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ON THE CONSTRUCTION OF THE EXTENSIONAL TOPOLOGICAL HULL

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Dedicated to the Memory of Honza Reiterman

Résumé. Les objets (X, α) de l'enveloppe topologique extensionelle, d'une catégorie **A** sont caractérisés, parmi les objets d'une extension quelconque de **A** qui est topologique, extensionelle, et engendrée finalement par **A**, au moyen d'un test essentiellement basé sur l'information intrinsèquement contenue dans (X, α). En tant qu'outil utile, on développe le concept d'une extension à points multiples qui, à beaucoup d'égards, est parallèle à celui de l'extension à un seul point. Le théorème de caractérisation est alors appliqué pour distinguer l'enveloppe topologique extensionelle de la catégorie des espaces de Cauchy (respectivement, des espaces aux limites uniformes) de la catégorie des espaces semi-Cauchy (respectivement, des espaces aux limites semi-uniformes).

1. Introduction

A number of properties that a topological category may possess have proved useful and convenient in a variety of connections: being cartesian closed, extensional, or a topological universe. Since many important categories lack one or the other of these desirable properties, extensions where such properties are present are of considerable interest, and among them, in particular, extensions of various types that are as small as possible: i.e., hulls.

If such hulls exist, they can be found as subcategories of the quasicategory of structured sinks [11]. However, the resulting internal descriptions are quite abstract and fairly complicated. A major step towards simplification was achieved by Adámek, Reiterman and Strecker in [7] for the cartesian closed topological hull and, with a similar approach, in [2] for the topological universe hull (an analogous description of the extensional topological hull is possible): the hulls are described as consisting of "closed collections", which are structured sinks of *inclusion maps*

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(subject to an appropriate closure condition). The construction principle of [7] is based on the existence of a finally dense subclass (subject to additional conditions) of the category **A** whose hull is to be determined. This assumption is fulfilled in the following situation: the category **A** (supposed to be topological for simplicity of exposition) is contained as a bireflective subcategory in a topological category **B** which has the desired convenience property, and **A** has a subclass **H** which is finally dense in **B** and closed under formation of quotients in **B**.

The main advantage of the method of closed collections is the fact that an object of the hull under consideration is described by *subsets* carrying **H**-structures. We concentrate, in this paper, on the extensional topological hull, and present an alternative approach which allows one to characterize its objects as infima (with respect to the usual categorical order) of objects closely related to subspaces. While a closed collection provides a substitute for a certain final structure, and hence an "approximation from below", infima are initial structures and approximate the objects of the extensional topological hull "from above". This theory, and the concept of a many-point extension as a useful tool, are developed in Section 3. By application of the main result (3.7), some counterexamples are obtained in Section 4.

2. Preliminaries

In this paper a *topological category* is taken to be one in the sense of [10], viz., a small-fibred category of structured sets such that there are unique initial structures, and the empty set and the singletons carry only one structure. If $\mathbf{A}$ is a topological category and X is a set, then $\mathbf{A}(X)$ will denote the set of all $\mathbf{A}$-structures on X. Pair notation will be used for $\mathbf{A}$-objects, e.g., (X,α), where $\alpha \in \mathbf{A}(X)$. No notational distinction will be made between $\mathbf{A}$-morphisms and their underlying maps; in particular, an identity map will always be written as 1, regardless of whether it is an identity in $\mathbf{A}$. The set $\mathbf{A}(X)$ can be partially ordered as follows: $\alpha \leq \beta$ if and only if the identity map $1 : (X,\alpha) \longrightarrow (X,\beta)$ is an $\mathbf{A}$-morphism. The infimum of a family $(\alpha_i)_{i\in I}$ of $\mathbf{A}$-structures on X, i.e., the initial structure with respect to $\big(1 : X \longrightarrow (X,\alpha_i)\big)_{i\in I}$, is denoted by $\bigwedge_I \alpha_i$. If $(X,\alpha) \in \mathbf{A}$ and $Z \subset X$, then the initial structure with respect to the inclusion $Z \longrightarrow (X,\alpha)$ is denoted by $\alpha|Z$, and $(Z,\alpha|Z)$ will be called a *subspace* of (X,α). Background on categorical topology can be found in [3] and [15].

