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INNER ACTIONS AND GALOIS H-OBJECTS IN A SYMMETRIC CLOSED CATEGORY

by J.N. Alonso ALVAREZ and J.M. Fernandez VILABOA

Résumé. Le but de cet article est de montrer que pour une algèbre de Hopf **H** dans une catégorie fermée symétrique $\mathscr{C}$, un **H**–module triple d' Azumaya $(\mathbf{A}, \varphi_{\mathbf{A}})$ a une action intérieure si et seulement si le **H**–object de Galois $(\mathbf{A}\#\mathbf{H})^{\mathbf{A}} = \Pi(\mathbf{A})$ est à base normale. En utilisant ce résultat nous montrons que le groupe de Brauer des **H**–modules triples d' Azumaya avec action intérieure est isomorphe à $\mathbf{B}(\mathscr{C}) \oplus \mathbf{N}_{\mathscr{C}}(\mathbf{H})$, où $\mathbf{N}_{\mathscr{C}}(\mathbf{H})$ est le groupe des classes d' isomorphismes des **H**–objects de Galois avec base normale et $\mathbf{B}(\mathscr{C})$ est le groupe de Brauer des triples d' Azumaya dans $\mathscr{C}$ défini par J. M. Fernández–Vilaboa dans [4].
En particulier, si R est un anneau commutatif avec unité, H une algèbre de Hopf projective de type fini, commutative et cocommutative sur R, alors toute extension H–Galois de R est à base normale si et seulement si toute H–module algèbre R–Azumaya a une H–action intérieure.

0. Introduction

In [4], for a Hopf algebra **H** in a symmetric closed category $\mathscr{C}$, Fernández Vilaboa defines the Brauer group of **H**–module Azumaya triples, $\mathbf{BM}(\mathscr{C},\mathbf{H})$, and the group of isomorphism classes of Galois **H**–objects, $\mathbf{Gal}_{\mathscr{C}}(\mathbf{H})$, and obtains the isomorphism: $\mathbf{BM}(\mathscr{C},\mathbf{H}) \cong \mathbf{B}(\mathscr{C}) \oplus \mathbf{Gal}_{\mathscr{C}}(\mathbf{H})$, where $\mathbf{B}(\mathscr{C})$ is the Brauer group of the symmetric closed category $\mathscr{C}$.

In this paper, we prove that the set $\mathbf{N}_{\mathscr{C}}(\mathbf{H})$, of isomorphism classes of Galois **H**–objects isomorphic to **H** as **H**–comodule is a subgroup of $\mathbf{Gal}_{\mathscr{C}}(\mathbf{H})$, and the set $\mathbf{BM}_{\mathrm{inn}}(\mathscr{C},\mathbf{H})$ of equivalence classes of **H**–module Azumaya triples with inner action is a subgroup of $\mathbf{BM}(\mathscr{C},\mathbf{H})$. Finally, we obtain the decomposition theorem: $\mathbf{BM}_{\mathrm{inn}}(\mathscr{C},\mathbf{H}) \cong \mathbf{B}(\mathscr{C}) \oplus \mathbf{N}_{\mathscr{C}}(\mathbf{H})$.

Section 1 contains a review of some background material that we will used freely in the sequel.

In the next section we describe the group of isomorphism classes of Galois **H**–objects with a normal basis.

We continue in section 3 giving a criterion for the action of an **H**–module Azumaya triple to be inner.

Finally, in section 4, we introduce $\mathbf{BM}_{inn}(\mathscr{C}, \mathbf{H})$ and we prove that it is isomorphic to $\mathbf{B}(\mathscr{C}) \oplus \mathrm{N}_{\mathscr{C}}(\mathbf{H})$.

1. Preliminary

A monoidal category $(\mathscr{C}, \otimes, K)$ consists of a category $\mathscr{C}$ with a bifunctor $-\otimes-: \mathscr{C} \times \mathscr{C} \longrightarrow \mathscr{C}$ and a basic object K, and with natural isomorphisms:

$a_{ABC}: A\otimes(B\otimes C) \cong (A\otimes B)\otimes C$

$l_A: K\otimes A \cong A$

$r_A: A\otimes K \cong A$

such that

$(A\otimes a_{BCD}) \circ a_{A(B\otimes C)D} \circ (a_{ABC}\otimes D) = a_{AB(C\otimes D)} \circ a_{(A\otimes B)CD}$

$(A\otimes l_B) \circ a_{AKB} = r_A \otimes B$

If there is a natural isomorphism $\tau_B^A : A\otimes B \cong B\otimes A$ such that $\tau_A^B \circ \tau_B^A = A\otimes B$, $\tau_{B\otimes C}^A = (B\otimes \tau_C^A) \circ (\tau_B^A \otimes C)$, then $\mathscr{C}$ is called a symmetric monoidal category.

A closed category is a symmetric monoidal category in wich each functor $-\otimes A: \mathscr{C} \longrightarrow \mathscr{C}$ has a specified right adjoint $[A,-]: \mathscr{C} \longrightarrow \mathscr{C}$ ([3], [5])

In what follows, $\mathscr{C}$ denotes a symmetric closed category with equalizers, co–equalizers and projective basic object K. We denote by α_M and β_M the unit and the co–unit, respectively, of the $\mathscr{C}$ -adjuntion $M\otimes - \dashv [M, -]: \mathscr{C} \longrightarrow \mathscr{C}$ wich exists for each object M of $\mathscr{C}$.

