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ACCESSIBLE EMBEDDINGS AND THE SOLUTION-SET CONDITION

by H. HU and M. MAKKAJ

Dedicated to the memory of Jan Reiterman

Résumé. Pour chaque catégorie localement présentable, on démontre qu'une sous-catégorie accessiblement plongée d'une catégorie localement présentable est accessible si et seulement si elle satisfait la condition de l'ensemble-solution.

Introduction

The connection between accessibility and the solution set condition has been pointed out in several papers, including [7] and [9]. J. Adámek and J. Rosický have recently proved that each locally presentable category has the following property: a full subcategory closed under products and κ -filtered limits (for some infinite regular cardinal κ) is accessible if the embedding satisfies the solution set condition, see [3]. It is well-known that a category is locally presentable iff it is accessible and complete, or equivalently cocomplete (cf. [7]), but no assumption is made of the existence of any particular limits for accessible categories. One is impelled to ask: does the above mentioned result hold for any full subcategory closed under κ -filtered colimits? In this paper we provide an affirmative answer to this question. We will show that, for any locally presentable category \mathbf{B} , a full subcategory \mathbf{A} of \mathbf{B} closed under κ -filtered colimits is accessible iff the embedding satisfies the solution set condition. The proof of this result is a modification of the proof of the above mentioned result of J. Adámek and J. Rosický in [3].

1 Generalities on Accessible Categories

Let κ be an infinite regular cardinal. Recall that a category \mathbf{A} is κ -filtered if for any graph \mathbf{G} of cardinality less than κ , any diagram $D : \mathbf{G} \rightarrow \mathbf{A}$ has a cocone on it. \mathbf{A} has κ -filtered colimits, if \mathbf{A} has colimits of all diagrams whose domain is a κ -filtered category. Another concept is limit of κ -diagram, where a κ -diagram is a diagram whose domain category is of size less than κ .

An object A of a category \mathbf{A} is said to be κ -presentable if the representable functor $\mathbf{A}(A, -)$ preserves κ -filtered colimits existing in \mathbf{A} . The full subcategory of \mathbf{A} whose objects are the κ -presentable ones is denoted by \mathbf{A}_κ . Note that for any category, a κ -colimit of a diagram in which the objects are κ -presentable is κ -presentable itself (see [5] and [7]).

Definition 1.1 ([7]) *A category \mathbf{A} is κ -accessible if:*

- (i) \mathbf{A} has κ -filtered colimits;
 - (ii) *There is a small full subcategory \mathbf{C} of \mathbf{A}_κ such that every object of \mathbf{A} is a κ -filtered colimit of a diagram of objects in \mathbf{C} .*
- A category is accessible if it is κ -accessible for some infinite regular cardinal κ .*

Recall from [4] that a category is called locally κ -presentable if it is locally small, cocomplete, and has a small strong generator consisting of κ -presentable objects. A theorem (Theorem 6.1.4. in [7]) says that an accessible category is complete iff it is cocomplete. P. Gabriel and F. Ulmer have shown in [4] that a category \mathbf{B} is locally κ -presentable if it is equivalent to the category of the form $L_\kappa(\mathbf{C}, \mathbf{Set})$, the category of all functors preserving κ -limits; here \mathbf{C} is a small category with κ -limits, and \mathbf{Set} denotes the category of small sets.

Let \mathbf{B} be a κ -accessible category, and \mathbf{A} a full subcategory of \mathbf{B} . \mathbf{A} is said to be κ -accessibly embedded if \mathbf{A} is closed under κ -filtered colimits in \mathbf{B} . \mathbf{A} is accessibly embedded if it is λ -accessibly embedded for some regular cardinal λ with $\lambda \geq \kappa$.

