)$ of all weak right ideals of R is closed under intersections and hence is a complete lattice.*

Proof. Easy $\square$

Lemma 3.2. *The suprema $\bigvee J_i$ of $J_i \in WRId(R)$ is calculated as follows: $\bigvee J_i = \{a \in R | \exists$ central elements u_l with $\sum u_l = 1$, such that $u_i a \in J_{i(l)}$ for some $J_{i(l)}\}$*

Proof. First check that $\bigvee J_i$ is a weak right ideal:

(1) If $a \in \bigvee J_i$, then there are central elements $\{u_k\}$ with $\sum u_k = 1$ and $u_k a \in J_{i(k)}$. Thus for any $x \in R$, one has $u_k ax \in J_{i(k)}$ and hence $ax \in \bigvee J_i$.

(2) If there are central elements z_k with $\sum z_k = 1$ so that $z_k a \in \bigvee J_i$, then there are centrals $u_{k,l}$ with $\sum_l u_{k,l} = 1$ such that $u_{k,l} z_k a \in J_{i(k,l)}$. Note that $u_{k,l} z_k$ are central elements with $\sum_{k,l} u_{k,l} z_K = 1$, so that $a \in \bigvee J_i$.

Then we note that $\cup J_i \subseteq \bigvee J_i$ and it is easy to check $\bigvee J_i$ is the least upper bound of $J_i's$. $\square$

Theorem 3.1. *$WRId(R)$ is a locale.*

Proof. It suffices to prove that $J \cap \bigvee I_i \subseteq \bigvee(J \cap I_i)$ for any $J, I_i \in WId(R)$. Let $x \in J \cap \bigvee I_i$. Then $x \in J$ and there are centrals u_k with $\sum u_k = 1$ such that $u_i x \in J_{i(k)}$. Moreover we have $u_i x = x u_i \in J$ since u_i are central and $JR \subseteq J$. Thus $u_i x \in J \cap I_{i(k)}$ and hence $x \in \bigvee J \cap I_i$. □

Note that a principal right ideal aR is compact in the lattice $WRId(R)$ in the usual sense and that each weak right ideal can be expressed as a join of aR's. So $WRId(R)$ is in fact an algebraic locale, and hence is spatial. Thus we have

Theorem 3.2. *For a general ring R with identity, $WRId(R)$ is a spatial locale.*

Remark 3.1. Note that the lattice of all right ideals of a ring R is not necessarily a locale, even for commutative ring R. For example, Let R be the polynomial (commutative) ring $k[X,Y]$, with two generators X and Y, over a field k. Then R is Noetherian, but is not Dedekind, since non-zero prime ideals are not necessarily maximal (for example, the prime ideal (X)). Thus IdR is not distributive by the fact that a Noetherian domain is Dedekind iff Id (R) is distributive.

4. The construction

Let R be a not-necessarily commutative ring with an identity and $\mathcal{V} = WRIdR$ the distributive monoidal category of weak right ideals of R.

Definition 4.1. If M is a right-R-module, then $\mathcal{V}$-category $L(M)$ is defined by
(1) the set of objects of $L(M)$ is M;
(2) if $m, m' \in M$ then the hom $d(m, m')$ is $ann_r(m - m')$,
where $ann_r(m) = \{r \in R | 0 = mr\}$ is a right ideal and hence is a weak right ideal.

Lemma 4.1. *Each $L(M)$ is a symmetric skeletal $\mathcal{V}$-category.*

Proof. The symmetry is obvious. If $d(m, m') = R$, then trivially $m = m'$ since R has an identity; so that $L(M)$ is skeletal. It remains to show that

$$ann_r(m' - m'') \cap ann_r(m - m') \subseteq ann_r(m - m'').$$

Let $x \in ann_r(m' - m'') \cap ann_r(m - m')$. Then

$$(m' - m'')x = 0 = (m - m')x$$

and hence $(m - m'')x = 0$; i.e., $x \in ann_r(m - m'')$. □

Lemma 4.2. *For any adjoint pair of $\mathcal{V}$-modules*

$$\phi : \{*\} \longrightarrow\!\!\!\!\!\!\!+\;\, M \qquad and \qquad \psi : M \longrightarrow\!\!\!\!\!\!\!+\;\, \{*\},$$

it is the case that $\phi(*, m) = \psi(m, *)$ for all $m \in M$.

Proof. First for any $m, m' \in M$, we have

$$\phi(*, m) \cap \phi(*, m') = R \cap \phi(*, m) \cap \phi(*, m') = \Big(\bigvee_{m'' \in M} \psi(m'', *)\Big) \cap \phi(*, m) \cap \phi(*, m')$$

$$= (\bigvee_{m''\in M} \psi(m'',*)\cap\phi(*,m))\cap\psi(m'',*)\cap\phi(*,m') \leq \bigvee_{m''\in M} ann_r(m''-m)\cap ann_r(m''-m')$$

$$\leq ann_r(m'-m).$$

Then

$$\phi(*,m) = R\cap\phi(*,m) = \bigvee_{m'}\phi(*,m')\cap\psi(m',*)\cap\phi(*,m)$$

$$= \bigvee_{m'}(\phi(*,m')\cap\phi(*,m))\cap\psi(m',*) \leq ann_r(m-m')\cap\psi(m',*) \leq \psi(m,*).$$

Here we use Theorem 3.1. Similarly we have $\psi(m,*) \leq \phi(*,m)$. □

Lemma 4.3. *Each $L(M)$ is cauchy complete.*

Proof. Consider adjoint pair of $\mathcal{V}$-modules

$$\phi : \{*\} \longrightarrow\!\!\!\!\!\!\!\!\mapsto M \qquad \text{and} \qquad \psi : M \longrightarrow\!\!\!\!\!\!\!\!\mapsto \{*\}.$$

That is, $\phi(*,m), \psi(m,*), m \in M$ are objects of $\mathcal{V}$ satisfying the following:

(1) $d(m',m)\cap\phi(*m') \leq \phi(*,m)$;
(2) $\psi(m',*)\cap d(m,m') \leq \psi(m,*)$;
(3) $R \leq \bigvee_m \psi(m,*)\cap\phi(*,m)$;
(4) $\phi(*,m')\cap\psi(m,*) \leq d(m,m')$.

