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EQUATIONALLY CLOSED SUBFRAMES AND REPRESENTATION OF QUOTIENT SPACES

by Aleš PULTR and Anna TOZZI

In memory of Jan Reiterman

Résumé : On dit que le subframe B du frame A est équationalement fermé si les relations $a \vee x, a \wedge x, a \in B$ entraînent $x \in A$. Entre autres on montre que pour une certaine classe d'espaces, les immersions des subframes équationalement fermés donnent une description des quotients topologiques, et on caractérise cette classe. On prouve aussi l'analogue du Théorème de diagonalization de Dikranjan - Giuli.

It is a well-known fact that in a distributive lattice A an element $x \in A$ is, for any $a \in A$, uniquely determined by the values $a, a \vee x$ and $a \wedge x$. Consequently, the values of a frame homomorphism $\phi : A \longrightarrow B$ determined on a subset $M \subseteq A$ are uniquely determined also on all the solutions x of the equation pairs

$$(*) \qquad a \vee x = b, \;\; a \wedge x = c$$

with $a, b, c \in M$.

In this paper we present a few facts on subframes closed under solutions of the equations $(*)$ (equationally closed subframes). In particular, we show that in a class of topological spaces containing all metrizable ones, these represent well the quotients, that is, a continuous map $X \longrightarrow Y$ is a quotient if and only if the induced frame homomorphism $\Omega(Y) \longrightarrow \Omega(X)$ is an embedding of an equationally closed subframe.

The article is divided into Preliminaries and four further sections. In Section 2, basics on equational closedness are proved. In particular, we show

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that all subframes that are Heyting subalgebras are closed (and consequently, on the one hand all Boolean subalgebras, and on the other hand images of open homomorphisms are), and that the images of closed homomorphisms are equationally closed. In Section 3, the notion is used for proving an analogon of the Dikranjan - Giuli diagonalization theorem. In Section 4, a necessary and sufficient condition for a space Y to have the quotients $X \longrightarrow Y$ represented by equationally closed subframes is found. The last Section 5 is devoted to more information on the resulting class of spaces. It is not yet satisfactorily understood ; we know that it includes, for instance, all Fréchet spaces, but also all the ordinals. For quasidiscrete spaces, a complete characterization is presented.

1 Preliminaries

1.1.1 A *frame* A is a complete lattice satisfying the distributivity law

$$(\bigvee_J a_i) \wedge b = \bigvee_J (a_i \wedge b),$$

a frame *homomorphism* $\phi : A \longrightarrow B$ preserves general joins and finite meets. A typical example is the lattice $\Omega(X)$ of open sets of a topological space, and if $f : X \longrightarrow Y$ is a continuous map, $\Omega(f) : \Omega(Y) \longrightarrow \Omega(X)$ defined by $\Omega(f)(U) = f^{-1}(U)$ is a frame homomorphism. Another class of examples is provided by complete Boolean algebras and complete Boolean homomorphism.

The category of frames and their homomorphisms will be denoted by

$$\mathbf{Frm}.$$

1.1.2. Because of the distributivity law, the mappings $x \mapsto x \wedge a$ preserve suprema and hence we have a uniquely defined binary operation $\rightarrow$ satisfying

$$a \wedge b \leq c \ \text{ iff } \ a \leq b \rightarrow c.$$

Thus, each frame is a Heyting algebra. Frame homomorphisms are not necessarily Heyting ones (that is, do not necessarily preserve $\rightarrow$), not to speak of complete Heyting (the infinite meets are not necessarily preserved). Frame homomorphisms between Boolean algebras are completely Boolean, though.

1.1.3. A frame homomorphism is said to be *open* if it is completely Heyting . This notion corresponds to the classical openness of continuous maps in the sense that $\Omega(f)$ is open in this new sense if and only if f is open (see [8],[2]) Similarly, a homomorphism $\phi : A \longrightarrow B$ is said to be *closed* if we have the implication

$$\phi(x) \leq \phi(y) \vee z \quad \Rightarrow \quad x \leq y \vee \phi_+(z)$$

for ϕ_+ the right adjoint to ϕ (that is, the mapping $\phi_+ : B \longrightarrow A$ satisfying $\phi(x) \leq y$ iff $x \leq \phi_+(y)$) and $\Omega(f)$ is closed in this sense if and only if f is closed in the usual one (see, e.g.[10], [11]).