1.1 An object M of $\mathscr{C}$ is called profinite in $\mathscr{C}$ if the morphism $[M, \beta_M(K)\otimes M] \circ \alpha_M(\hat{M}\otimes M): \hat{M}\otimes M \longrightarrow [M,M] = E(M)$ is an isomorphism, where $\hat{M} = [M,K]$. If, moreover, the factorization of $\beta_M(K): M\otimes \hat{M} \longrightarrow K$ through the co–equalizer of the morphisms $\beta_M(M)\otimes \hat{M}$ and $M\otimes([M, \beta_M(K) \circ (\beta_M(M)\otimes \hat{M})] \circ \alpha_M(E(M)\otimes \hat{M})) : M\otimes E(M)\otimes \hat{M} \longrightarrow M\otimes \hat{M}$ is an isomorphism, we say that M is a progenerator in $\mathscr{C}$.

1.2 A triple $\mathbf{A} = (A, \eta_A, \mu_A)$ is an object A in $\mathscr{C}$ together with two morphisms $\eta_A: K \longrightarrow A$, $\mu_A: A\otimes A \longrightarrow A$, such that $\mu_A \circ (A\otimes \eta_A) = A = \mu_A \circ (\eta_A \otimes A)$, $\mu_A \circ (\mu_A \otimes A) = \mu_A \circ (A\otimes \mu_A)$. If $\mu_A \circ \tau_A^A = \mu_A$, then we will say that $\mathbf{A}$ is a commutative triple. Given two triples $\mathbf{A} = (A, \eta_A, \mu_A)$ and $\mathbf{B} = (B, \eta_B, \mu_B)$ in $\mathscr{C}$,

$f:A \longrightarrow B$ is a triple morphism if $\mu_B \circ (f\otimes f) = f \circ \mu_A$ and $f \circ \eta_A = \eta_B$.

A cotriple (cocommutative), $\mathbf{D} = (D, \varepsilon_D, \delta_D)$ is an object D in $\mathscr{C}$ together with two morphisms $\varepsilon_D:D \longrightarrow K$, $\delta_D:D \longrightarrow D\otimes D$, such that $(\delta_D\otimes D) \circ \delta_D = (D\otimes\delta_D) \circ \circ \delta_D$ and $(\varepsilon_D\otimes D) \circ \delta_D = 1_D = (D\otimes\varepsilon_D) \circ \delta_D$ ($\tau_D^D \circ \delta_D = \delta_D$). If $\mathbf{D} = (D, \varepsilon_D, \delta_D)$ and $\mathbf{E} = (E, \varepsilon_E, \delta_E)$ are cotriples, $f:D \longrightarrow E$ is a cotriple morphism if $(f\otimes f) \circ \circ \delta_D = \delta_E \circ f$ and $\varepsilon_E \circ f = \varepsilon_D$.

1.3 For a triple $\mathbf{A} = (A, \eta_A, \mu_A)$ and a cotriple $\mathbf{D} = (D, \varepsilon_D, \delta_D)$ in $\mathscr{C}$, we denote by Reg(D,A) the group of invertible elements in $\mathscr{C}$ (D,A) (morphisms in $\mathscr{C}$ from D to A) with the operation " convolution " given by: $f \wedge g = \mu_A \circ (f\otimes g) \circ \delta_D$. The unit element is $\varepsilon_D\otimes\eta_A$.

Observe that Reg(D,A) is an abelian group when **D** is cocommutative and **A** is commutative.

1.4 Definition Let $\Pi=(C, \eta_C, \mu_C)$ be a triple and $C=(C, \varepsilon_C, \delta_C)$ a cotriple in $\mathscr{C}$ and let $\lambda:C \longrightarrow C$ be a morphism. Then $\mathbf{H} = (C=(C, \varepsilon_C, \delta_C), \Pi=(C, \eta_C, \mu_C), \tau^C, \lambda)$ is a Hopf algebra in $\mathscr{C}$ with respect to the cotriple C if ε_C and δ_C are triple morphisms (equivalently, η_C and μ_C are cotriple morphisms) and λ is the inverse of $1_C:C \longrightarrow C$ in Reg(C,C). The Hopf algebra is commutative (cocommutative) if Π (C) is.

