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THE LINTON THEOREM REVISITED

by Hans-Eberhard PORST

Dedicated to the memory of Honza Reiterman

Résumé. Les catégories essentiellement algébriques avec la propriété supplémentaire que leur (*Epi, Mono-source*)-factorisations soient obtenues d'une certaine qualité soulevante de leur foncteurs d'oubli, sont examinés et, aussi, sont caractérisés comme une classe spécifique de sous-catégories pleines et epi-réflexives des catégories monadiques. Les cas particuliers de ce resultat comprennent la caractérisation de Linton pour les catégories quasi-primitives d'algèbres et tous ses généralisations. Ceci est apparenté à l'observation que, en catégories essentiellement algébriques d'algèbres partielles, l'image d'un homomorphisme n'a pas besoin d'être un sous-algèbre de son but.

Introduction

Linton's monadicity theorem for Set-based functors [8], stating that a faithful functor $U: \mathbf{A} \to \mathbf{Set}$ is monadic if and only if (0) $\mathbf{A}$ has coequalizers, (1) U has a left adjoint, (2) U preserves and reflects regular epimorphisms, and (3) U preserves and reflects congruence relations, is often considered to be of less importance than the Beck-Paré Theorem. For example, MacLane's standard textbook on category theory [7] doesn't contain Linton's theorem at all, and the recent book [2] treats it as a consequence of the Beck-Paré Theorem. The reason is probably, that Linton's characterization cannot be generalized to arbitrary base categories, which however, from the point of view of algebra, is not a strong argument, since monadic functors over categories different from Set might badly fail to enjoy properties common to underlying functors of categories of algebras (see e.g. [2, p. 321 f]).

In fact, from the point of view of algebra, Linton's theorem has quite a number of advantages:

- it comprises two theorems: while, as a whole, it characterizes varieties, conditions (0), (1), and (2) only characterize quasivarieties (up to rank considerations);

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- all its conditions are extremely natural from an algebraic viewpoint.

Correspondingly, Linton's theorem served as a starting point for an axiomatic theory of algebraic functors on various levels of generality (see e.g. [4], [5], [6], [9], [10]). Moreover, it could be generalized to a wide range of base categories — see the section on applications in this note.

What only gradually became clear is the fact, that a functor is "algebraic" in any reasonable way (from the point of view of total algebras) iff it enjoys a certain — extremely weak — lifting property with respect to factorizations of sources, as it is rudimentarily expressed by Linton's condition (2). In its most general form this will be developed in the main part of this paper. Finally we will comment on the question, why one cannot expect a lifting property of this kind on the most general level in the hierarchy of algebraic functors, viz. for essentially algebraic functors.

We will use the following terminology: a right adjoint functor U will be called *monadic* (resp. *premonadic, of descent type*) iff the corresponding comparison functor is an equivalence (resp. a fully faithful right adjoint, fully faithful). A functor U will be called *essentially algebraic* if it has a left adjoint, reflects isomorphisms, and its domain has (*Epi, Mono-source*)-factorizations, while U will be called *algebraic* (resp. *regular*) if it is essentially algebraic and preserves extremal epimorphisms (resp. preserves regular epimorphisms and, in addition, its codomain has (*Regular Epi, Mono-source*)-factorizations) — for equivalent descriptions of these classes of functors see [2].

1 Functors lifting (E, M)-factorizations

Throughout this section let $\mathbf{X}$ be an (E, M)-category in the sense of [2]; that is, every source $(X, X \xrightarrow{f_i} X_i)_I$ of morphisms in $\mathbf{X}$ can be factored as $f_i = m_i \circ e$ with $X \xrightarrow{e} Y$ belonging to a prescribed class E of $\mathbf{X}$-morphisms and the source $(Y, Y \xrightarrow{m_i} X_i)_I$ belonging to a prescribed class M of sources in $\mathbf{X}$, subject to the usual diagonal-fill-in condition.