A topological category $\mathbf{B}$ is called *extensional* provided that each $(Y,\beta) \in \mathbf{B}$ is a subspace of an object $(Y^\sharp,\beta^\sharp)$, where $Y^\sharp = Y \cup \{\infty\}$ and $\infty \notin Y$, with the property that for every *partial morphism* $f : (Z,\alpha|Z) \longrightarrow (Y,\beta)$ from (X,α) to (Y,β), the map $f^X : (X,\alpha) \longrightarrow (Y^\sharp,\beta^\sharp)$, defined by $f^X(x) = f(x)$ if $x \in Z$, $f^X(x) = \infty$ if $x \in X - Z$, is a $\mathbf{B}$-morphism. The object $(Y^\sharp,\beta^\sharp)$ is called a *one-point extension* of (Y,β) [1], [12]. If $\mathbf{A}$ is a subcategory of $\mathbf{B}$, then $\mathbf{A}^\sharp$ denotes the class $\{(X^\sharp,\alpha^\sharp) \mid (X,\alpha) \in \mathbf{A}\}$. A topological category $\mathbf{B}$ is said to be an *extensional topological hull* of a topological category $\mathbf{A}$ (in symbols: $\mathbf{B} = \mathrm{ETH}\mathbf{A}$) if $\mathbf{B}$ is extensional, $\mathbf{A}$ is finally dense in $\mathbf{B}$, and $\mathbf{A}^\sharp$ is initially dense in $\mathbf{B}$ [12].

3. Construction of the extensional topological hull

For this entire section it is assumed that **A** is a topological category which is finally dense, and hence bireflective, in an extensional topological category **B**, the bireflector $\mathbf{B} \longrightarrow \mathbf{A}$ being denoted by $(\tilde{-})$. (These assumptions may strike the reader as being rather restrictive. One could, in fact, just assume that **A** is a topological category which is contained in an extensional topological category **C**, and is closed under subspaces in **C**. If **B** denotes the final hull of **A** in **C**, then **A** and **B** satisfy the conditions stated above. One could even weaken the assumption that **A** is topological.)

By the final density of **A** in **B**, the extensional topological hull of **A** is the bireflective hull of $\mathbf{A}^\sharp$ in **B**, i.e., the objects of ETH**A** are given as the domains of inital sources in **B** of the form $(f_i : (X,\alpha) \longrightarrow (Y_i^\sharp, \beta_i^\sharp))_{i\in I}$ with all (Y_i, β_i) in **A**. In order to transform this description into a simpler one that is mainly based on (X,α), we introduce the concept of a many-point extension of (X,α), which is derived from that of the one-point extension of (X,α) and shares many of its nice properties. In particular, it can be characterized as carrying the largest **B**-structure which makes (X,α) a subspace. Cf. results 3.3 and 3.4.

3.1 Definition. For $(X,\alpha) \in \mathbf{B}$ and $Y \supset X$, we denote by $[\alpha]_{XY}$ the initial structure with respect to $q : Y \longrightarrow (X^\sharp, \alpha^\sharp)$, where q is defined by $q(y) = y$ if $y \in X$, $q(y) = \infty$ if $y \in Y - X$. The object $(Y, [\alpha]_{XY})$ is called a *many-point extension* of (X,α).

The map q is, of course, on the level of sets, just the map 1^Y given by the universal property of the one-point extension of (X,α). In the case of $Y = X^\sharp$, it is an identity map, and $(Y, [\alpha]_{XY}) = (X^\sharp, \alpha^\sharp)$.

3.2 Proposition. *Suppose that $(X,\alpha) \in \mathbf{B}$, and $(X_i)_{i\in I}$ is a family of subsets of X. The following are equivalent:*

(1) *α is the largest **B**-structure on X for which each $(X_i, \alpha|X_i)$ is a subspace.*

(2) *The source $(q_i : (X,\alpha) \longrightarrow (X_i^\sharp, (\alpha|X_i)^\sharp))_{i\in I}$ is initial (where each q_i is defined, as in 3.1, by $q_i(x) = x$ if $x \in X_i$, $q_i(x) = \infty_i$ if $x \in X - X_i$.)*