We say that **H** is a finite Hopf algebra if C is profinite in $\mathscr{C}$. In this case, $\widehat{\mathbf{H}} = (\widehat{\Pi}, \widehat{C}, \widehat{\tau^C}, \widehat{\lambda})$ will denote the dual Hopf algebra of **H**. $\widehat{\Pi} = (\widehat{C}, \varepsilon_{\widehat{C}}, \delta_{\widehat{C}})$ stands for $\big([C,K], \varepsilon_{\widehat{C}} = \bar{b}_C \circ (\eta_C\otimes\widehat{C}), \delta_{\widehat{C}} = ((\widehat{C}\otimes\widehat{C})\otimes(\bar{b}_C \circ ((\mu_C \circ \tau_C^C) \otimes\widehat{C}))) \circ \circ (\widehat{C}\otimes\bar{a}_C \otimes C\otimes\widehat{C}) \circ (\bar{a}_C\otimes\widehat{C})\big)$ a cotriple in $\mathscr{C}$; in a similar manner, $\widehat{C} = (\widehat{C}, \eta_{\widehat{C}}, \mu_{\widehat{C}})$ denotes the triple structure $\big([C,K], \eta_{\widehat{C}} = (\widehat{C}\otimes\varepsilon_C) \circ \bar{a}_C , \mu_{\widehat{C}} = (\widehat{C}\otimes\bar{b}_C) \circ \circ (\widehat{C}\otimes C\otimes\bar{b}_C\otimes\widehat{C}) \circ (\widehat{C}\otimes(\tau_C^C \circ \delta_C)\otimes\widehat{C}\otimes\widehat{C}) \circ (\bar{a}_C\otimes\widehat{C}\otimes\widehat{C})\big)$; and $\widehat{\lambda}$, the antipode, is $(\widehat{C}\otimes\bar{b}_C) \circ (\widehat{C}\otimes\lambda\otimes\widehat{C}) \circ (\bar{a}_C\otimes\widehat{C})$, where $\bar{a}_C$ and $\bar{b}_C$ represent the unit and co–unit of the $\mathscr{C}$-adjuntion $C\otimes- \,\big|\, \widehat{C}\otimes-:\mathscr{C} \longrightarrow \mathscr{C}$.

To prove the final an main theorem of this work, we will need the Hopf algebra to be finite, commutative and cocommutative. For this reason, although for some of the results used in the proof of the main theorem these hypothese are not needed we will assume them from now on.

2. Galois H–objects with a normal basis.

2.1 Definition $(\mathbf{B}, \rho_B) = (B, \eta_B, \mu_B ; \rho_B)$ is a right **H**–comodule triple if :

i) $\mathbf{B} = (B, \eta_B, \mu_B)$ is a triple in $\mathscr{C}$

ii) (B, ρ_B) is a right **H**–comodule $((\rho_B \otimes C) \circ \rho_B = (B \otimes \delta_C) \circ \rho_B$; $(B \otimes \varepsilon_C) \circ \rho_B = B)$

iii) $\rho_B : B \longrightarrow B \otimes C$ is a triple morphism from (B, η_B, μ_B) to the product triple $\mathbf{B}\Pi = (B \otimes C, \eta_B \otimes \eta_C, (\mu_B \otimes \mu_C) \circ (B \otimes \tau_B^C \otimes C))$ (that is, $\rho_B \circ \eta_B = \eta_B \otimes \eta_C$ and $\rho_B \otimes \mu_B = (\mu_B \otimes \mu_C) \circ (B \otimes \tau_B^C \otimes C) \circ (\rho_B \otimes \rho_B)$).

Similarly, $(\mathbf{A}, \varphi_A) = (A, \eta_A, \mu_A ; \varphi_A)$ is a left **H**–module triple if :

i) $\mathbf{A} = (A, \eta_A, \mu_A)$ is a triple in $\mathscr{C}$

ii) (A, φ_A) is a left **H**–module $(\varphi_A \circ (C \otimes \varphi_A) = \varphi_A \circ (\mu_C \otimes A)$, $\varphi_A \circ (\eta_C \otimes A) = A)$

iii) η_A, μ_A are morphisms of left **H**–modules $(\varphi_A \circ (C \otimes \mu_A) = \mu_A \circ \varphi_{A \otimes A}$ and $\varphi_A \circ (C \otimes \eta_A) = \eta_A \otimes \varepsilon_C$, where $\varphi_{A \otimes A} = (\varphi_A \otimes \varphi_A) \circ (C \otimes \tau_A^C \otimes A) \circ$ $\circ (\delta_C \otimes A \otimes A))$

We say that the action φ_A of **H** in **A** is inner if there exists a morphism f in Reg(C,A) such that $\varphi_A = \mu_A \circ (A \otimes (\mu_A \circ \tau_A^A)) \circ (f \otimes f^{-1} \otimes A) \circ$ $\circ (\delta_C \otimes A) : C \otimes A \longrightarrow A$, where f^{-1} is the convolution inverse of f.

2.2 For each Hopf algebra **H** and each **H**–module triple $(\mathbf{A}, \varphi_A)$, the triple smash product **A#H** is defined as follows:

$\mathbf{A\#H} := (A \otimes C, \eta_{A\#H}, \mu_{A\#H})$, where

$\eta_{A\#H} = \eta_A \otimes \eta_C$ and

$\mu_{A\#H} = (\mu_A \otimes \mu_C) \circ (A \otimes \varphi_A \otimes C \otimes C) \circ (A \otimes C \otimes \tau_A^C \otimes C) \circ (A \otimes \delta_C \otimes A \otimes C)$

2.3 Definition A right **H**–comodule triple $(\mathbf{B}, \rho_B)$ is said to be a Galois **H**–object if and only if :

i) The morphism $\gamma_B := (\mu_B \otimes C) \circ (B \otimes \rho_B) : B \otimes B \longrightarrow B \otimes C$ is an isomorphism

ii) B is a progenerator in $\mathscr{C}$

If a Galois **H**–object is isomorphic to **H** as an **H**–comodule then we say that it has a normal basis.