We slightly generalize [2, 20.23] by saying

1.1 Definition A functor $U: \mathbf{A} \to \mathbf{X}$ *lifts* (E, M)*-factorizations* provided that, for any source $(A, A \xrightarrow{f_i} A_i)_I$ in $\mathbf{A}$ and any (E, M)-factorization of its underlying source in $\mathbf{X}$

$$Uf_i = UA \xrightarrow{e} X \xrightarrow{m_i} UA_i$$

- there exists a factorization $f_i = A \xrightarrow{\tilde{e}} B \xrightarrow{\tilde{m}_i} A_i$ in $\mathbf{A}$ and an $\mathbf{X}$-isomorphism $k: X \to UB$ with $U\tilde{e} = k \circ e$ and $U\tilde{m}_i \circ k = m_i$ for all $i \in I$, such that

- the U-lifts are unique in the sense that, given two such factorizations $f_i = A \xrightarrow{\tilde{e}^j} B^j \xrightarrow{\tilde{m}_i{}^j} A_i$ with isomorphisms $h_j: X \to UB^j$ (j = 1, 2), there exists an isomorphism $\varphi: B^1 \to B^2$ with $\varphi \circ \tilde{e}^1 = \tilde{e}^2$ and $\tilde{m}_i^1 = \tilde{m}_i^2 \circ \varphi$ for all $i \in I$.

The terms *E-monad* and *E-monadic functor* are used as in [2].

1.2 Examples 1. Every regular functor lifts (*Regular Epi, Mono-Source*)-factorizations [2, 23E].

2. Every algebraic functor over an (*Extremal Epi, Mono-source*)-category lifts (*Extremal Epi, Mono-source*)-factorizations [2, 23.31].

3. Every T-regular functor with respect to some monotopological functor T (c.f. [10]) lifts *T-regular*, i.e., (T^{-1}[*Regular Epi*], *T-initial Mono-source*)-factorizations by definition; in particular

4. Every underlying space functor V: **TopAlg** $\to$ **Top** of a category of algebra-objects (the algebras from some variety) in the category **Top** of topological spaces lifts (*continuous surjections, initial point-separating source*)-factorizations.

5. For any E-monad **T** on **X** the Eilenberg-Moore functor $U^{\mathbf{T}}: \mathbf{X}^{\mathbf{T}} \to \mathbf{X}$ lifts (E, M)-factorizations [2, 20.24].

1.3 Lemma *Every functor $U: \mathbf{A} \to \mathbf{X}$ which lifts (E, M)-factorizations reflects isomorphisms.*

Proof *1.* If for some **A**-morphism $f: A \to B$ the morphism Uf is an isomorphism, then $UA \xrightarrow{1_{UA}} UA \xrightarrow{Uf} UB$ and $UA \xrightarrow{Uf} UB \xrightarrow{1_{UB}} UB$ are (E, M)-factorizations of Uf with U-lifts $A \xrightarrow{1_A} A \xrightarrow{f} B$ and $A \xrightarrow{f} B \xrightarrow{1_B} B$ respectively. By the second condition of Definition 1.1 there is an isomorphism $\varphi: B \to A$ with $f \circ \varphi = 1_B$. Hence f is an isomorphism. ◇

1.4 Proposition *Let $U: \mathbf{A} \to \mathbf{X}$ lift (E, M)-factorizations. Then each of the following statements implies the next one:*

1. *U is faithful.*

2. *Every **A**-morphism f with $Uf \in E$ is a U-final epimorphism.*

3. ***A** is a $(U^{-1}[E], U^{-1}[M])$-category.*

If U has a left adjoint, all the conditions above are equivalent.

Proof *1.* $\Longrightarrow$ *2.* Our proof is a modification of the proof of [2, 23.23]. Let $f: A \to B$ be an **A**-morphism with $Uf \in E$ and $h: UB \to UC$ an **X**-morphism with $h \circ Uf = Ug$ for some **A**-morphism $g: A \to C$. Consider the 2-source

$$UC \xleftarrow{Ug} UA \xrightarrow{Uf} UB$$

and its (E, M)-factorization

$$UC \xleftarrow{\tilde{n}} X \xleftarrow{e} UA \xrightarrow{e} X \xrightarrow{\tilde{m}} UB$$