1.1.4. A *sublocale* of a frame A is a surjective frame homomorphism $\gamma : A \longrightarrow C$.

1.1.5. Monomorphisms in **Frm** are exactly the one-one frame homomorphisms.

1.1.6. Recall that $A \oplus B$, the coproduct of frames A, B, is generated by the $a \oplus b = \nu_A(a) \wedge \nu_B(b)$ where ν_A, ν_B are the coproduct injections.

A reader wishing for more details on frames may consult, e.g., [7].

1.2.1. As usual, a continuous map $f : X \longrightarrow Y$ will be called *quotient* (or, a *quotient map*) if U is open in Y iff $f^{-1}(U)$ is open in X.

1.2.2. A topological space X is said to satisfy T_D (see [1]) if

each $x \in X$ has an open neighbourhood U such that $U \setminus \{x\}$ is open.

This separation axiom allows easy algebraic interpretations of various topological facts (see, e.g., [12],[2],[11]). In particular we have

Proposition : *A space X satisfies T_D if and only if, for any space Y, the continuous onto maps $f : Y \longrightarrow X$ are exactly those continuous maps for which $\Omega(f)$ are monomorphisms in* **Frm**.

Proof : Trivially, if f is onto, $\Omega(f)$ is one-one. If f is not onto and if X satisfies T_D, consider an $x \in X \setminus f[Y]$ and an open $U \ni x$ such that $V = U \setminus \{x\}$ is open. Then $U \neq V$ and $f^{-1}(U) = f^{-1}(V)$.

Now let T_D be not satisfied, let x_0 be such that for no open $U \ni x_0$, $U \setminus \{x_0\}$ is open. Put $Y = X \setminus \{x_0\}$ and let $f : Y \longrightarrow X$ be the embedding of the subspace. Then $\Omega(f)$, sending U to $U \setminus \{x_0\}$, is one-one but f is not onto. □

1.2.3. Ordinals will be viewed as usual with the interval topology.

1.2.4. Speaking of *quasidiscrete spaces* (cf. [3]) we will have in mind those connected with partially ordered sets (that is, we really have in mind the T_0-quasidiscrete spaces) : Given a poset $(X, \leq)$ we denote, for a set $M \subseteq X$, $\downarrow M = \{x | \exists y \in M, x \leq y\}$; a set is declared open if $M = \downarrow M$.

We write $\downarrow x$ for $\downarrow \{x\}$ and $\uparrow x$ for $\{y | y \geq x\}$. Obviously $\uparrow x$ is the closure of $\{x\}$.

1.3.1. From category theory only basics are assumed (the reader is referred to the general chapters of [9] or [6]). A coreflective subcategory is said to be *extremally monocoreflective* if the coreflection morphisms are extremal monomorphisms. For a subcategory $\mathcal{A}$ of $\mathcal{C}$,

$$Epi(\mathcal{A})$$

is the full subcategory of $\mathcal{C}$ generated by the objects b such that there is an epimorphism $\varepsilon : a \longrightarrow b$ with $a \in \mathcal{A}$. It is easy to see that

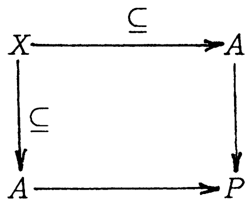
for an extremally monoreflective $\mathcal{A}$, $Epi(\mathcal{A}) = \mathcal{A}$.

(Indeed, if μ is the coreflection and $\varepsilon : a \longrightarrow b$ is an epimorphism with $a \in \mathcal{A}$, we have $\varepsilon = \mu_b \circ \varepsilon'$, hence μ_b is an epimorphism and hence an isomorphism.)

1.3.2. The Dikranjan-Giuli diagonalization theorem, the analogon of which we will prove in 3.5 below, is the following fact [4]):

Theorem : *Let $\mathcal{A}$ be an extremally epireflective subcategory of the category of topological spaces. Let a space A be in $\mathcal{A}$, let X be a subspace of A. Then X is $\mathcal{A}$-closed (that is, an equalizer of morphisms in $\mathcal{A}$) if an only if $A \sqcup_X A$ is in $\mathcal{A}$.*

(The vertex P in the pushout



is denoted by $A \sqcup_X A$.)