By (3) we have

$$R = \bigvee_m \psi(m,*)\cap\phi(*,m)$$

So there are central elements $\epsilon_i \in \phi(*,m_i)\cap\psi(m_i,*)$ such that $\sum_{i=1}^n \epsilon_i = 1$ by Lemma 3.2.

In particular, we have

$$R = \bigvee_i \psi(m_i,*)\cap\phi(*,m_i).$$

Let $m_0 = \sum_{i=1}^n m_i\epsilon_i$. We shall prove that

$$\psi(m,*) \leq ann_r(m_0 - m) \leq \phi(*,m), \text{for all} \quad m \in M.$$

To this end, we want to show

$$\psi(m_i,*) \leq ann_r(m_0 - m_i) \quad \text{for all} \quad i.$$

In fact, for any $t \in \psi(m_i,*)$, we have

$$(m_0 - m_i)t = (\sum_j^n m_j\epsilon_j - \sum_j^n m_i\epsilon_j)t = \sum_{j=1}^n (m_j - m_i)\epsilon_j t = 0$$

since ϵ_j is central, and hence $\epsilon_j t \in \phi(*,m_j)\cap\psi(m_i,*) \leq ann_r(m_j - m_i)$. So $\psi(m_i,*) \leq ann_r(m_0 - m_i)$.

Moreover, we have, by (2)

$$\phi(*, m_0) \geq \bigvee_i \phi(*, m_i) \cap ann_r(m_0 - m_i) \geq \bigvee_i \phi(*, m_i) \cap \psi(m_i, *) = R.$$

Thus we have, by (2) again,

$$\phi(*, m_0) \cap ann_r(m - m_0) \leq \phi(*, m)$$

and hence

$$ann_r(m - m_0) \leq \phi(*, m).$$

On the other hand, by (4), we have

$$\phi(*, m) \cap \psi(m_0, *) \leq ann_r(m_0 - m)$$

and hence

$$\psi(m, *) \leq ann_r(m_0 - m) \leq \phi(*, m)$$

for all $m \in M$.
Similarly, we have

$$\phi(m, *) \leq ann_r(m_0 - m) \leq \psi(*, m)$$

(it also follows from Lemma 4.2). Thus we finally have

$$\phi(*, m) = ann_r(m_0 - m) = \psi(m, *), \qquad \text{for all} \qquad m \in M.$$

and so that this adjoint pair is induced from the point $m_0 \in M$. □

Now let $f : M_1 \longrightarrow M_2$ for a left R- and right R-module morphism and let $Lf = f$. Then we have,

Lemma 4.4. *Lf is a $\mathcal{V}$-functor from $\mathcal{V}$-category $\mathcal{V}_1$ to the $\mathcal{V}$-category $\mathcal{V}_2$.*

Proof. It suffices to show that $d_1(m, m') \leq d_2(fm, fm')$ but this is obvious. □

We have thus proved the following

Theorem 4.1. *L is a functor from the category MOD-R to the category $\mathcal{V} - Cat_{cc}$ of symmetric skeletal cauchy complete $\mathcal{V}$-categories and $\mathcal{V}$-functors.*

5. Remarks

Remark 5.1. The questions of how characterize those symmetric skeletal cauchy complete $\mathcal{V}$-categories which is isomorphic to some $L(M)$,and of the existence of a left adjoint of the functor L, will be considered in the first author's forthcoming paper.

Remark 5.2. It is possible to establish a counterpart of the main results in §3 and §4 for the case that weak right ideals are replaced by weak ideals (i.e., those weak right ideals I satisfying $RI \subseteq I$). However, the proofs are similar to the previous sections and left to the readers.

Lemma 5.1. *The set $WId(R)$ of all weak ideals of R is closed under intersections and hence is a complete lattice.*

Theorem 5.1. *The suprema $\bigvee J_i$ of $J_i \in WId(R)$ is calculated as follows: $\bigvee J_i = \{a \in R | \exists$ central elements u_l with $\sum u_l = 1$, such that $u_i a \in J_{i(l)}$ for some $J_{i(l)}\}$ and hence $WId(R)$ is a locale.*

Let R be a not-necessarily commutative ring with an identity and $\mathcal{W} = WIdR$ the distributive monoidal category of weak ideals of R.

Definition 5.1. If M is a right-R-module, then $\mathcal{W}$-category $L(M)$ is defined by

(1) the set of objects of $L(M)$ is M;

(2) if $m, m' \in M$ then the hom $d(m, m')$ is $Ann_r(m - m')$,

where $Ann_r(m) = \{r \in R | \{0\} = mRr\}$ is a two sided ideal and hence is a weak ideal.

Theorem 5.2. *Each $L(M)$ is a symmetric skeletal cauchy complete $\mathcal{W}$-category.*

Now let $f : M_1 \longrightarrow M_2$ for a left R- and right R-module morphism and let $Lf = f$. Then we have,

Theorem 5.3. *L is a functor from the category MOD-R to the category $\mathcal{W} - Cat_{cc}$ of symmetric skeletal cauchy complete $\mathcal{W}$-categories and $\mathcal{W}$-functors.*

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