If $\mathbf{B_1}$ and $\mathbf{B_2}$ are Galois **H**–objects, then $f:B_1 \longrightarrow B_2$ is a morphism of Galois **H**–objects if it is a morphism of **H**–comodules ($\rho_{B_2} \circ f = (f\otimes C) \circ \rho_{B_1}$) and of triples ($f \circ \eta_{B_1} = \eta_{B_2}$; $f \circ \mu_{B_1} = \mu_{B_2} \circ (f \otimes f)$).

We denote by $\mathbf{Gal}_{\mathscr{C}}(\mathbf{H})$ the set of isomorphism classes of Galois **H**–objects and by $\mathbf{N}_{\mathscr{C}}(\mathbf{H})$ the set of isomorphism classes of Galois **H**–objects with a normal basis.

2.4 Proposition *If (A, ρ_A) and (B, ρ_B) are H–comodule triples, then $A \circ B$, defined by the following equalizer diagram*

$$A\circ B \xrightarrow{i_{AB}} A\otimes B \underset{\partial^2_{AB}}{\overset{\partial^1_{AB}}{\rightrightarrows}} A\otimes B\otimes C$$

where

$$\partial^1_{AB} = (A\otimes\tau^C_B) \circ (\rho_A \otimes B), \text{ and}$$

$$\partial^2_{AB} = A\otimes \rho_B$$

is an H–comodule triple to be denoted by $(A\circ B, \rho_{AB})$.

If moreover (A, ρ_A) and (B, ρ_B) are Galois H–objects, then $(A\circ B, \rho_{AB})$ is also a Galois H–object, where ρ_{AB} is the factorization of the morphism $\partial^1_{AB} \circ i_{AB}$ (or $\partial^2_{AB} \circ i_{AB}$) *through the equalizer $i_{AB}\otimes C$*

The set $\mathbf{Gal}_{\mathscr{C}}(\mathbf{H})$ *with this operation is an abelian group.The unit element is the class of the Galois* **H**–*object* (Π, δ_C), *and the opposite of $[(A, \rho_A)] \in Gal_{\mathscr{C}}(H)$ is $[(A^{op}, (A\otimes\lambda) \circ \rho_A)]$* .[4]

2.5 Proposition *$N_{\mathscr{C}}(H)$ is a subgroup of $Gal_{\mathscr{C}}(H)$*

Proof.:

Let $(\mathbf{A}, \rho_A)$ and $(\mathbf{B}, \rho_B)$ be Galois **H**–objects with a normal basis and let $f:C \longrightarrow A$, $g:C \longrightarrow B$ be isomorphisms of **H**–comodules.

The morphism $(f\otimes g) \circ \delta_C$ factors through the equalizer i_{AB} :

$$(A\otimes\tau^C_B) \circ (\rho_A\otimes B) \circ (f\otimes g) \circ \delta_C =$$

$$= (f\otimes\tau^C_B) \circ (\delta_C\otimes g) \circ \delta_C =$$

$= (f\otimes\tau_B^C) \circ (C\otimes C\otimes g) \circ (C\otimes(\tau_C^C \circ \delta_C)) \circ \delta_C =$
$= (f\otimes g\otimes C) \circ (C\otimes\delta_C) \circ \delta_C =$
$= (A\otimes\rho_B) \circ (f\otimes g) \circ \delta_C$

because **H** is a cocommutative Hopf algebra

Let $r:C \longrightarrow A^\circ B$ be the factorization of $(f\otimes g) \circ \delta_C$

The morphism $(f^{-1} \otimes g^{-1}) \circ i_{AB}$ factors through the equalizer

$$C \xrightarrow{\delta_C} C\otimes C \overset{\partial^1_{CC}}{\underset{\partial^2_{CC}}{\rightrightarrows}} C\otimes C\otimes C$$

$(C\otimes\tau_C^C) \circ (\delta_C\otimes C) \circ (f^{-1} \otimes g^{-1}) \circ i_{AB} =$
$= (f^{-1} \otimes\tau_C^C) \circ (\rho_A\otimes g^{-1}) \circ i_{AB} =$
$= (f^{-1} \otimes g^{-1}\otimes C) \circ (A\otimes\rho_B) \circ i_{AB} =$
$= (C\otimes\delta_C) \circ (f^{-1} \otimes g^{-1}) \circ i_{AB}$

Let $s:A^\circ B \longrightarrow C$ be the factorization of $(f^{-1} \otimes g^{-1}) \circ i_{AB}$

The morphisms r and s are inverse to each other. Moreover, r is a morphism of **H**–comodules:

$(i_{AB}\otimes C) \circ \rho_{AB} \circ r =$
$= (A\otimes\rho_B) \circ i_{AB} \circ r =$
$= (A\otimes\rho_B) \circ (f \otimes g) \circ \delta_C =$
$= (f \otimes g\otimes C) \circ (\delta_C\otimes C) \circ \delta_C =$
$= (i_{AB}\otimes C) \circ (r\otimes C) \circ \delta_C$

and as $i_{AB}\otimes C$ is a monomorphism, $\rho_{AB} \circ r = (r \otimes C) \circ \delta_C$

Moreover, if $f:(C, \delta_C) \longrightarrow (A, \rho_A)$ is an isomorphism of **H**–comodules, then $f \circ \lambda:(C, \delta_C) \longrightarrow (A, (A\otimes\lambda) \circ \rho_A)$ is an isomorphism of **H**–comodules.