2 Equationally closed subframes

2.1. A subframe B of a frame A is said to be *equationally closed* (abbreviated, EQC) if

for any $a \in B$, $a \vee x \in B$ and $a \wedge x \in B$ imply that $x \in B$.

That is, an equationally closed subframe is one closed with respect to solutions x of the systems of equations

$$a \vee x = b,$$

$$a \wedge x = c.$$

A frame *homomorphism* $\phi : B \longrightarrow A$ is said to be *equationally closed* if $\phi[B]$ is EQC in A.

Obviously, an intersection of EQC subframes is EQC. Hence, for any subset M of A we have the least EQC subframe B of A containing M. It will be called the *equational closure* of B and denoted by $\mathcal{E}(M)$. More exactly, one should write $\mathcal{E}^A(M)$. Note that

(2.1.1) *if $M \subseteq B \subseteq A$ and B is EQC in A then $\mathcal{E}^B(M) = \mathcal{E}^A(M)$.*

2.2. For a subset M of a frame A put

$$eq(M) = \{x | \exists a \in M \text{ s.t. } a \vee x, a \wedge x \in M\}$$

and denote by

$$sf(M)$$

the subframe generated by M. Further, put

$$\mathcal{E}_1(M) = sf(eq(M)),$$

and define $\mathcal{E}_\alpha$ for ordinals α by

$$\mathcal{E}_{\alpha+1}(M) = \mathcal{E}_1(\mathcal{E}_\alpha(M)), \text{and}$$

$$\mathcal{E}_\alpha(M) = \bigcup_{\beta<\alpha} \mathcal{E}_\beta(M) \text{if } \alpha \text{ is a limit ordinal}.$$

Obviously,
$\mathcal{E}(M) = \mathcal{E}_\alpha(M)$ *such that* $\mathcal{E}_{\alpha+1}(M) = \mathcal{E}_\alpha(M)$.

Note : For any α there is an example of a frame A and a subframe $B \subseteq A$ such that $\mathcal{E}_\gamma(M)$ do not saturate before α. Consider the embedding $A \longrightarrow N(A)$ from [7], Chapter II and iterate the construction transfinitely. By II.2.10 in [7], if A is a free frame with infinitely many generators, the procedure never stops. This example was pointed to us by B.Banaschewski.

2.3. It is a well- known fact that in a distributive lattice,

(2.3.1) an element x is uniquely determined by the values $a, a \vee x$ and $a \wedge b$.

(Indeed, let the values for x and y coincide. We have $x = x \wedge (a \vee x) = x \wedge (a \vee y) = (x \wedge a) \vee (x \wedge y) = (y \wedge a) \vee (y \wedge x) = y \wedge (a \vee x) = y \wedge (a \vee y) = y$.)
Consequently, if the values of a frame homomorphism $\phi : A \longrightarrow C$ are determined on $M \subseteq A$, they are determined on $eq(M)$. Since trivially a homomorphism determined on M is determined on $sf(M)$, we obtain by transfinite induction

Proposition : *A homomorphism* $\phi : B \longrightarrow A$ *such that* $\mathcal{E}(\phi[B]) = A$ *is an epimorphism.*

2.4. Lemma : *Let $H \subseteq A$ be a Heyting algebra, $a, a \vee x, a \wedge x \in H$. Then*

$$x = (a \vee x) \wedge (a \rightarrow (a \wedge x)).$$

Proof : Put $b = a \vee x$, $c = a \wedge x$, $y = b \wedge (a \rightarrow c)$. We obviously have $x \leq b$ and by the second equation $x \leq a \rightarrow c$ so that $x \leq y$. As $a \leq b$ we have $b = a \vee x \leq a \vee y \leq b$, that is,

$$a \vee x = a \vee y.$$

As $a \rightarrow c \leq a \rightarrow c$ we have $a \wedge (a \rightarrow c) \leq c$ so that

$$c = a \wedge x \leq a \wedge y = a \wedge (b \wedge (a \rightarrow c)) = a \wedge (a \rightarrow c) \leq c$$

and hence also

$$a \wedge x = a \wedge y.$$

Thus, by (2.3.1), $x = y$. □

As a consequence we immediately obtain

Proposition : *Let a subframe $B \subseteq A$ be a Heyting subalgebra. Then it is EQC.*