The lift of this factorization is

$$UC \xleftarrow{Un} UD \xleftarrow{e} UA \xrightarrow{e} X \xrightarrow{Um} UB$$

with some isomorphism $\varphi: X \to UD$. From $\tilde{n} \circ e = Ug = h \circ Uf = h \circ \tilde{m} \circ e$ one concludes $\tilde{n} = h \circ \tilde{m}$, hence $\tilde{m} \circ (h, 1_{UB}) = (h\tilde{m}, \tilde{m}) = (\tilde{n}, \tilde{m})$. From this $Um \in M$ follows. But $Um \in E$, too, since $Um \circ (\varphi \circ e) = Uf$ and $Uf,\ \varphi \circ e \in E$. Hence Um is an isomorphism. By Lemma 1.3 m is an isomorphism, too. Now $\bar{h} = n \circ m^{-1}$ is the desired morphism $B: \to C$.
2. $\Longrightarrow$ *3.* is straightforward.
Assume now that U has a left adjoint and that *3.* holds. The counit ϵ of the adjunction for U is pointwise in $U^{-1}[E]$ since $U\epsilon$ is pointwise a retraction, hence in E. By [2, 15.5 (3)] ϵ is pointwise epic; therefore U is faithful. ◇

1.5 Proposition *Let $U: \mathbf{A} \to \mathbf{X}$ be a right adjoint which lifts (E, M)-factorizations. Then the following hold:*

1. *The monad $\mathrm{T} = (UF, \eta, U\epsilon F)$ induced by U and its left adjoint F is an E-monad; hence, in particular*
   (a) *the Eilenberg-Moore functor $U^{\mathbf{T}}: \mathbf{X}^{\mathbf{T}} \to \mathbf{X}$ lifts (E, M)-factorizations;*
   (b) *$\mathbf{X}^{\mathbf{T}}$ is a $(U^{\mathbf{T}-1}[E], U^{\mathbf{T}-1}[M])$-category.*
2. *The comparison functor $K: \mathbf{A} \to \mathbf{X}^{\mathbf{T}}$ lifts $(U^{\mathbf{T}-1}[E], U^{\mathbf{T}-1}[M])$-factorizations.*

Proof *1.* To show that $UFe \in E$ for every $e \in E$ consider, for any such $e: X \to Y$, the (E, M)-factorization

$$UFX \xrightarrow{UFe} UFY = UFX \xrightarrow{q} Z \xrightarrow{m} UFY$$

and its U-lift

$$UFX \xrightarrow{UFe} UFY = UFX \xrightarrow{U\tilde{q}} UA \xrightarrow{U\tilde{m}} UFY$$

with isomorphism $\varphi: Z \to UA$. The commutative square

$$\begin{array}{ccc} X & \xrightarrow{e} & Y \\ {\scriptstyle \tilde{q}\eta_X}\downarrow & & \downarrow{\scriptstyle \eta_Y} \\ UA & \xrightarrow[U\tilde{m}]{} & UFY \end{array}$$

admits a diagonal $d: Y \to UA$. By adjointness there results a morphism $d^{\prime}: FY \to A$. From $U(\tilde{m} \circ d^{\prime}) \circ \eta_Y \circ e = U\tilde{m} \circ d \circ e = \eta_Y \circ e$ one concludes (e is epic !) $\tilde{m} \circ d^{\prime} = 1_{FY}$, hence $m \circ \varphi^{-1} \circ Ud^{\prime} = 1_{UFY}$. It follows $m \in E \cap M$ and therefore $UFe \in E$.
2. Given a source $(A, f_i: A \to A_i)_I$ in **A**, let

$$KA \xrightarrow{Kf_i} KA_i = KA \xrightarrow{e} (X, x) \xrightarrow{m_i} KA_i$$

be the factorization in $\mathbf{X}^{\mathbf{T}}$ with $U^{\mathbf{T}}e \in E$ and $(X, U^{\mathbf{T}}m_i)_I \in M$. Then $U^{\mathbf{T}}(m_i \circ e)$ is the (E, M)-factorization of Uf_i, such that there are an **X**-isomorphism $\varphi: X \to UB$ and U-lifts $\tilde{e}, \tilde{m}_i$ with $U\tilde{e} = \varphi \circ U^{\mathbf{T}}e$, $U\tilde{m}_i = U^{\mathbf{T}}m_i \circ \varphi$, $f_i = A \xrightarrow{\tilde{e}} B \xrightarrow{\tilde{m}_i} A_i$ for all $i \in I$. Since then $K\tilde{m}_i \circ K\tilde{e}$ is a $(U^{\mathbf{T}-1}[E], U^{\mathbf{T}-1}[M])$-factorization of $(KA, Kf_i)_I$, too, φ lifts to an $\mathbf{X}^{\mathbf{T}}$- isomorphism $(X, x) \longrightarrow KB$ as required by Definition 1.1.
It is an easy exercise, using *1.(a)* and Proposition 1.4, to show that K also meets the second condition of Definition 1.1. ◇

1.6 Corollary *Every faithful right adjoint functor U:* **A** $\to$ **X** *which lifts (E, M)-factorizations is of descent type.*

Proof The counit ϵ of the adjunction for U is pointwise U-final by Proposition 1.4, since $U\epsilon$ belongs pointwise (being a retraction) to E. Hence U, being faithful, is of descent type (see [13]). ◇

From now on we will work under the following additional hypothesis:
Assumption: The class M of the factorization system on **X** consists of mono-sources only, i.e., $M \subset$ *Mono-source*(**X**).