Proof: (1) $\Rightarrow$ (2): Each $q_i : (X,\alpha) \longrightarrow (X_i^\sharp, (\alpha|X_i)^\sharp)$ is a morphism by extensionality. If β is initial with respect to $(q_i : X \longrightarrow (X_i^\sharp, (\alpha|X_i)^\sharp))_{i\in I}$ then $\alpha \leq \beta$, so each inclusion $j_i : (X_i, \alpha|X_i) \longrightarrow (X,\beta)$ is a morphism. Since each composition $(X_i, \alpha|X_i) \xrightarrow{j_i} (X,\beta) \xrightarrow{q_i} (X_i^\sharp, (\alpha|X_i)^\sharp)$ is initial, it follows that each $j_i : (X_i, \alpha|X_i) \longrightarrow (X,\beta)$ is initial. Hence the assumption in (1) yields $\beta \leq \alpha$.

(2) $\Rightarrow$ (1): Suppose $\beta \in \mathbf{B}(X)$ is such that each inclusion $j_i : (X_i, \alpha|X_i) \longrightarrow (X,\beta)$ is initial. Then each $q_i : (X,\beta) \longrightarrow (X_i^\sharp, (\alpha|X_i)^\sharp)$ is a morphism, by extensionality. From the initiality assumed in (2) it follows that $\beta \leq \alpha$. ■

In the following characterization of many-point extensions, observe that condition (2) is an analogue of the universal property which defines one-point extensions.

3.3 Proposition (cf. 2.6 of [17]). *For* **B**-*objects* (X,α) *and* (Y,μ) *with* $X \subset Y$*, the following are equivalent:*

(1) (Y,μ) *is a many-point extension of* (X,α).

(2) (X,α) *is a subspace of* (Y,μ)*; and given a partial morphism* $f : (Z,\beta|Z) \longrightarrow (X,\alpha)$ *from* (W,β) *to* (X,α)*, every map* $\bar{f} : W \longrightarrow Y$ *with* $\bar{f}|Z = f$ *and* $\bar{f}(W - Z) \subset Y - X$ *is a morphism* $\bar{f} : (W,\beta) \longrightarrow (Y,\mu)$.

(3) μ *is the largest* **B**-*structure on* Y *such that* (X,α) *is a subspace of* (Y,μ).

Proof: (1) $\Rightarrow$ (2): Since (X,α) is a subspace of $(X^\sharp,\alpha^\sharp)$, the initiality of $q : (Y,\mu) \longrightarrow (X^\sharp,\alpha^\sharp)$ implies that the inclusion $(X,\alpha) \longrightarrow (Y,\mu)$ is an initial **B**-morphism. Further, let $f,\bar{f}$ be as described in (2). Then $f^W = q\circ\bar{f}$, so the fact that $\bar{f}$ is a **B**-morphism is, again, a consequence of the initiality of $q : (Y,\mu) \longrightarrow (X^\sharp,\alpha^\sharp)$.

(2) $\Rightarrow$ (3): Suppose $\mu' \in \mathbf{B}(X)$ such that (X,α) is a subspace of (Y,μ'). Now take f in condition (2) to be the identity $1 : (X,\alpha) \longrightarrow (X,\alpha)$. Then $1 : (Y,\mu') \longrightarrow (Y,\mu)$ is a morphism.

(3) $\Rightarrow$ (1): This follows from 3.2. ∎

Obviously, a necessary (but not sufficient) condition for (Y,μ) with $Y \supset X$ to be a many-point extension of (X,α) is that $\mu|(Y-X)$ is indiscrete. The next lemma collects together further basic properties of the many-point extension.

3.4 Lemma. *Suppose* $X \subset Y$.

(1) (cf. 2.13 of [17]) *If* $\alpha_i \in \mathbf{B}(X)$ *for each* $i \in I$ *and* $\alpha = \bigwedge_I \alpha_i$*, then* $[\alpha]_{XY} = \bigwedge_I [\alpha_i]_{XY}$.

(2) *If* $\beta \in \mathbf{B}(Y)$ *then* $\beta \leq [\beta|X]_{XY}$.

(3) (cf. 2.13 of [17]) *If* $\alpha,\beta \in \mathbf{B}(X)$ *and* $\alpha \leq \beta$*, then* $[\alpha]_{XY} \leq [\beta]_{XY}$.