2.5. A homomorphism is open if and only if it is a complete Heyting homomorphism (recall 1.1.3). Thus, we have

Corollary : *Each open homomorphism is EQC.*

2.6. Proposition : *Each closed homomorphism is EQC.*

Proof : Let $\phi : B \longrightarrow A$ be a closed homomorphism (recall 1.1.3). Let $\phi(a) \vee u = \phi(b)$ and $\phi(a) \wedge u = \phi(c)$. We want to prove that $u \in \phi[B]$. We have

$$\phi(a) \wedge u = \phi(c) = \phi\phi_+\phi(c) = \phi(\phi_+(\phi(a)) \wedge \phi_+(u)) \leq$$

$$\leq \phi\phi_+\phi(a) \wedge \phi\phi_+(u) = \phi(a) \wedge \phi\phi_+(u) \leq \phi(a) \wedge u.$$

As ϕ is closed, we have the implication

$$\phi(x) \leq \phi(y) \vee z \quad \Rightarrow \quad x \leq y \vee \phi_+(z),$$

hence in particular $b \leq a \vee \phi_+(u)$ so that

$$\phi(a) \vee u = \phi(b) \leq \phi(a) \vee \phi\phi_+(u) \leq \phi(a) \vee u.$$

Thus,

$$\phi(a) \vee u = \phi(a) \vee \phi\phi_+(u) \quad \textit{and} \quad \phi(a) \wedge u = \phi(a) \wedge \phi\phi_+(u)$$

so that, by (2.3.1), $u = \phi\phi_+(u)$. □

2.7. Let A be a subcategory of **Frm**. We will denote by $Equi(\mathcal{A})$ the class of all frames B such that there is a homomorphism $\varepsilon : A \longrightarrow B$ with $A \in \mathcal{A}$ and $\mathcal{E}(\varepsilon[A]) = B$.

3 EQC, congruences, and the diagonal theorem

3.1. Lemma : *Let A be a frame. Then the congruences $C \subseteq A \times A$ are exactly the equationally closed subframes of $A \times A$ containing the diagonal Δ.*

Proof : I. Let $C \supseteq \Delta$ be an EQC subframe. Since it is a subframe, it suffices to show that it is an equivalence.

Let $(x, y) \in C$. We have $(y, x) \wedge (x, y) \in C$ and $(y, x) \vee (x, y) \in C$. By the EQC, $(y, x) \in C$.

Let $(x, y), (y, z) \in C$. We have

$$(x, z) \wedge (y, y) = (x \wedge y, y \wedge z) = (x, y) \wedge (y, z),$$

$$(x, z) \vee (y, y) = (x \vee y, y \vee z) = (x, y) \vee (y, z).$$

Hence also $(x, z) \in C$.

II. Let C be a congruence. Then it is, trivially, a subframe containing the diagonal. Let $(x, y) \vee (a, b)$, $(x, y) \wedge (a, b)$ and (a, b) be in C. Write uCv for $(u, v) \in C$. Thus, we want to prove that xCy. We have

$$x = x \wedge (x \vee a) \; C \; x \wedge (y \vee b) =$$

$$= (x \wedge y) \vee (x \wedge b) C (x \wedge y) \vee (x \wedge a)\ C\ (x \wedge y) \vee (y \wedge b) =$$

$$= y \wedge (x \vee b)\ C\ y \wedge (v \vee a)\ C\ y \wedge (y \vee b) = y.$$

□

3.2. Corollary : *Let* $\Delta \subseteq M \subseteq A \times A$. *Then* $\mathcal{E}(M)$ *is the congruence generated by* M.

3.3. Let $\mathcal{A}$ be a subcategory of **Frm**, $A \in \mathcal{A}$. A sublocale $\gamma : A \longrightarrow C$ is said to be $\mathcal{A}$-*closed* if it is a coequalizer of two morphisms from A.