Recall from [3] that a full subcategory \mathbf{A} of \mathbf{B} is said to be cone-reflective if the inclusion functor $\mathbf{A} \rightarrow \mathbf{B}$ satisfies the solution set condition, i.e., for each object B of \mathbf{B} there exists a small cone $\langle r_i : B \rightarrow A_i \rangle_{i \in I}$ with $A_i \in \mathbf{A}$ such that for any $A \in \mathbf{A}$, every morphism $B \rightarrow A$ factors through some r_i . Let \mathbf{D} be a set of objects of \mathbf{A} . We say that \mathbf{D} weakly reflects B (in \mathbf{A}) if for every $A \in \mathbf{A}$ and $f : B \rightarrow A$ there is $D \in \mathbf{D}$ and a factorization

$$\begin{array}{ccc}
 B & \xrightarrow{m} & D \\
 \downarrow f & & \swarrow f' \\
 & & A
 \end{array}$$

where m and f' are some morphisms. Note that \mathbf{A} is cone-reflective in \mathbf{B} iff for every $B \in \mathbf{B}$, there is a small set $\mathbf{D} \subset \mathbf{A}$ weakly reflecting B . If \mathbf{B} is accessible, then this is equivalent to saying that for every $B \in \mathbf{B}$ there is κ such that $\mathbf{D}_\kappa = \mathbf{A} \cap \mathbf{B}_\kappa$ weakly reflects B . Note that, of course, if $\kappa < \kappa'$ and \mathbf{D}_κ weakly reflects B , so does $\mathbf{D}_{\kappa'}$.

The following lemma can be found in [7] (Lemma 1.1.2.).

Lemma 1.2 *Suppose that \mathbf{J} is κ -filtered and the functor $F : \mathbf{I} \rightarrow \mathbf{J}$ satisfies that for every $J \in \mathbf{J}$, there exists I in \mathbf{I} and a morphism $J \rightarrow F(I)$. If F is full and faithful, then \mathbf{I} is κ -filtered and F is final, i.e., for any diagram $\Sigma : \mathbf{J} \rightarrow \mathbf{A}$, $\text{colim}\Sigma$ exists if and only if $\text{colim}\Sigma \circ F$ exists and the canonical morphism $\text{colim}\Sigma(F) \rightarrow \text{colim}\Sigma$ is an isomorphism.*

Proposition 1.3 *Let \mathbf{B} be a κ -accessible category, and \mathbf{A} a κ -accessibly embedded subcategory of \mathbf{B} . If every $B \in \mathbf{B}_\kappa$ is weakly reflected in \mathbf{A} by $\mathbf{D} = \mathbf{A} \cap \mathbf{B}_\kappa$, then \mathbf{A} is κ -accessible.*

Proof: \mathbf{A} has κ -filtered colimits, by assumption. For any $A \in \mathbf{A}$, we have a canonical diagram $G : \mathbf{B}_\kappa/A \rightarrow \mathbf{B}$, and $A = \text{colim}G$, by Proposition 2.1.5. in [7]. This colimit is κ -filtered. Let $\mathbf{D} = \mathbf{A} \cap \mathbf{B}_\kappa$. Since \mathbf{A} is closed under κ -filtered colimits in \mathbf{B} , all objects in \mathbf{D} are κ -presentable in \mathbf{A} . We have a full and faithful functor $F : \mathbf{D}/A \rightarrow \mathbf{B}_\kappa/A$. Let $G' : \mathbf{D}/A \rightarrow \mathbf{B}$ be the canonical diagram. Given an object $f : B \rightarrow A$ in \mathbf{B}_κ/A , by assumption, there is a factorization $f = f' \circ m$, with $f' : D \rightarrow A$ in \mathbf{D}/A . By Lemma 1.2, A is the κ -filtered colimit $\text{colim}G'$ in \mathbf{B} , and as a consequence, also in \mathbf{A} . Thus, \mathbf{A} is κ -accessible.

2 Main Theorem and Some Remarks

In what follows, κ , λ and subscripted variants of them always denote infinite cardinals.

The proof of the following theorem follows closely the lines of the corresponding proof in [3].