Proof: (1) The equality $\alpha^\sharp = \bigwedge_I \alpha_i^\sharp$ follows from 2.13 of [17]. For each $i \in I$ it holds that

$$(Y,[\alpha]_{XY}) \xrightarrow{q} (X^\sharp,\alpha^\sharp) \xrightarrow{1} (X^\sharp,\alpha_i^\sharp) = (Y,[\alpha]_{XY}) \xrightarrow{1} (Y,[\alpha_i]_{XY}) \xrightarrow{q} (X^\sharp,\alpha_i^\sharp),$$

so $\left(1 : (Y,[\alpha]_{XY}) \longrightarrow (Y,[\alpha_i]_{XY})\right)_{i\in I}$ is initial, being the first factor of an initial source.

(2) is an immediate consequence of 3.3(1),(3).

(3) By extensionality it is obtained from $\alpha \leq \beta$ that $\alpha^\sharp \leq \beta^\sharp$. But

$$(Y,[\alpha]_{XY}) \xrightarrow{1} (Y,[\beta]_{XY}) \xrightarrow{q} (X^\sharp,\beta^\sharp) = (Y,[\alpha]_{XY}) \xrightarrow{q} (X^\sharp,\alpha^\sharp) \xrightarrow{1} (X^\sharp,\beta^\sharp),$$

and the inequality $[\alpha]_{XY} \leq [\beta]_{XY}$ is then a consequence of initiality. ∎

The following proposition shows that the **B**-objects (X,α) which are domains of initial sources with all codomains of the type $(Y^\sharp,\beta^\sharp)$ can be described as being domains of initial sources of the more restricted form $\left(q_i : (X,\alpha) \longrightarrow (X_i^\sharp,(\alpha|X_i)^\sharp)\right)_{i\in I}$ with $X_i \subset X$ for all $i \in I$. This result and Proposition 3.6 constitute the crucial ingredients in the proof of the Characterization Theorem 3.7.

3.5 Proposition. *If $(f_i : (X,\alpha) \longrightarrow (Y_i^{!}, \beta_i^{!}))_{i\in I}$ is initial, then $(q_i : (X,\alpha) \longrightarrow (X_i^{!}, (\alpha|X_i)^{!}))_{i\in I}$ is initial, where $X_i = f_i^{-1}(Y_i)$ and q_i is defined as in 3.2 for each $i \in I$.*

Proof: Let $S = (f_i : (X,\alpha) \longrightarrow (Y_i^{!}, \beta_i^{!}))_{i\in I}$ be initial. Proposition 3.2 will be applied. Assume that $\beta \in \mathbf{B}(X)$ is such that each $(X_i, \alpha|X_i)$ is a subspace of (X,β). From the initiality of the inclusions $(Y_i,\beta_i) \longrightarrow (Y_i^{!}, \beta_i^{!})$ it follows that the restrictions $f_i|X_i : (X_i, \alpha|X_i) \longrightarrow (Y_i, \beta_i)$ are morphisms. Extensionality arguments then yield the conclusion that each $f_i : (X,\beta) \longrightarrow (Y_i^{!}, \beta_i^{!})$ is a morphism. Since S is initial, we obtain $\beta \leq \alpha$. ∎

3.6 Proposition. *Suppose that $(X,\alpha) \in \mathbf{B}$, and $(X_i)_{i\in I}$ is a family of subsets of X. If $\alpha = \bigwedge_I [\beta_i]_{X_i X}$, where all $\beta_i \in \mathbf{A}(X_i)$, then $\alpha = \bigwedge_I [\widetilde{\alpha|X_i}]_{X_i X}$.*

Proof: Let $i \in I$. Since $\alpha \leq [\beta_i]_{X_i X}$ we have $\alpha|X_i \leq [\beta_i]_{X_i X}|X_i = \beta_i$. Then $\alpha|X_i \leq \widetilde{\alpha|X_i} \leq \beta_i$ because $\beta_i \in \mathbf{A}(X)$. The chain of inequalities

$$\alpha \leq \bigwedge_I [\alpha|X_i]_{X_i X} \leq \bigwedge_I [\widetilde{\alpha|X_i}]_{X_i X} \leq \bigwedge_I [\beta_i]_{X_i X} = \alpha$$

now results from application of Lemma 3.4(2),(3). ∎

3.7 Theorem (Characterization of the objects of the extensional topological hull). *If $\mathbf{A}$ is a finally dense subcategory of an extensional topological category $\mathbf{B}$, then for $(X,\alpha) \in \mathbf{B}$, the following are equivalent:*