3.4. For a sublocale $\gamma : A \longrightarrow C$ denote

$$A \sqcap_\gamma A = \{(x,y) | \gamma(x) = \gamma(y)\} \subseteq A \times A.$$

Obviously,

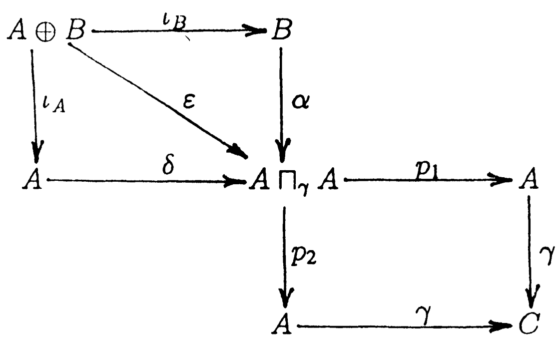
$$\begin{array}{ccc} A \sqcap_\gamma A & \xrightarrow{p_1} & A \\ \downarrow{\scriptstyle p_2} & & \downarrow{\scriptstyle \gamma} \\ A & \xrightarrow{\gamma} & C \end{array}$$

with $p_i(x_1, x_2) = x_i$, is a pullback in **Frm**.

3.5. Theorem (analogon of the Dikranjan-Giuli diagonalization theorem) : *Let* $\mathcal{A}$ *be a coreflective subcategory of* **Frm**, *let* $A \in \mathcal{A}$ *and let* $\gamma : A \longrightarrow C$ *be a sublocale of* A. *Then the following statements are equivalent :*

(i) γ *is* $\mathcal{A}$*-closed,*
(ii) $A \sqcap_\gamma A$ *is in* $Equi(\mathcal{A})$,
(iii) $A \sqcap_\gamma A$ *is in* $Epi(\mathcal{A})$.

Proof : (i) ⇒ (ii) : Let $\gamma = \text{coequ}(\alpha_1, \alpha_2) : A \longrightarrow B$. Consider the following diagram



with $p_i \circ \delta = id, p_i \circ \alpha = \alpha_i, \iota_A, \iota_B : A, B \longrightarrow A \oplus B$ the coproduct, and ε given by

$$\varepsilon \circ \iota_A = \delta \text{ and } \varepsilon \circ \iota_B = \alpha.$$

As $\gamma = \text{coequ}(\alpha_1, \alpha_2)$, $A \sqcap_\gamma A = \{(x, y) | \gamma(x) = \gamma(y)\}$ is the congruence generated in $A \times A$ by

$$N = \{(\alpha_1(b), \alpha_2(b)) | b \in B\}.$$

We easily see (recall 1.1.6) that $\varepsilon(a \oplus b) = \varepsilon(\iota_A(a) \wedge \iota_B(b)) = \delta(a) \wedge \alpha(b) = (a \wedge \alpha_1(b), a \wedge \alpha_2(b))$. Thus, $M = \varepsilon(A \oplus B)$ contains both N (put $a = 1$) and the diagonal (put $b = 1$) and hence, by 3.2, $A \sqcap_\gamma A = \mathcal{E}(M)$ (in $A \times A$, but by (2.1.1) also in $A \sqcap_\gamma A$). Thus, by 3.2 and (2.1.1), $\mathcal{E}(\varepsilon(A \oplus B)) = A \sqcap_\gamma A$.

(ii) $\Rightarrow$ (iii) by 2.3.

(iii) $\Rightarrow$ (i) quite analogously with the proof of 1.12 in [4] :

Obviously $\gamma = \text{coequ}(p_1, p_2)$. If $\eta : B \longrightarrow A \sqcap_\gamma A$ is an epimorphism, we have also $\gamma = \text{coequ}(p_1\eta, p_2\eta)$. By(iii) we can choose an η with $B \in \mathcal{A}$. □

3.6 Notes : 1. If A is extremally-monocoreflective, $Epi(\mathcal{A}) = \mathcal{A}$ and hence we have that

γ is A-closed iff $A \sqcap_\gamma A$ is in A.

2. The extremal monocoreflexivity is not necessary for this stronger statement. It holds for instance also for $\mathcal{A}$ the subcategories of regular or completely regular frames.

4 Equational closedness and algebraic description of quotient maps

4.1 A topological space X is said to satisfy the condition of *approximation by closed sets* (abbreviated, AC) if

for each non-open $M \subseteq X$ there is a closed F and open U, V such that

$$U \cup (M \cap V) = U \cup (F \cap V)$$

and this set is not open.