Theorem 2.1 *Let \mathbf{B} be a locally presentable category, and \mathbf{A} an accessibly embedded subcategory of \mathbf{B} . If \mathbf{A} is cone-reflective, then it is accessible.*

Proof: We may assume that \mathbf{B} is a functor category $(\mathbf{C}, \mathbf{Set})$, for some small category \mathbf{C} . The reason is that every locally presentable category is a reflective subcategory of a functor category $(\mathbf{C}, \mathbf{Set})$ with \mathbf{C} small, and the inclusion functor is accessibly embedded. If λ is a regular cardinal bigger than the cardinality of \mathbf{C} and \aleph_0 , then a functor $F \in \mathbf{B}$ is λ -presentable in \mathbf{B} iff the cardinality of $\coprod_{C \in \mathbf{C}} F(C)$ is less than λ . It easily follows that if $\mu = \sup_{i < \nu} \kappa_i$ with $\kappa_i \leq \kappa_j$ for $i < j < \nu$, and $B \in \mathbf{B}_{\mu+}$, then we can write B as a colimit of a ν -chain, $B = \text{colim}_{i < \nu} B_i$, with $B_i \in \mathbf{B}_{\kappa_i}$.

Let κ be a regular cardinal such that \mathbf{A} is closed under κ -filtered colimits in \mathbf{B} . Let us define κ_i for $i < \kappa$ by transfinite induction. Let $\kappa_0 = \kappa$. Given $0 < i < \kappa$, having defined κ_j for all $j < i$, for i limit, let κ_i be a regular cardinal bigger than κ_j

for all $j < i$, and for $i = j + 1 < \kappa$, let κ_{j+1} be a regular cardinal $\geq \kappa_j$ such that all objects in \mathbf{B}_{κ_j} have a weak reflection in $\mathbf{D}_{\kappa_{j+1}}$; since each $\mathbf{B}_{\kappa'}$ is small, and since every $B \in \mathbf{B}$ has a weak reflection in $\mathbf{D}_{\kappa'}$, for some κ' , such κ_{j+1} clearly exists.

Let $\mu = \sup_{i < \kappa} \kappa_i$, and $\lambda = \mu^+$. We claim that every $B \in \mathbf{B}_\lambda$ has a weak reflection in \mathbf{D}_λ . Since \mathbf{B} is clearly κ' -accessible for all $\kappa' \geq \aleph_0$, in particular, for $\lambda = \kappa'$, and $\lambda > \kappa$, by Proposition 1.3, this will suffice for the proof of the theorem.

Let $B \in \mathbf{B}_\lambda$. As above, let us represent B as the colimit of a κ -chain $(b_{i,j} : B_i \rightarrow B_j)_{i < j < \kappa}$, with $B_i \in \mathbf{B}_{\kappa_i}$. Let $\phi_i : B_i \rightarrow B$ be the colimit coprojection. Let $A \in \mathbf{A}$ and $f : B \rightarrow A$ be arbitrary; we want to find $A^* \in \mathbf{D}_\lambda$ with a factorization

$$\begin{array}{ccc}
 B & \xrightarrow{f^*} & A^* \\
 \downarrow f & \searrow a^* & \\
 A & &
 \end{array}$$

By transfinite induction on $i < \kappa$, we will define objects $A_i \in \mathbf{D}_{\kappa_{i+1}}$ and morphisms $a_{i,j} : A_i \rightarrow A_j$, $f_i : B_i \rightarrow A_i$, $\psi_i : A_i \rightarrow A$ such that the morphisms $a_{i,j}$ form a chain, the morphisms ψ_i form a compatible cocone, and the following diagram

$$\begin{array}{ccccc}
 B_i & \xrightarrow{b_{i,j}} & B_j & \xrightarrow{\phi_j} & B \\
 \downarrow f_i & & \downarrow f_j & & \downarrow f \\
 A_i & \xrightarrow{a_{i,j}} & A_j & \xrightarrow{\psi_i} & A
 \end{array}$$

commutes.

For $i = 0$, we let $A_0 \in \mathbf{D}_{\kappa_1}$, $f_0 : B_0 \rightarrow A_0$ and $\psi_0 : A_0 \rightarrow A$ such that

$$\begin{array}{ccc}
 B_0 & \xrightarrow{\phi_0} & B \\
 \downarrow f_0 & & \downarrow f \\
 A_0 & \xrightarrow{\psi_0} & A
 \end{array}$$

commutes; these items are obtained from a suitable factorization of the morphism $f \circ \phi_0 : B_0 \rightarrow A$, possible by the choice of κ_1 and $B_0 \in \mathbf{D}_{\kappa_0}$.