(1) *(X,α) belongs to the extensional topological hull of $\mathbf{A}$.*

(2) *There exists a family $(X_i)_{i\in I}$ of subsets of X such that $\alpha = \bigwedge_I [\widetilde{\alpha|X_i}]_{X_i X}$.*

(3) *$\alpha = \bigwedge_{Z\subset X} [\widetilde{\alpha|Z}]_{ZX}$.*

Moreover, the equality sign in (2) and (3) can be replaced by $\geq$.

Proof: (1) ⇒ (2): Assume that $(f_i : (X,\alpha) \longrightarrow (Y_i^{!}, \beta_i^{!}))_{i\in I}$ is initial in $\mathbf{B}$, where each $(Y_i, \beta_i) \in \mathbf{A}$. For each $i \in I$ take α_i to be initial with respect to $f_i : X \longrightarrow (Y_i^{!}, \beta_i^{!})$. Then $\alpha = \bigwedge_I \alpha_i$. With $X_i = f_i^{-1}(Y_i)$, it follows from Proposition 3.5 that every $q_i : (X,\alpha_i) \longrightarrow (X_i^{!}, (\alpha_i|X_i)^{!})$ is initial. Hence $\alpha_i = [\alpha_i|X_i]_{X_i X}$, and $\alpha = \bigwedge_I [\alpha_i|X_i]_{X_i X}$. Initiality of $f_i : (X,\alpha_i) \longrightarrow (Y_i^{!}, \beta_i^{!})$ yields initiality of the domain-range restriction $(X_i, \alpha_i|X_i) \longrightarrow (Y_i, \beta_i)$ for each $i \in I$. Thus $(Y_i, \beta_i) \in \mathbf{A}$ implies $(X_i, \alpha_i|X_i) \in \mathbf{A}$, and application of Proposition 3.6 completes the proof.

(2) ⇒ (1): Condition (2) means that the source

$$S = (1 : (X,\alpha) \longrightarrow (X, [\widetilde{\alpha|X_i}]_{X_i X}))_{i\in I}$$

is initial. By composing each member of S with the initial morphism given by the definition of the many-point extension, one obtains an initial source with domain (X,α) and all codomains in $\mathbf{A}^{!}$.

The fact that conditions (2) and (3) above are equivalent to the respective modifications where the equality sign is replaced by $\geq$ follows immediately from 3.4(2),(3). The remaining implication (2) ⇒ (3) is then trivial. ∎

4. Some applications

We now apply the Characterization Theorem from Section 3 to construct two counterexamples. For the convenience of the reader, we collect in 4.1 the definitions of the categories we will be considering; more information and further references can be found in [16], for example.

$\mathbb{F}(X)$ will denote the set of all filters on a given set X; note that the power set $\mathcal{P}(X)$ is also considered to be a filter. Lattice operations on $\mathbb{F}(X)$ are understood to be with respect to the ordering by set inclusion. The filter generated by a non-empty family $\mathcal{A} \subset \mathbb{F}(X)$ is written $[\mathcal{A}]$, with the usual abbreviations $[A]$ if $\mathcal{A} = \{A\}$ for some $A \subset X$ and $\dot{x}$ if $\mathcal{A} = \{\{x\}\}$ for some $x \in X$. Given a map $f : X \longrightarrow Y$ and $\mathcal{F} \in \mathbb{F}(X)$, the filter generated by $\{ f(F) \mid F \in \mathcal{F} \}$ is denoted by $f(\mathcal{F})$.