3. H–module Azumaya triples with inner action

3.1 Definition A triple **A** $=(A, \eta_A, \mu_A)$ is said to be Azumaya if and only if :

i) A is a progenerator in $\mathscr{C}$

ii) The morphism of triples χ_A: $A\otimes A \longrightarrow [A,A]$; $\chi_A := [A, \mu_A \circ (A\otimes\mu_A) \circ$
$\circ (\tau_A^A\otimes A)] \circ \alpha_A(A\otimes A)$ is an isomorphism.

3.2 Definition If (A, ρ_A) is an **H**-module triple, we define the object $\Pi(A) := Ig(m_{A\#H}, n_{A\#H})$

$$\Pi(A) \xrightarrow{j_{A\#H}} A\otimes C \overset{m_{A\#H}}{\underset{n_{A\#H}}{\rightrightarrows}} [A, A\otimes C]$$

where

$$m_{A\#H} = [A, \mu_{A\#H} \circ (A\otimes\eta_C\otimes A\otimes C)] \circ \alpha_A(A\otimes C) =$$
$$= [A, \mu_A\otimes C] \circ \alpha_A(A\otimes C)$$
$$n_{A\#H} = [A, \mu_{A\#H} \circ (A\otimes\tau_C^A\otimes C) \circ (\tau_A^A\otimes C\otimes\eta_C)] \circ \alpha_A(A\otimes C) =$$
$$= [A, (\mu_A\otimes C) \circ (A\otimes(\varphi_A \circ \tau_C^A)\otimes C) \circ (\tau_A^A\otimes\delta_C)] \circ \alpha_A(A\otimes C)$$

3.3 Proposition *If (A, ρ_A) is an **H**–module Azumaya triple, then $\Pi(A)$ is a Galois **H**–object.*

Proof :

$(\Pi(A), \rho_{\Pi(A)}) = (\Pi(A), \eta_{\Pi(A)}, \mu_{\Pi(A)}; \rho_{\Pi(A)})$ is a Galois **H**–object, where $\eta_{\Pi(A)}$ and $\mu_{\Pi(A)}$ are the factorizations of $\eta_{A\#H}$ and $\mu_{A\#H} \circ (j_{A\#H} \otimes j_{A\#H})$ respectively, through the equalizer $j_{A\#H}$, and $\rho_{\Pi(A)}$ is the factorization of $(A\otimes\delta_C) \circ j_{A\#H}$ through the equalizer $j_{A\#H}\otimes C$.

3.4 Proposition *If (A, φ_A) is an **H**–module Azumaya triple, then φ_A is inner if and only if $\Pi(A)$ is a Galois **H**–object with a normal basis.*

Proof :

"$\Rightarrow$" Let $g := (\mu_A\otimes C) \circ (A\otimes f\otimes C) \circ (A\otimes\delta_C) : A\otimes C \longrightarrow A\otimes C$, where f is the morphism in Reg(C,A) such that $\varphi_A = \mu_A \circ (A\otimes(\mu_A \circ \tau_A^A)) \circ (f\otimes f^{-1}\otimes A) \circ (\delta_C\otimes A)$.

Since **A** is an Azumaya triple and **H** a progenerator in $\mathscr{C}$

$$C \xrightarrow{\eta_A\otimes C} A\otimes C \overset{\nu_1\otimes C}{\underset{\nu_2\otimes C}{\rightrightarrows}} [A,A]\otimes C$$

is an equalizer diagram, where

$\upsilon_1 = [A, \mu_A] \circ \alpha_A(A)$, and

$\upsilon_2 = [A, \mu_A \circ \tau_A^A] \circ \alpha_A(A)$.

$(\mu_A\otimes C)\circ(A\otimes g)\circ(A\otimes j_{A\#H})=$

$=(\mu_A\otimes C)\circ(\mu_A\otimes f\otimes C)\circ(A\otimes\varphi_A\otimes\delta_C)\circ(A\otimes\tau_C^A\otimes C)\circ(\tau_A^A\otimes\delta_C)\circ$
$\circ(A\otimes j_{A\#H})=$

$=(\mu_A\otimes C)\circ(A\otimes\mu_A\otimes C)\circ(A\otimes\mu_A\otimes f\otimes C)\circ(A\otimes A\otimes(\mu_A\circ\tau_A^A)\otimes\delta_C)\circ$
$\circ(A\otimes f\otimes f^{-1}\otimes A\otimes C)\circ(A\otimes\delta_C\otimes A\otimes C)\circ(A\otimes\tau_C^A\otimes C)\circ(\tau_A^A\otimes\delta_C)\circ$
$\circ(A\otimes j_{A\#H})=$