1.7 Proposition *Let U:* **A** $\to$ **X** *be a faithful functor lifting (E, M)-factorizations. Then the following hold:*

1. **A** *has coequalizers.*
2. *$Ue \in E$ for every regular epimorphism e in* **A**.

Proof Due to our assumption on M this is a consequence of Proposition 1.4 in connection with [2, 15.7]. ◇

1.8 Proposition *If U:* **A** $\to$ **X** *is a faithful right adjoint functor which lifts (E, M)-factorizations, then U is premonadic, and the unit of the adjunction for the comparison K functor of U is pointwise in $U^{\mathbf{T}-1}[E]$.*

Proof It is well known that, in the presence of coequalizers in **A**, the unit of the adjunction for the comparison functor K at some algebra (X, x) is the unique $X^{\mathbf{T}}$-morphism $\rho_{(X,x)}: (X, x) \to KL(X, x)$ with $\rho_{(X,x)} \circ x = Kc$, where $c: FX \to L(X, x)$

is a coequalizer of the pair (Fx, ϵ_{FX}) with $x: F^{\mathbf{T}}X \to (X, x)$ the canonical morphism and F the adjoint of U. It follows $U^{\mathbf{T}}\rho_{(X,x)} \circ U^{\mathbf{T}}x = Uc$, and therefore $U^{\mathbf{T}}\rho_{(X,x)} \in E$, since $Uc \in E$ by Proposition 1.7. ◇

We are now in the position to state our main result.

1.9 Theorem *For any functor $U: \mathbf{A} \to \mathbf{X}$ into some (E, M)-category $\mathbf{X}$ with $M \subset$ Mono-source$(\mathbf{X})$ the following are equivalent:*

1. *U is a faithful right adjoint which lifts (E, M)-factorizations.*
2. *Up to concrete equivalence, $(\mathbf{A}, U)$ is a full $U^{\mathbf{T}^{-1}}[E]$-reflective concrete subcategory of some E-monadic category over $\mathbf{X}$.*

Proof *1.* ⇒ *2.* is clear by Proposition 1.8; *2.* ⇒ *1.* follows from [2, 20.24, 20.26]. ◇

2 Applications

2.1 Example (Linton's Theorem) For $\mathbf{X}$ = Set with its (*surjections, point-separating source*) factorization system, a right adjoint functor $U: \mathbf{A} \to$ Set lifts these factorizations iff $\mathbf{A}$ has coequalizers and U preserves and reflects regular epimorphisms (c.f. [5]). Clearly the functor part of every monad on Set preserves surjections. Hence our theorem yields Linton's result.

2.2 Example (Regular functors) Let $\mathbf{X}$ be any regular category, i.e., (E, M) = (*Regular Epi, Mono-source*). Now the theorem specializes to [9, Theorem 2]: A functor $U: \mathbf{A} \to \mathbf{X}$ is regular iff $(\mathbf{A}, U)$ is, up to concrete equivalence, a full regular epireflective concrete subcategory of some regular monadic category over $\mathbf{X}$ (see also [2, 24.2]). Clearly, this contains Linton's result as a special case.

2.3 Example (Algebraic functors) Let $\mathbf{X}$ be any (*Extremal Epi, Mono-source*)-category. Now, using [2, 23.31 ff] our theorem yields the following: A functor $U: \mathbf{A} \to \mathbf{X}$ is algebraic iff $(\mathbf{A}, U)$ is, up to concrete equivalence, a full extremally epireflective concrete subcategory of some extremally monadic (resp. algebraic and monadic) category over $\mathbf{X}$.