4.1 Definition. A *semiuniform limit space* [18] is a set X equipped with a non-empty family $\alpha \subset \mathbb{F}(X \times X)$ subject to the following conditions:
(1) If $\mathcal{G} \in \mathbb{F}(X \times X)$ with $\mathcal{G} \supset \mathcal{F}$ for some $\mathcal{F} \in \alpha$, then $\mathcal{G} \in \alpha$.
(2) $\mathcal{F}, \mathcal{G} \in \alpha$ implies $\mathcal{F} \cap \mathcal{G} \in \alpha$.
(3) $\dot{x} \times \dot{x} \in \alpha$ for all $x \in X$.
(4) $\mathcal{F}^{-1} \in \alpha$ whenever $\mathcal{F} \in \alpha$ (where $\mathcal{F}^{-1} = \{ F^{-1} \mid F \in \mathcal{F} \}$ and $F^{-1} = \{ (y,x) \mid (x,y) \in F \}$).
A map $f : (X,\alpha) \longrightarrow (Y,\beta)$ between two semiuniform limit spaces is *uniformly continuous* if $\mathcal{F} \in \alpha$ implies $(f \times f)(\mathcal{F}) \in \beta$. The category of semiuniform limit spaces and uniformly continuous maps is denoted by **SULim**.

$(X,\alpha) \in$ **SULim** is called a *uniform limit space* [18] if α satisfies, in addition, the following condition:
(5) $\mathcal{F} \circ \mathcal{G} \in \alpha$ whenever $\mathcal{F}, \mathcal{G} \in \alpha$ (where $\mathcal{F} \circ \mathcal{G} = [\{ F \circ G \mid F, G \in \alpha \}]$ and $F \circ G = \{ (x,y) \mid$ there is a $z \in X$ with $(x,z) \in F$ and $(z,y) \in G \}$).
The resulting full subcategory of **SULim** is denoted by **ULim**.

A *filter space* (or *filter-merotopic space*, [13]) is a set X, together with a non-empty family $\alpha \subset \mathbb{F}(X)$ which contains all principal ultrafilters on X and is closed under finer filters. If the structure α is induced by a semiuniform limit structure on X, i.e., if α satisfies
(6) $\mathcal{G} \in \alpha$ whenever there are $\mathcal{F}_1, \ldots, \mathcal{F}_n \in \alpha$ such that $(\mathcal{F}_1 \times \mathcal{F}_1) \cap \ldots \cap (\mathcal{F}_n \times \mathcal{F}_n) \subset \mathcal{G} \times \mathcal{G}$,
then (X,α) is called a *semi-Cauchy space* [16]. A *Cauchy space* [14] is a filter space (X,α) such that
(7) $\mathcal{F} \cap \mathcal{G} \in \alpha$ whenever $\mathcal{F}, \mathcal{G} \in \alpha$ with $\mathcal{F} \vee \mathcal{G} \neq \mathcal{P}(X)$.
A map $f : (X,\alpha) \longrightarrow (Y,\beta)$ between (semi-)Cauchy spaces is *continuous* if $\mathcal{F} \in \alpha$ implies $f(\mathcal{F}) \in \beta$; the resulting categories are denoted by **SChy** and **Chy**, respectively.

It is known from [16] that the Cauchy spaces form a finally dense (hence bireflective) subcategory of the topological universe **SChy**, i.e., $\mathbf{A} = \mathbf{Chy}$ and $\mathbf{B} = \mathbf{SChy}$ fulfill the assumptions of Section 3. An application of Theorem 3.7 to this situation yields the description of **ETHChy** as given in 3.6 of [8]. Since the topological universe hull (TUH) of a category can be obtained in two steps, by first forming the extensional topological hull of the given category, and then the cartesian closed topological hull of the resulting category [6], the following result gives a partial answer to the question [16] whether **SChy** = **TUHChy**.

4.2 Theorem. *The extensional topological hull of* **Chy** *is strictly contained in* **SChy**.

Proof: Put $X = \mathbb{R}$ (the reals) and

$$\alpha = \left\{ \mathcal{F} \in \mathbb{F}(\mathbb{R}) \mid \mathcal{F} \supset [C] \text{ for some open interval C of length } 1 \right\}.$$

Then (X, α) is a semi-Cauchy space which is not in **ETHChy**.

To prove the latter fact, we consider the filter $\mathcal{H} = [A]$ generated by the half-open interval $A =]\,0,1]$, and show that $\mathcal{H} \in [\widetilde{\alpha|Z}]_{ZX}$ for every $Z \subset \mathbb{R}$. Since, obviously, $\mathcal{H} \notin \alpha$, application of 3.7 then completes the proof.