$=(\mu_A\otimes C)\circ(A\otimes\mu_A\otimes C)\circ(A\otimes(\mu_A\circ\tau_A^A)\otimes A\otimes C)\circ(\tau_A^A\otimes f\otimes(f^{-1}\wedge f)\otimes C)\circ$
$\circ(A\otimes A\otimes\delta_C\otimes C)\circ(A\otimes A\otimes\delta_C)\circ(A\otimes j_{A\#H})=$

$=(\mu_A\otimes C)\circ(A\otimes\mu_A\otimes C)\circ(A\otimes f\otimes A\otimes C)\circ(A\otimes\tau_C^A\otimes C)\circ(\tau_A^A\otimes\delta_C)\circ$
$\circ(A\otimes j_{A\#H})=$

$=((\mu_A\circ\tau_A^A)\otimes C)\circ(A\otimes g)\circ(A\otimes j_{A\#H})$

and then, there exists a morphism $r:\Pi(A)\longrightarrow C$ such that $(\eta_A\otimes C)\circ r=g\circ j_{A\#H}$.

Moreover, the morphism $(f^{-1}\otimes C)\circ\delta_C$ factors through the equalizer $j_{A\#H}$. Indeed :

$(\mu_A\otimes C)\circ(A\otimes\varphi_A\otimes C)\circ(A\otimes\tau_C^A\otimes C)\circ(\tau_A^A\otimes\delta_C)\circ(A\otimes f^{-1}\otimes C)\circ(A\otimes\delta_C)=$

$=(\mu_A\otimes C)\circ(A\otimes\mu_A\otimes C)\circ(A\otimes A\otimes(\mu_A\circ\tau_A^A)\otimes C)\circ(f^{-1}\otimes f\otimes f^{-1}\otimes$
$\otimes A\otimes C)\circ(C\otimes\delta_C\otimes A\otimes C)\circ(C\otimes\tau_C^A\otimes C)\circ(\tau_C^A\otimes\delta_C)\circ(A\otimes\delta_C)=$

$=(\mu_A\otimes C)\circ(A\otimes(\mu_A\circ\tau_A^A)\otimes C)\circ((f^{-1}\wedge f)\otimes f^{-1}\otimes A\otimes C)\circ(\delta_C\otimes A\otimes C)\circ$
$\circ(\tau_C^A\otimes C)\circ(A\otimes\delta_C)=$

$=(\mu_A\otimes C)\circ(A\otimes f^{-1}\otimes C)\circ(A\otimes\delta_C)$

Then, there is a morphism $h:C\longrightarrow\Pi(A)$ satisfying $j_{A\#H}\circ h=(f^{-1}\otimes C)\circ\delta_C$

Trivially, the morphism r is of **H**–comodules, with inverse h and thus $\Pi(A)$ is a Galois **H**–object with a normal basis.

"$\Leftarrow$" There is an **H**–comodule isomorphism $r:C\longrightarrow\Pi(A)$.

Moreover, since A is an Azumaya triple, the morphism $g=(\mu_A\otimes C)\circ$ $\circ(A\otimes j_{A\#H}):A\otimes\Pi(A)\longrightarrow A\otimes C$ is an isomorphism.

The morphism $v:=(A\otimes\varepsilon_C)\circ(A\otimes r^{-1})\circ g^{-1}\circ(\eta_A\otimes C)\in Reg(C,A)$ with inverse $u=(A\otimes\varepsilon_C)\circ j_{A\#H}\circ r$. Indeed :

$v\wedge u=(\mu_A\otimes\varepsilon_C)\circ(A\otimes\varepsilon_C\otimes j_{A\#H})\circ(A\otimes r^{-1}\otimes r)\circ(A\otimes\rho_{\Pi(A)})\circ g^{-1}\circ(\eta_A\otimes C)=$

$=(A\otimes\varepsilon_C)\circ g\circ(A\otimes[(\varepsilon_C\otimes r)\circ\delta_C\circ r^{-1}])\circ g^{-1}\circ(\eta_A\otimes C)=$

$= \varepsilon_C \otimes \eta_A$

because r^{-1} is a morphism of **H**–comodules and the equality

$$(g^{-1} \otimes C) \circ (A \otimes \delta_C) = (A \otimes \rho_{\Pi(A)}) \circ g^{-1}$$

Similarly, by the equality $g^{-1} \circ (\mu_A \otimes C) = (\mu_A \otimes \Pi(A)) \circ (A \otimes g^{-1})$, we have that $u \wedge v = \varepsilon_C \otimes \eta_A$.