2.4 Example (T-regular functors) Let $T: \mathbf{X} \to \mathbf{Y}$ be a monotopological functor (e.g. the underlying functor of the category of Hausdorff spaces). Consider the T-regular factorization on $\mathbf{X}$. Now our theorem reduces to [10, Theorem 2.10]: A functor $U: \mathbf{A} \to \mathbf{X}$ is T-regular iff $(\mathbf{A}, U)$ is, up to concrete equivalence, a full $(TU^{\mathbf{T}})^{-1}$[*Regular Epi* $(\mathbf{Y})$]-reflective concrete subcategory of some T^{-1}[*Regular Epi* $(\mathbf{Y})$]-monadic category over $\mathbf{X}$. (Note that, in the instance of $\mathbf{X}$ = Hausdorff spaces T^{-1}[*Regular Epi* $(\mathbf{Y})$] is simply the class of continuous surjections.)

3 Some remarks on essentially algebraic functors

From our previous results it is clear that the following holds

3.1 Proposition *For any functor $U: \mathbf{A} \to \mathbf{X}$ into some (E, M)-category $\mathbf{X}$ with $M \subset$ Mono-source$(\mathbf{X})$ the following are equivalent:*

1. *U is a faithful right adjoint which lifts (E, M)-factorizations.*
2. *U is essentially algebraic in such a way that the (Epi, Mono-source)-factorization of a source $(A, A \xrightarrow{f_i} A_i)_I$ in $\mathbf{A}$ can be obtained as a U-lift of the (E, M)-factorization of the source $(UA, UA \xrightarrow{Uf_i} UA_i)_I$.*

Moreover, if U fulfills the above conditions, so does the Eilenberg-Moore functor $U^{\mathbf{T}}: \mathbf{X}^{\mathbf{T}} \to \mathbf{X}$ and the comparison functor $K: \mathbf{A} \to \mathbf{X}^{\mathbf{T}}$, which, up to equivalence, is a full $U^{\mathbf{T}-1}[E]$-reflective embedding.

Observe that, in general, the comparison functor of an essentially algebraic functor U fails to be full (see e.g. [2, 20.41 (3)]).

Also in general, an essentially algebraic functor $U: \mathbf{A} \to \mathbf{X}$ will not lift (E, M)-factorizations, which means that the (*Epi*, *Mono-source*)-facto rizations in $\mathbf{A}$ are not necessarily carried by (E, M)-factorizations in $\mathbf{X}$. In fact, there are well known examples for this effect, e.g., the category **Cat** of small categories, Set-based by the morphism functor (see also the example of partially ordered sets below). While from a formal point of view this might appear as a shortcoming of the con cept of "algebraic functor" there is good reason for this proper extension of the class of "faithful right adjoints lifting (E, M)-factorizations": Over Set the essentially algebraic categories are precisely the essentially equational categories of partial algebras (see [1]), and here even (*surjections, injections*)-factorizations of single homomorphisms cannot be lifted, since in these categories the set-theoretical image of a homomorphi sm might fail to be a subalgebra of its codomain as already was pointed out in [12].

3.2 Example A simple illustration of this effect is the following. The relation functor R: POS $\to$ Set, assigning to a partially ordered set its relation, and to a monotone map f the map f^2 restricted and corestricted to the corresponding relations, is essentially algebraic. Consider the monotone map $f: M_1 \to M_2$, where M_1 consists of two disjoint copies $\{0, 1\}$ and $\{0', 1'\}$ of the 2-chain, where M_2 is the 3-chain on $M = \{0, 1, 2\}$, and where $f(0) = 0$, $f(1) = f(0') = 1$, $f(1') = 2$. f has an (*Epi*, *Mono*)-factorization $M_1 \xrightarrow{f} M_2 \xrightarrow{1_{M_2}} M_2$ in POS, which is not an R-lift of the factorization of Rf in Set. For the latter is $RM_1 \xrightarrow{Rf} Rf[RM_1] \xrightarrow{in} RM_2$ with *in* th e inclusion, and the set-theoretical image $Rf[RM_1]$ is not an order on M, since it contains $(0, 1)$ and $(1, 2)$, but not $(0, 2)$. Though somewhat hidden this indeed is an example based on the existence of a (essentially equational) partial operation (see [11]).

Hence, the existence of essentially equational partial operations can prevent the underlying functor U from lifting (E, M)-factorizations, or its comparison functor K from being full. On the other hand there might be such operations, and nevertheless U will lift factorizations and K will be full: quasivarieties are an example (following [1] one replaces every implication by an essentially equational partial operation and two additional equations).