It is well-known that $\alpha|Z = \left\{ \mathcal{F}|Z \mid \mathcal{F} \in \alpha \right\}$ and

$$[\widetilde{\alpha|Z}]_{ZX} = \left\{ \mathcal{F} \in \mathbb{F}(\mathbb{R}) \mid \mathcal{F} \supset [\mathcal{G}] \cap [\mathbb{R} - Z] \text{ for some } \mathcal{G} \in \widetilde{\alpha|Z} \right\}.$$

We distinguish the following cases:

Case 1: $1 \notin Z$. With $C =]\,0,1\,[$, we have $[C] \in \alpha$, and consequently, $\mathcal{G} = [C]|Z \in \alpha|Z \subset \widetilde{\alpha|Z}$. Since $\mathcal{H} \supset [\mathcal{G}] \cap [\mathbb{R} - Z]$, it follows that $\mathcal{H} \in [\widetilde{\alpha|Z}]_{ZX}$.

Case 2: $1 \in Z$. If $Z \subset \mathbb{R} -]\,0,1\,[$, we have $\mathcal{H} \supset \dot{1} \cap [\mathbb{R} - Z] \in [\widetilde{\alpha|Z}]_{ZX}$. If not, there is a $z \in]\,0,1\,[\cap Z$. Choose some x with $0 < x < z$. Putting $A =]\,x, x+1\,[$ and $B =]\,0,1\,[$, we obtain $z \in A \cap B \cap Z \neq \emptyset$, and consequently, $[A]|Z \vee [B]|Z \neq \mathcal{P}(Z)$. It follows that $\mathcal{G} = [A]|Z \cap [B]|Z \in \widetilde{\alpha|Z}$. Finally, observe that $\mathcal{H} \supset [\mathcal{G}] \cap [\mathbb{R} - Z] \in [\widetilde{\alpha|Z}]_{ZX}$. ∎

Adámek and Reiterman recently gave a description of the topological universe hull of the category **Unif** of uniform spaces [4], [5]. In the course of their investigations, the natural conjecture arose that the category of semiuniform spaces, which is an extensional topological category containing **Unif** as a finally dense, bireflective subcategory, might constitute the extensional topological hull of **Unif**. This conjecture was disproved by Behling [9]. The semiuniform space given by Behling as a counterexample can also be used to distinguish between **ETHULim** and the category **SULim** (which is a topological universe containing **ULim** as a bireflective subcategory). As in 4.2, our proof is an application of 3.7.

4.3 Theorem. *The extensional topological hull of* **ULim** *is strictly contained in* **SULim**.

Proof: If it were true that **SULim** = **ETHULim**, then every semiuniform limit space (X, α) would have a representation according to 3.7. Now put $X = \mathbb{R}$, $N = \left\{ (x,y) \in \mathbb{R} \times \mathbb{R} \mid |x-y| < 1 \right\}$, $\mathcal{N} = [N]$ and $\alpha = \left\{ \mathcal{F} \in \mathbb{F}(\mathbb{R} \times \mathbb{R}) \mid \mathcal{F} \supset \mathcal{N} \right\}$.

Then (X,α) is a semiuniform limit space. We will show that $\bigcap\{\,[\widetilde{\alpha|Z}]_{ZX} \mid Z \subset \mathbb{R}\,\} \not\subset \alpha$. This means we have to find a filter $\mathcal{F} \in \mathbb{F}(\mathbb{R}\times\mathbb{R})$ such that $\mathcal{F} \notin \alpha$, but $\mathcal{F} \in [\widetilde{\alpha|Z}]_{ZX}$ for every $Z \subset \mathbb{R}$. We define

$$\mathcal{F} = \{\, F \subset \mathbb{R}\times\mathbb{R} \mid F \supset N \cup P_E \text{ for some finite } E \subset \mathbb{R}\,\},$$

where $P_E = \{\,(x, x-1) \mid x \in]1,2[-E\,\}$. Obviously, $N \notin \mathcal{F}$ and consequently, $\mathcal{F} \notin \alpha$.