Moreover,

$$\begin{aligned}
\varphi_A &= \mu_A \circ ((v \wedge u) \otimes \varphi_A) \circ (\delta_C \otimes A) = \\
&= \mu_A \circ (\mu_A \otimes \varepsilon_C \otimes \varphi_A) \circ (A \otimes \varepsilon_C \otimes j_{A\#H} \otimes C \otimes A) \circ (A \otimes r^{-1} \otimes \rho_{\Pi(A)} \otimes A) \circ \\
&\circ (g^{-1} \otimes r \otimes A) \circ (\eta_A \otimes \delta_C \otimes A) = \\
&= \mu_A \circ (\mu_A \otimes \varphi_A) \circ [A \otimes (\varepsilon_C \circ r^{-1}) \otimes A \otimes ((\varepsilon_C \otimes C) \circ \delta_C) \otimes A] \circ (g^{-1} \otimes (j_{A\#H} \circ \\
&\circ r) \otimes A) \circ (\eta_A \otimes \delta_C \otimes A) = \\
&= (\mu_A \otimes \varepsilon_C) \circ (A \otimes \varepsilon_C \otimes [(\mu_A \otimes C) \circ (A \otimes (\varphi_A \circ \tau_C^A) \otimes C) \circ (\tau_A^A \otimes \delta_C) \circ \\
&\circ (A \otimes j_{A\#H})]) \circ (A \otimes r^{-1} \otimes A \otimes r) \circ (g^{-1} \otimes \tau_A^C) \circ (\eta_A \otimes \delta_C \otimes A) = \\
&= \mu_A \circ (A \otimes (\mu_A \circ \tau_A^A)) \circ (v \otimes u \otimes A) \circ (\delta_C \otimes A).
\end{aligned}$$

because r is a morphism of **H**–comodules and $j_{A\#H}$ an equalizer. Thus, we conclude that the action φ_A of **H** in **A** is inner.

Be aware that in the proof of the previous proposition it is not necessary that **H** be either commutative nor cocommutative.

4. Normal basis and inner actions

4.1 Definition On the set of **H**–module triple isomorphisms classes of **H**–module Azumaya triples we define the following equivalence relation :

$$(\mathbf{A}, \varphi_A) \sim (\mathbf{B}, \varphi_B) \Leftrightarrow \mathbf{A}\ E(M)^{op} \cong \mathbf{B}\ E(N)^{op}$$

for some progenerators **H**–modules (M, φ_M) and (N, φ_N).

The set of equivalence classes of **H**–module Azumaya triples forms a group under the operation induced by the tensor product, $(\mathbf{AB}, \varphi_{A \otimes B} = (\varphi_A \otimes \varphi_B) \circ (C \otimes \tau_A^C \otimes B) \circ$ $\circ (\delta_C \otimes A \otimes B))$. The unit element is the class of the **H**–module Azumaya triple

$$\big(E(M)^{op}, \varphi_{E(M)} = [M, \varphi_M \circ (C \otimes \beta_M(M)) \circ (\tau_C^M \otimes [M,M]) \circ (\varphi_M \otimes C \otimes [M,M]) \circ \circ (\tau_C^M \otimes C \otimes [M,M]) \circ (M \otimes \tau_C^C \otimes [M,M]) \circ (M \otimes C \otimes \lambda \otimes [M,M]) \circ (M \otimes \delta_C \otimes [M,M]] \circ \circ \alpha_M(C \otimes [M,M])\big)$$

for some progenerator H–module (M, φ_M), and the opposite of (A, φ_A) is (A^{op}, φ_A). This group is denoted by **BM**$(\mathscr{C}, \mathbf{H})$.

If $\mathbf{1} = (1, 1, \tau^K, 1)$ is the trivial Hopf algebra in $\mathscr{C}$, then we define the Brauer group of Azumaya triples in $\mathscr{C}$ as **BM**$(\mathscr{C}, \mathbf{1})$ and we will denote it by **B**$(\mathscr{C})$.

4.2 Proposition *$BM(\mathscr{C}, H) \cong B(\mathscr{C}) \oplus Gal_{\mathscr{C}}(H)$*

Proof :

There is an epimorphism of abelian groups

$$\Pi : \mathbf{BM}(\mathscr{C}, \mathbf{H}) \longrightarrow \mathbf{Gal}_{\mathscr{C}}(\mathbf{H})$$

given by $\Pi([(A, \varphi_A)]) := (\Pi(A), \rho_{\Pi(A)})$

If $[(\mathbf{B}, \rho_B)] \in \mathbf{Gal}_{\mathscr{C}}(\mathbf{H})$, then $[(\mathbf{B\#\widehat{H}}, \varphi_{B\#\widehat{H}} = (B\otimes\widehat{C}\otimes\bar{b}_C) \circ (B\otimes\tau^C_{\widehat{C}}\otimes\widehat{C}) \circ$ $\circ (\tau^C_B\otimes\delta_{\widehat{C}}))] \in \mathbf{BM}(\mathscr{C}, \mathbf{H})$ and there is an **H**–comodule triple isomorphism $\Pi(B\#\widehat{H}) \cong B$.

The sequence

$$1 \longrightarrow B(\mathscr{C}) \xrightarrow{\ \ i\ \ } BM(\mathscr{C}, H) \xrightarrow{\ \ \Pi\ \ } Gal_{\mathscr{C}}(H) \longrightarrow 1$$

is split exact, where the morphism i is given by $i[(A)] = [(A, \varepsilon_C\otimes A)]$ and the homomorphism

$$j : \mathbf{BM}(\mathscr{C}, \mathbf{H}) \longrightarrow \mathbf{B}(\mathscr{C})$$

defined by $j([(A, \varphi_A)]) = [A]$ is a retraction.

([4], (4.3))

In particular, if $\mathscr{C}$ is the category of R–modules over a commutative ring R, every Galois H–object has a normal basis if and only if every Azumaya H–module triple has inner action.