Fix k , $0 < k < \kappa$, and assume that all items with indices $< k$ have been defined. Let $C = \text{colim}(a_{i,j} : A_i \rightarrow A_j)_{i < j < k}$ with coprojections $a_i : A_i \rightarrow C$, and $B^* = \text{colim}(b_{i,j} : B_i \rightarrow B_j)_{i < j < k}$ with coprojections $b_i^* : B_i \rightarrow B^*$.

Since $A_i \in \mathbf{B}_{\kappa_i} \subset \mathbf{B}_{\kappa_k}$, and \mathbf{B}_{κ_k} is closed under $< \kappa \leq \kappa_k$ -sized colimits, $C \in \mathbf{B}_{\kappa_k}$. Similarly, $B_i \in \mathbf{B}_{\kappa_{i+1}} \subset \mathbf{B}_{\kappa_k}$, and so $B^* \in \mathbf{B}_{\kappa_k}$.

By the universal property of B^* , we have $b^* : B^* \rightarrow B_k$ such that

$$\begin{array}{ccc} B_i & \xrightarrow{b_i^*} & B^* \\ \downarrow b_{i,k} & \searrow b^* & \\ B_k & & \end{array}$$

commute, and $c : B^* \rightarrow C$ such that

$$\begin{array}{ccc} B_i & \xrightarrow{b_i^*} & B^* \\ \downarrow f_i & & \downarrow c \\ A_i & \xrightarrow{a_i} & C \end{array}$$

commute, for all $i < k$.

By the universal property of C , we have $a : C \rightarrow A$ such that

$$\begin{array}{ccc} A_i & \xrightarrow{a_i} & C \\ \downarrow \psi_i & \searrow a & \\ A & & \end{array}$$

commute for all $i < k$. We form the pushout of c and b^* :

$$\begin{array}{ccc}
 B^* & \xrightarrow{b^*} & B_k \\
 \downarrow c & & \downarrow h \\
 C & \xrightarrow{g} & D
 \end{array}$$

Since $B^*, B_k, C \in \mathbf{B}_{\kappa_k}$, we have $D \in \mathbf{B}_{\kappa_k}$. We prove that $a \circ c = f \circ \phi_k \circ b^*$ by showing that

$$\begin{aligned}
 a \circ c \circ b_i^* &= a \circ a_i \circ f_i \\
 &= \psi_i \circ f_i \\
 &= f \circ \phi_i \\
 &= f \circ \phi_k \circ b^* \circ b_i^*
 \end{aligned}$$

for each projection b_i^* . By using the universal property of pushout D , we have a unique morphism $l : D \rightarrow A$ such that $a = l \circ g$ and $f \circ \phi_k = l \circ h$. Since $D \in \mathbf{B}_{\kappa_k}$, and every object in \mathbf{B}_{κ_k} is weakly reflected by $\mathbf{D}_{\kappa_{k+1}}$, there is $A_k \in \mathbf{D}_{\kappa_{k+1}}$ with $\psi_k : A_k \rightarrow A$ such that the diagram

$$\begin{array}{ccc}
 D & \xrightarrow{m} & A_k \\
 \downarrow l & \searrow \psi_k & \\
 A & &
 \end{array}$$

commutes. We have defined the items A_k and ψ_k .