Now consider $Z \subset \mathbb{R}$. We have:

$$\alpha|Z = \{\, \mathcal{G} \in \mathbb{F}(Z\times Z) \mid \mathcal{G} \supset [N_Z]\,\},$$

where $N_Z = N \cap (Z\times Z)$,

$$\widetilde{\alpha|Z} = \{\, \mathcal{G} \in \mathbb{F}(Z\times Z) \mid \mathcal{G} \supset [{N_Z}^k] \text{ for some } k \geq 1\,\},$$

where ${N_Z}^k = N_Z \circ \ldots \circ N_Z$ (k times), and

$$[\widetilde{\alpha|Z}]_{ZX} = \{\, \mathcal{G} \in \mathbb{F}(\mathbb{R}\times\mathbb{R}) \mid \mathcal{G} \supset \big[{N_Z}^k \cup ((\mathbb{R}\times\mathbb{R}) - (Z\times Z))\big] \text{ for some } k \geq 1\,\}.$$

As an abbreviation, put $M_{Z,k} = {N_Z}^k \cup \big((\mathbb{R}\times\mathbb{R}) - (Z\times Z)\big)$. To prove $\mathcal{F} \in [\widetilde{\alpha|Z}]_{ZX}$, it is now sufficient to show that there is a natural number $k \geq 1$ with $M_{Z,k} \in \mathcal{F}$, i.e., with $P_E \subset M_{Z,k}$ for some finite $E \subset \mathbb{R}$. This is done by classifying Z according to its large gaps intersecting $]1,2[$.

We call a non-empty subset $A \subset \mathbb{R}$ a *gap of Z* if

$$A = \bigcup\{\, C \mid C \text{ is an interval of } \mathbb{R} \text{ with } x \in C \text{ and } C \cap Z = \emptyset\,\}$$

for some (and hence for all) $x \in A$. (We do not require an interval to be bounded; i.e., the intervals are exactly the convex subsets of $\mathbb{R}$.) Obviously, every gap A of Z is an interval with $A \cap Z = \emptyset$. A gap A is said to be *large* iff its length is at least 1, i.e. if A is unbounded or $\bigvee A - \bigwedge A \geq 1$.

Now the following cases can occur:

Case 1: $A \cap]1,2[= \emptyset$ for every large gap A of Z. Considering $(x, x-1)$ with $x \in]1,2[$, we have $(x, x-1) \in (\mathbb{R}\times\mathbb{R}) - (Z\times Z)$ whenever $x \notin Z$ or $x - 1 \notin Z$. If both x and $x-1$ belong to Z, then $Z \cap]x-1, x[$ cannot be empty, because otherwise, there would be a large gap A of Z with $]x-1,x[\subset A$, whence $A \cap]1,2[\neq \emptyset$, contradicting the assumption. Take $z \in Z$ with $x-1 < z < x$. Then $(x,z), (z, x-1) \in N_Z$, and $(x, x-1) \in N_Z \circ N_Z$. It follows that $P_\emptyset \subset M_{Z,2}$.

Case 2: There is a large gap A of Z with $]1,2[\subset A$. Then $P_\emptyset \subset (\mathbb{R}-Z)\times\mathbb{R} \subset M_{Z,1}$.

Case 3: There is a large gap A of Z with $A \cap]1,2[\neq \emptyset$ and $\bigvee A < 2$. Put $s = \bigvee A$ and $r = \bigwedge A$ (if $\bigwedge A$ exists; otherwise, put $r = 0$). Then $r < 1 < s < 2$.

Now consider $(x, x-1)$ with $x \in]1,2[-\{s\}$. If $x < s$, then $x \in A$, and $(x, x-1) \in (\mathbb{R} - Z) \times \mathbb{R}$. If $x > s$, then $r \leq s-1 < x-1 < 1 < s$ implies $x-1 \in A$, and $(x, x-1) \in \mathbb{R} \times (\mathbb{R} - Z)$. Hence $P_{\{s\}} \subset M_{Z,1}$.

Case 4: There is a large gap A of Z with $A \cap]1,2[\neq \emptyset$ and $\bigwedge A > 1$, but $B \cap]1,2[= \emptyset$ for any large gap $B \neq A$ of Z. Then $P_{\emptyset} \subset M_{Z,2}$ can be shown like in Case 1. ∎

Since **Unif** is a bireflective subcategory of **ULim**, and the filter $\mathcal{N}$ in the proof of 4.3 is a semiuniformity on $\mathbb{R}$, Behling's result is now obtained as an immediate consequence.

4.4 Corollary [9]. *The extensional topological hull of the category of uniform spaces is strictly contained in the category of semiuniform spaces.*

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