4.2. Proposition : *A space X satisfies AC if and only if for each non-open $M \subseteq X$ there is a closed F and open U_1, V_1, U_2, V_2 such that $U_1 \cup (M \cap V_1) = U_2 \cup (F \cap V_2)$ and this set is not open.*

Proof : Let $U_1 \cup (M \cap V_1) = U_2 \cup (F \cap V_2)$. Put $U = U_1 \cup U_2, V = V_1 \cap V_2$. We have

$$U \cup (M \cap V) = U_2 \cup U_1 \cup (U_1 \cap V_2) \cup (M \cap V_1 \cap V_2) =$$

$$= U_2 \cup U_1 \cup ((U_1 \cup (M \cap V_1)) \cap V_2) = U_2 \cup U_1 \cup (U_2 \cup (F \cap V_2)) \cap V_2 =$$

$$= U_1 \cup U_2 \cup (F \cap V_2) = U_1 \cup (M \cap V_1),$$

and similarly $U \cup (F \cap V) = U_2 \cup (F \cap V_2)$. □

4.3. Theorem : *Let Y be a space satisfying T_D and AC. Then a continuous $f : X \longrightarrow Y$ is a quotient map if and only if $\Omega(f)$ is an EQC monomorphism.*

Note : As we will see in the next section, this concerns, for instance, all metrizable spaces.

Proof : We easily check that the equations

$$f^{-1}(U) \cup M = f^{-1}(V),$$

$$f^{-1}(U) \cap M = f^{-1}(W)$$

are equivalent with

$$M = f^{-1}(W \cup ((Y \setminus U) \cap V)).$$

Thus, $\Omega(f)$ is EQC if and only if
$(*)$ for U, V, W open, $f^{-1}(W \cup ((Y \setminus U) \cap V)$ open implies $W \cup ((Y \setminus U) \cap V)$ open.
Now let $f : X \longrightarrow Y$ be a quotient map. Then, first, since f is onto, $\Omega(f)$ is a monomorphism. As it is a quotient, we have generally the implication $f^{-1}(N)$ open $\Rightarrow N$ open. Thus, in particular, $(*)$.

Let Y be T_D and AC and let $\Omega(f)$ be a monomorphism. Then, by 1.2.2, f is onto. Let $\Omega(f)$ be EQC and suppose it is not a quotient. Thus, there is a non-open M with $f^{-1}(M)$ open. Consider the U, V, F from the definition of AC, $N = U \cup (F \cap V) = U \cup (M \cap V)$ non-open. Then, by $(*)$, $f^{-1}(N)$ is not open. But $f^{-1}(U) \cup (f^{-1}(M) \cap f^{-1}(V))$ is. □

4.4. Theorem : *A space Y satisfies T_D and AC if and only if the quotients $f : X \longrightarrow Y$ are exactly those continuous maps for which $\Omega(f)$ are equationally closed monomorphisms.*

Proof : $\Rightarrow$ by 2.3.

$\Leftarrow$: If Y does not satisfy T_D there is an equationally closed monomorphism, indeed an isomorphism $\Omega(f)$ with f not being onto at all (see1.2.2).

Now let Y violate AC. Thus, we have a non-open M such that whenever $U \cup (M \cap V) = U_1 \cup (F \cap V_1)$ with U, V, U_1, V_1 open and F closed, $U \cup (M \cap V)$ is open. Define X as the space carried by the same set as Y and with

$$\Omega(X) = \{U \cup (M \cap V) |\ U, V \text{ open in } Y\}.$$

Obviously $f : X \longrightarrow Y$ carried by the identity is not a quotient. $\Omega(f)(\Omega(Y)) = \Omega(Y)$ is EQC in $\Omega(X)$, though : If $W_1 \cup (U \cup (M \cap V)) = V_1$ and $W_1 \cap (U \cup (M \cap V)) = U_1$ with U_1, V_1, W_1 open in Y, we compute easily that $U \cup (M \cap V) = U_1 \cup ((Y \setminus W_1) \cap V_1)$ and hence $U \cup (M \cap V)$ is open by the assumption. □

5 More about AC

5.1. Let us say that a space X has the property of *approximation of points* (abbreviated, AP) if for any $x \in M \setminus \operatorname{int} M$ in X there is a $C \subseteq X$ such that

$$C \cap M = \emptyset \quad \text{and} \quad \overline{C} \cap M = \{x\}.$$

Thus, e.g., each metrizable space, or more generally each Fréchet space, has AP.