Next, we define $f_k = m \circ h : B_k \rightarrow A_k$ and $a_{i,k} = m \circ g \circ a_i : A_i \rightarrow A_k$. Note that the diagrams

$$\begin{array}{ccccc}
 B_i & \xrightarrow{b_i^*} & B^* & \xrightarrow{b^*} & B_k \\
 \downarrow f_i & & \downarrow c & & \downarrow h \\
 A_i & \xrightarrow{a_i} & C & \xrightarrow{g} & D
 \end{array}$$

commute for all $i < k$; and $b_{i,k} = b^* \circ b_i^*$. Then the diagrams

$$\begin{array}{ccc}
 B_i & \xrightarrow{b_{i,k}} & B_k \\
 \downarrow f_i & & \downarrow f_k \\
 A_i & \xrightarrow{a_{i,k}} & A_k
 \end{array}$$

commute for all $i < k$, and the diagram

$$\begin{array}{ccc}
 B_k & \xrightarrow{\phi_k} & B \\
 \downarrow f_k & & \downarrow f \\
 A_k & \xrightarrow{\psi_k} & A
 \end{array}$$

commutes. It is clear that $\psi_i = a_{i,k} \circ \psi_k$ and $a_{i,k} = a_{i,j} \circ a_{j,k}$ hold for all $i < j < k$. This completes the construction.

Put $A^* = \text{colim}(a_{i,j} : A_i \rightarrow A_j)_{i < j < \kappa}$ with coprojections $p_i : A_i \rightarrow A^*$. Since \mathbf{A} is closed under κ -filtered colimits in \mathbf{B} , $A^* \in \mathbf{A}$. Also, since $A_i \in \mathbf{B}_{\kappa_{i+1}} \subset \mathbf{B}_\lambda$ and $\kappa < \lambda$, we have that $A^* \in \mathbf{B}_\lambda$; that is, $A^* \in \mathbf{D}_\lambda$. By the construction above, we have $f^* : B \rightarrow A^*$ such that the diagrams

$$\begin{array}{ccc}
 B_i & \xrightarrow{\phi_i} & B \\
 f_i \downarrow & & \downarrow f^* \\
 A_i & \xrightarrow{p_i} & A^*
 \end{array}$$

commute for all $i < \kappa$; also, we have $a^* : A^* \rightarrow A$ such that the diagrams

$$\begin{array}{ccc}
 A_i & \xrightarrow{p_i} & A^* \\
 \psi_i \downarrow & \searrow a^* & \\
 A & &
 \end{array}$$

commute for all $i < \kappa$; hence we have that

$$f \circ \phi_i = \psi_i \circ f_i = a^* \circ p_i \circ f_i = a^* \circ f^* \circ \phi_i$$

for all $i < \kappa$. Since $\langle \phi_i \rangle_{i < \kappa}$ is a colimit cocone, we conclude that the diagram

$$\begin{array}{ccc}
 B & \xrightarrow{f^*} & A^* \\
 f \downarrow & \searrow a^* & \\
 A & &
 \end{array}$$

commutes. This completes the proof.

Definition 2.2 For any locally presentable category \mathbf{B} , an accessibly embedded subcategory \mathbf{A} of \mathbf{B} is accessible iff \mathbf{A} is cone-reflective.

Proof: By Theorem 2.1 and Proposition 2.1.8. in [7].

Remark 2.3 *Let \mathbf{A} be an accessibly embedded subcategory of \mathbf{B} closed under limits. J. Adámek and J. Rosický have shown in [2] that \mathbf{A} is a reflective subcategory of \mathbf{B} . Thus \mathbf{A} is locally presentable.*

Remark 2.4 *Recall from [1] that Vopěnka's principle is the following statement: the category \mathbf{Gra} of graphs does not have a large discrete full subcategory. It has been shown in [8] that Vopěnka's principle is equivalent to the following statement: every accessibly embedded subcategory of a locally presentable category is accessible. Thus, assuming Vopěnka's principle, every accessible embedding of a locally presentable category satisfies the solution set condition.*

Remark 2.5 *Let \mathbf{A} be an accessible full subcategory of an accessible category \mathbf{B} . Suppose that the inclusion functor from \mathbf{A} to \mathbf{B} satisfies the solution-set condition. J. Rosický and W. Tholen have recently proved that the inclusion functor is accessible (see Theorem 3.10 in [9]). Also, they have proved in [9] that Vopěnka's principle is equivalent to the the following statement: a functor between accessible categories is accessible if and only if it satisfies the solution-set condition.*

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