4.3 Definition We denote by **BM**$_{inn}(\mathscr{C}, \mathbf{H})$ the set of **BM**$(\mathscr{C}, \mathbf{H})$ built up with the equivalence classes that can be represented by an **H**–module Azumaya triple with inner action.

4.4 Proposition *$BM_{inn}(\mathscr{C}, H)$ is a subgroup of $BM(\mathscr{C}, H)$*

Proof :

If $(\mathbf{A}, \varphi_A = \mu_A \circ (A\otimes(\mu_A \circ \tau^A_A)) \circ (f\otimes f^{-1} \otimes A) \circ (\delta_C\otimes A))$ and $(\mathbf{B}, \varphi_B = \mu_B \circ$ $\circ (B\otimes(\mu_B \circ \tau^B_B)) \circ (g\otimes g^{-1} \otimes B) \circ (\delta_C\otimes B))$ are **H**–module Azumaya triples with inner actions, then $(A\otimes B, \varphi_{A\otimes B})$ is an **H**–module Azumaya triple with inner action. Indeed :

$$\varphi_{A\otimes B} = (\varphi_A \otimes \varphi_B) \circ (C \otimes \tau_A^C \otimes B) \circ (\delta_C \otimes A \otimes B) =$$

$$= \mu_{A\otimes B} \circ (A\otimes B \otimes (\mu_{A\otimes B} \circ \tau_{A\otimes B}^{A\otimes B})) \circ (u \otimes u^{-1} \otimes A \otimes B) \circ (\delta_C \otimes A \otimes B)$$

where $u := (f\otimes g) \circ \delta_C$ and $u^{-1} = (f^{-1} \otimes g^{-1}) \circ \delta_C . \in \mathrm{Reg}(C, A\otimes B)$

Moreover, if (M, φ_M) is an **H**–module progenerator in $\mathscr{C}$, then $(E(M)^{op}, \varphi_{E(M)})$ is an **H**–module Azumaya triple with inner action

$$\varphi_{E(M)} = \mu_{E(M)^{op}} \circ (E(M) \otimes (\mu_{E(M)^{op}} \circ \tau_{E(M)}^{E(M)})) \circ (v \otimes v^{-1} \otimes E(M)) \circ (\delta_C \otimes E(M))$$

where $v = [M, \varphi_M \circ \tau_C^M] \circ \alpha_M(C) : C \longrightarrow E(M)$ is in $\mathrm{Reg}(C, E(M)^{op})$ with inverse $v^{-1} = [M, \varphi_M \circ \tau_C^M \circ (M\otimes\lambda)] \circ \alpha_M(C)$.

The inverse of $[(A, \varphi_A)] \in \mathbf{BM}_{inn}(\mathscr{C}, \mathbf{H})$ is $[(A^{op}, \varphi_A = \mu_{A^{op}} \circ (A \otimes (\mu_{A^{op}} \circ \tau_A^A)) \circ (f^{-1} \otimes f \otimes A) \circ (\delta_C \otimes A))]$.

Remark Let $[(A, \varphi_A)] = [(B, \varphi_B)] \in \mathbf{BM}(\mathscr{C}, \mathbf{H})$ where the action φ_A of **H** in **A** is inner. Then the action φ_B of **H** in **B** is inner.

Indeed, there is an isomorphism of **H**–comodule triples $\omega : \Pi(A) \longrightarrow \Pi(B)$ and an isomorphism of **H**–comodules $f : C \longrightarrow \Pi(A)$ and thus $\Pi(B)$ is a Galois **H**–object with a normal basis and then, by Proposition 3.4, the action φ_B of **H** in **B** is inner.

4.5 Theorem $\mathit{BM}_{inn}(\mathscr{C}, H) \cong B(\mathscr{C}) \oplus N_{\mathscr{C}}(H)$

Proof :

If $[A] \in \mathbf{B}(\mathscr{C})$, then $[(A, \varphi_A = \varepsilon_C \otimes A)] \in \mathbf{BM}_{inn}(\mathscr{C}, \mathbf{H})$, since $\varepsilon_C \otimes A = \mu_A \circ (A \otimes (\mu_A \circ \tau_A^A)) \circ ((\eta_A \circ \varepsilon_C) \otimes (\eta_A \circ \varepsilon_C) \otimes A) \circ (\delta_C \otimes A)$.

If $[(A, \varphi_A)] \in \mathbf{BM}_{inn}(\mathscr{C}, \mathbf{H})$ then, by Proposition 3.4, $\Pi(A)$ is a Galois **H**–object with a normal basis. Moreover, if $(\mathbf{B}, \rho_B)$ is a Galois **H**–object with a normal basis, then there are **H**–comodule isomorphisms $\Pi(B\#\widehat{H}) \cong B \cong C$ and therefore $\varphi_{B\#\widehat{H}}$ is an inner action of **H** in $\mathbf{B\#\widehat{H}}$ and $[(B\#\widehat{H}, \varphi_{B\#\widehat{H}})] \in \mathbf{BM}_{inn}(\mathscr{C}, \mathbf{H})$ and so, by Proposition 4.2, we have $\mathbf{BM}_{inn}(\mathscr{C}, \mathbf{H}) \cong \mathbf{B}(\mathscr{C}) \oplus \mathbf{N}_{\mathscr{C}}(\mathbf{H})$.

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