5.2. Proposition : *Each T_D space with AP satisfies AC.*

Proof : Take a non-open M, choose $x \in M \setminus \operatorname{int} M$, and take the C from the definition of of AP. We obviously have

$$(X \setminus \overline{C}) \cup M = (X \setminus \overline{C}) \cup \{x\}.$$

By T_D we have an open V such that $\{x\} = V \cap \overline{\{x\}}$. Thus, it suffices to show that $(X \setminus \overline{C}) \cup \{x\}$ is not open. If U is an open neighbourhood of x, by definition of C we have an element $y \neq x, y \in U \cap C$. Thus, $U \not\subseteq (X \setminus \overline{C}) \cup \{x\}$. □

Corollary : *For a metrizable (or, more generally, Fréchet) spaces Y, a continuous map $f : X \longrightarrow Y$ is a quotient if and only if $\Omega(f)$ is an equationally closed monomorphism.*

5.3. Example : $\omega_1 + 1$ *does not have AP.*

Proof : Take $M = \{\alpha \leq \omega_1 | \alpha \text{ limit}\}, x = \omega_1$. Suppose $C \not\ni \omega_1, \overline{C} \ni \omega_1$. Thus, for each $\alpha < \omega_1$ there is an $\alpha' < \omega_1$ such that $\alpha < \alpha' \in C$. Choose $\alpha_0 \in C$ and, by induction, $\alpha_{n+1} = \alpha'_n$, and put $\beta = \sup\{\alpha_n | n \in \omega_0\}$. Then $\beta \in M \cap \overline{C}$. Thus, if $C \cap M = \emptyset$ and $M \cap \overline{C} \ni x$,we have $M \cap \overline{C} \neq \{x\}$. □

5.4. Proposition : *Let $X = \bigcup_{i \in J} X_i$, X_i open, let X_i satisfy AC. Then X does.*

Proof : Let $M \subseteq X$ not be open. Then,. for some $j \in J$, $M_j = M \cap X_j$ is not open and hence there are U_j, V_j open and F_j closed in X_j such that

$$U_j \cup (M_j \cap V_j) = U_j \cup (F_j \cap V_j)$$

and this set is not open. Take an F closed in X such that $F_j = X_j \cap F$ and put $U = U_j, V = V_j \cap X_j$. □

5.5. Proposition : *Let X be a disjoint union of X_1, X_2 with X_1 open(and hence X_2 closed). If both X_1 and X_2 satisfy AC, so does X.*

Proof : Let M be non-open.

I. If $M \cap X_1$ is not open we find U, V and F as in the proof of 5.4.

II. If $M \cap X_1$ is open and if $M \cap X_2$ is open in X_2, we have

$$M = \emptyset \cup (M \cap X) = (M \cap X_1) \cup (X_2 \cap V)$$

for some V open in X. Use 4.2.

III. If $M \cap X_1$ is open and $M \cap X_2$ is not open in X_2, we have $N = (U \cap X_2) \cup (M \cap X_2 \cap V) = (U \cap X_2) \cup (F \cap X_2 \cap V)$ not open in X_2, with U, V open and F closed. We have

$$U \cup N = U \cap ((M \cap X_2) \cap V) = U \cup (F \cap V).$$

Since $(M \cap X_1) \cup (M \cap X_2 \cap V) = M \cap (X_1 \cup V)$ we obtain, for $V_1 = X_1 \cup V, U_1 = U \cup (M \cap X_1)$,

$$U \cup (M \cap V_1) = U \cup (M \cap X_1) \cup (M \cap X_2 \cap V) =$$

$$= U \cup (M \cap X_1) \cup (F \cap V) = U_1 \cup (F \cap V),$$

and this set is not open in X since obviously its meet with X_2 is N. Use 4.2 again. □