Finally, the comparison functor of an essentially algebraic functor U might be full without U lifting (E, M)-factorizations (even over Set !). Th e functor R: **POS** $\to$ Set discussed above provides an example. Since R reflects regular epimorphisms (if, for some monotone map f, Rf is surjective, f is surjective and R-final, hence a regular epimorphism in **POS** by [2, 8.O]), R is premonadic. This discussion contains most of the arguments needed to prove more in general:

3.3 Proposition *The following are equivalent for an essentially algebraic functor* U: **A** $\to$ Set.

1. *U is premonadic,*
2. *U is of descent type,*
3. *U reflects regular epimorphisms,*
4. *an* **A**-*morphism f is U-final, if Uf is surjective.* ◇

The functor R: POS $\to$ Set also is an example of an essentially algebraic functor which reflects regular epimorphisms without preserving them. Concerning these properties one has however (see also [3, 2.1]) the following:

3.4 Proposition *Let* U: **A** $\to$ **X** *be an essentially algebraic functor. If U maps regular epimorphisms to epimorphisms (hence in particular, if U preserves regular epimor phisms), then U reflects regular epimorphisms (and therefore is of descent type).*

Proof Let $f: A \to B$ be an **A**-morphism such that Uf is a coequalizer of a pair of **X**-morphisms $r, s: X \to UA$. Let $g: A \to C$ be a coequalizer of the **A**-morphisms $r^{\prime}, s^{\prime}: FX \to A$ with $r^{\prime}\eta_X = r$, $s^{\prime}\eta_X = s$ (where $\eta_X: X \to UFX$ is the unit of the adjunction for U). The coequalizer properties of g and Uf imply that there exists a unique **A**-morphism $h: C \to B$ with $h \circ g = f$ as well as a unique X-morphism $k: UB \to X$ with $k \circ Uf = Ug$. It follows $Uh \circ k = 1_{UB}$ by the coequalizer property of Uf, and $k \circ Uh = 1_{UC}$ since Ug is an epimorphism by assumption. Since U reflects is omorphisms h is an isomorphism and therefore $f = h \circ g$ is a regular epimorphism. ◇

Finally, for the sake of completeness, we add the following combination of Proposition 1.5 and Theorem 1.9 (see also [2, 23.32]):

3.5 Proposition *Let* U: **A** $\to$ **X** *be an essentially algebraic functor over an* (E, M)-*category* **X** *with* $M \subset$ *Mono-source*(**X**), *and* **T** *the monad on* **X** *induced by* U. *The following are equivalent:*

1. *U lifts (E, M)-factorizations,*

2. *Up to equivalence, $(\mathbf{A}, U)$ is a full reflective concrete subcategory of the Eilenberg-Moore category $(\mathbf{X}^{\mathbf{T}}, U^{\mathbf{T}})$ which is closed w.r.t. $(U^{\mathbf{T}-1}[E], U^{\mathbf{T}-1}[M])$-factorizations, i.e., which contains, with every source $(A, A \xrightarrow{f_i} A_i)_I$ in $\mathbf{A}$, its $(U^{\mathbf{T}-1}[E], U^{\mathbf{T}-1}[M])$-factorization taken in $\mathbf{X}^{\mathbf{T}}$.* ◇

The added closure condition on the reflective subcategory **A** in the above proposition can again be nicely illustrated by means of the functor R: **POS** → **Set**. The Eilenberg-Moore category of R is the category **DGra** of directed graphs, i.e. of algebras (A, c, d) with two unary operations c, d subjects to the equations $cc = dc = c$ and $dd = cd = d$. The comparison functor K: **POS** → **DGra** assigns to a partially ordered set $(M, \leq)$ its directed graph $(\leq, \bar{c}, \bar{d})$ with $\bar{c}(x, y) = (y, y)$ and $\bar{d}(x, y) = (x, x)$ (see [11] for details). The monotone map f considered in Example 3.2 has in **DGra** the regular factorization $KM_1 \xrightarrow{Rf} (G, c, d) \xrightarrow{in} KM_2$ with $in: G = Rf[RM_1] \to RM_2$ the inclusion map and c, d the restrictions of the operations of KM_2. Since $(0,0)$, $(1,1)$, $(2,2) \in G$, (G, c, d) is a directed graph but, as seen in Example 3.2, not the graph of an order. Hence, the full reflective embedding of **POS** into **DGra** is not closed w.r.t. the factorizations in question.

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