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The Zariski category of graded commutative rings

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RÉSUMÉ : La théorie des catégories de Zariski donne un cadre général pour le développement de l'algèbre commutative et la géométrie algébrique. On montre ici comment cette théorie générale s'applique avec succès à l'algèbre commutative graduée, par exemple à l'étude des corps gradués algébriquements clos.

The theory of Zariski categories was introduced in the book : "Categories of commutative algebras" (Oxford University Press, 1992) in order to provide a general framework for commutative algebra and algebraic geometry. We show that this general theory applies with some success to graded commutative algebra. For example, on algebraically closed graded fields, it gives better results than the classical theory.

We focus on the category **GradCRng** of graded commutative rings since other categories of graded commutative algebras can be derived directly from it. We prove that **GradCRng** is a Zariski category which is linked by many morphisms of Zariski categories to the Zariski category **CRng** of commutative rings. The category **GradCRng** is isomorphic to the category of coactions of a cogroup in **CRng**, while **CRng** is equivalent to a coslice category of **GradCRng**.

The general theory defines the notions of simple objects and algebraically closed simple objects, predicts that any simple object has an algebraically closed simple extension, and states a good Nullstellensatz with respect to any algebraically closed simple object. On the other hand in their book : "Graded ring theory", Năstăsescu and Van Oystaeyen describe graded fields and their graded algebraic closure, and state a weak Nullstellensatz. It turns out that the two notions of algebraic closeness are different. We prove that

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our notion of algebraically closed simple object is more adequate for algebraic geometry.

Any algebraically closed simple object can be considered as a ground graded field in which we take values. Then any adequately finitely presentable object A in **GradCRng** naturally defines the algebraic set $\Sigma(A)$ of points of A with values in the ground gradued field. Any homogeneous ideal J of A naturally defines a zero set $Z(J)$ in $\Sigma(A)$, and any subset X of $\Sigma(A)$ naturally defines an homogeneous ideal $I(X)$ in A. Our graded Nullstellensatz asserts that $I(Z(J)) = rad(J)$. Subsets of $\Sigma(A)$ of the form $Z(J)$ are called algebraic subsets. They are the closed sets of a topology on $\Sigma(A)$ naturally called the Zariski topology, and are in one-to-one correspondence with radical homogeneous ideals of A. We prove that the graded prime spectrum of A, $Spec_{gr}(A)$, is precisely the universal sober space associated to the space $\Sigma(A)$, and that the projective space, $Proj(A)$, of the graded ring A is precisely the universal sober space associated to the open subspace $\Sigma(A)\backslash Z(I_0)$ of $\Sigma(A)$ where I_0 is the ideal of A generated by all the non zero degree homogeneous elements of A. The scheme structures on $Spec_{gr}(A)$ and $Proj(A)$ are investigated by using the geometrical morphisms of Zariski categories between **GradCRng** and **CRng**.

A paper with full details on this matter will appear in Canadian Mathematical Society Proceedings.

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