5.6. Proposition : *Closed subspaces inherit AC.*

Proof : Let X be closed in an AC space Y. Put $W = Y \setminus X$ Let $M \subseteq X$ be non-open. Then $M' = M \cup W$ is not open in Y and hence there are open U', V' in Y, and a closed F' such that

$$U' \cup (M' \cap V') = U' \cup (F' \cap V')$$

is not open in Y. Put $U = U' \cap X, V = V' \cap X, F = F' \cap X$. We have $U \cup (M \cap V) = U \cup (F \cap V)$ and it remains to be shown that this set is not open in X. Suppose it is. Thus, $U \cup (M \cap V) = T \cap X$ for some T open in Y. But then $U' \cup (M' \cap V') = (T \cup W) \cap (U' \cup V')$ is open. □

5.7. Examples : 1. All countable ordinals have AC (being metrizable). Now suppose all ordinals $< \alpha$ have AC. Then α has : if it is a limit ordinals, the fact follows from 5.4 ; if $\alpha = \beta + 1$ this is a consequence of 5.5. Thus, by induction

each ordinal (with the interval topology) has AC.

2. Quite analogously one can prove that

for any space X with AC, and any ordinal α, $\alpha \times X$ has AC.

3. Recall 5.3. $\omega_1 + 1$ is an example of a space with AC and without AP.

4. By 5.5 and 5.7.1 also the standard example of a non-normal space

$$(\omega_1 + 1) \times (\omega_0 + 1) \setminus \{(\omega_1, \omega_0)\}$$

satisfies AC : removing the closed $\{\omega_1\} \times \omega_0$, which is discrete and hence has AC, we get an open set which has AP.

5. The topology of complements of finite sets on an infinite set X does not satisfy AC. Indeed, consider an infinite M with $X \setminus M$ infinite. Suppose

$$(*) \quad U \cup (M \cap V) = U \cup (F \cap V)$$

with open U, V and closed F. We have to have $U = \emptyset$ and $V \neq \emptyset$ or else the set$(*)$ is open Then $M \cap V = F \cap V$ is infinite. But this is possible only for $F = X$ and $M \cap V = V$ is open again.

5.8. Now we will present a criterion for AC in among the quasidiscrete spaces (recall1.2.4).

An *interval* in $(X, \leq)$ is a subset $J \subseteq X$ such that $x, y \in J$ and $x \leq z \leq y$ imply $z \in J$. In particular, $\uparrow x \cap \downarrow y$ are intervals and we will denote them by$< x, y >$.

Proposition : *A quasidiscrete space $(X, \leq)$ satisfies AC if an only if for each M which is not open there is an $a \in M$ and a $b \notin M$ such that $b < a$ and $< b, a > \cap M$ is an interval.*

Proof : I. Let the condition be satisfied. Then it is easy to check that $U = X \setminus \uparrow b, V = \downarrow a$ and $F = \bigcup \{\uparrow c | c \in < b, a > \cap M\}$ witness for the condition AC.

II. Now let AC be satisfied, let $M \neq \downarrow M$. We have open U, V and closed F such that

$$(*) \quad U \cup (M \cap V) = U \cup (F \cap V)$$

is not open, Thus, in particular there is an $a \in M \cap V$ and $b < a$ with $b \notin U \cup (M \cap V)$. Intersecting (*) with $\downarrow a$ we obtain $(U \cap \downarrow a) \cup (M \cap \downarrow a) = (U \cap \downarrow a) \cup (F \cap \downarrow a)$ and hence, for $G = X \setminus (U \cap \downarrow a)$,

$$M \cap G \cap \downarrow a = F \cap G \cap \downarrow a.$$

As G is closed, for each $x \in G$, $M \cap \uparrow x \cap \downarrow a = F \cap \uparrow x \cap \downarrow a$. In particular, $b \notin U$ and hence $b \in G$ so that

$$M \cap < b, a >= F \cap < b, a >$$

Since F is closed, $F \cap < b, a >$ is an interval. $\square$

5.9. Notes : 1. Thus, e.g. each $(X, \leq)$ such that for every non-void $M \subseteq X$ there is a minimal $a \in M$ (in particular, each finite $(X, \leq)$) satisfies AC. (Indeed, if M is not open, choose $x \in M$ and $b < x, b \notin M$, and take for a a minimal element in $M \cap < b, x >$. Then $M \cap < b, a >= \{a\}$.) On the other hand, no linear $(X, \leq)$ containing an order-dense interval satisfies AC.

2. The condition AP is for quasidiscrete spaces of little interest since it is obviously violated whenever there are a, b, c with $a < b < c$.

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