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NOTES ON KURATOWSKI-MRÓWKA THEOREMS IN POINT-FREE CONTEXT

by A. PULTR[1] *and A. TOZZI*[1]

Résumé. Le fameux thórème de Kuratowski-Mrówka dit qu'un espace topologique X est compact si et seulement si la projection $X \times Y \longrightarrow Y$ est fermée pour tous les espaces Y. Nous démontrons le théorème de Kuratowski dans le domaine des locales. En particulier nous démontrons qu'un locale A est compact si et seulement si la projection $A \times B \longrightarrow B$ est fermée pour tous les locales B. Pour un numbre cardinal infini α le résultat que nous obtenons n'est pas si satisfaisant. Nous pouvons prouver seulement qu'un locale A est α-compact si et seulement si $A \times B \longrightarrow B$ est fermée pour tous les locales spatiaux α-discrets.

Introduction

The famous Kuratowski theorem characterizes compact spaces X by the fact that for each Y the projection $X \times Y \longrightarrow Y$ is closed. Precisely, Kuratowski [7] proved that, in the realm of metric spaces, if X is compact then the projection π_Y is closed for any metric space Y, Bourbaki [1] proved the same property in the category of Hausdorff spaces and Mrówka [9] established the converse property so that a space X is compact iff the projection π_Y is closed for any space Y. Similarly, by results of Noble [10], Vaughan [11] and Giuli [3], a space X is α-compact ($[\alpha,\beta]$-compact) iff the projection $\pi_Y : X \times Y \longrightarrow Y$ is closed for each α-discrete (α-discrete with character β) space Y. In this paper we consider these phenomena in the pointfree context.

In particular, we prove the Kuratowski theorem in the form that a locale A is compact iff the natural projection $A \times B \longrightarrow B$ is closed for each locale B. For general α the result we present is not so satisfactory: We are able to prove only that a locale A is α-compact iff $A \times B \longrightarrow B$ is closed for all α-discrete spatial B. Thus, the question whether in this case $A \times B \longrightarrow B$ is closed for all α-discrete B remains open. Still, the result answers the question whether there is a class $\mathcal{C}(\alpha)$ of locales such that the α-compactness is characterized by the closedness of the projections $A \times B \longrightarrow B$ with $B \in \mathcal{C}(\alpha)$. More-

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over, the negative part of the statement presented, namely that for an A which is not α-compact there is a *spatial* α-discrete B such that $A \times B \longrightarrow B$ is not closed, is in fact stronger that the respective part of the desired statement.

Finally, we consider a "dual" of the Kuratowski's characterization, namely the question as to which B have the property that the projections $A \times B \longrightarrow B$ are closed for all A . In the classical context this characterizes the quasidiscrete spaces, hence, in the regular case, the discrete spaces, not a very colourful class. In the pointfree context, however, this requirement characterizes the complete Boolean algebras in among regular frames, which is perharps more interesting.

Only basic knowledge of category theory (as e.g. in the introductory chapters of [8]) is assumed. All the necessary facts of pointfree topology are presented in Section 1. For more detail, the reader can consult, e.g., [5].

1. Preliminaries

1.1. Basic conventions: The cardinality of a set X will be indicated as $|X|$. The identity mapping of a set (object) X onto itself will be denoted by id_X or simply id . If $p_i : X_1 \times X_2 \longrightarrow X_i$ is a (categorial) product, the morphisms p_i will be referred to as the ***natural projections***, similarly the coproduct morphisms $X_i \longrightarrow X_1 \oplus X_2$ as the ***natural injections***.

If X is a partially ordered set and $x \in X$, $\uparrow x = \{y | x \leq y\}$.

1.2.Frames and locales: A *frame* is a complete lattice satisfying the distributive law $(\bigvee a_i) \wedge b = \bigvee(a_i \wedge b)$. If X is a topological space, the lattice

$$\Omega(X)$$

of all open sets of X is a frame. Another example is a Boolean algebra. If A, B are frames, a *(frame) homomorphism* $\phi : A \longrightarrow B$ is a mapping preserving *all* joins and *finite* meets. If $f : X \longrightarrow Y$ is a continuous mapping, $\Omega(f) : \Omega(Y) \longrightarrow \Omega(X)$ defined by $\Omega(f)(U) = f^{-1}(U)$ is a frame homomorphism. If A, B are Boolean algebras, the frame homomorphisms $A \longrightarrow B$ coincide with complete Boolean homomorphisms. Denote by **Frm** the category of frames and frame homomorphisms. The correspondence Ω above constitutes a contravariant functor

$$\Omega : \mathbf{Top} \longrightarrow \mathbf{Frm} .$$

The dual of **Frm** is called the *category of locales* and denoted by **Loc** . This makes Ω a covariant functor. In the "localic point of view" one thinks of frames (locales) as generalized spaces; technically, however, we will stay in **Frm** . Thus, e.g., statements on products of generalized spaces will appear as statements on coproducts of frames. A frame (locale) is said to be *spatial* if it is isomorphic to $\Omega(X)$.

The top (bottom) of A will be denoted by 1_A or simply 1 (0_A or 0). The two-element Boolean algebra will be denoted by **2** .

A cover of a frame A is a subset $U \subseteq A$ such that $\bigvee U = 1$.

1.3. Regularity: The pseudocomplement of $a \in A$ is

$$a^* = \bigvee \{x | x \wedge a = 0\} .$$

By the distributivity, $a^* \wedge a = 0$; hence, a^* is the largest element meeting a in zero. We have

$$(\bigvee a_i)^* = \bigwedge a_i^*$$

but the other De Morgan formula does not generally hold. We write

$$a \triangleleft b \quad \textit{for } a^* \vee b = 1.$$

A frame A is said to be *regular* if for each $a \in A$, $a = \bigvee \{x | x \triangleleft a\}$. Note that, trivially, each Boolean algebra is regular.

1.4. Sublocales: A sublocale (cf [4]) of a frame A is a surjective homomorfism $\phi : A \longrightarrow B$ (as, e.g., $\Omega(j)$ for an embedding $j : Y \subseteq X$ of a space). Sublocales are, obviously, in a one-one correspondence (up to isomorphism) with congruences (with respect to general joins and finite meets) on A and will be often dealt with as such. Note that:

(CR) for regular A , a congruence is determined by the set of elements congruent to 1.

Closed sublocales are those given by the congruences

$$x \sim y \text{ iff } x \vee a = y \vee a \text{ (a fixed).}$$

If $\phi : A \longrightarrow B$ is a homomorfism and $\gamma : B \longrightarrow C$ a sublocale, the *image of* γ *under* ϕ , denoted $\phi[\gamma]$ is given by the congruence

$$x \sim y \text{ iff } \gamma\phi(x) = \gamma\phi(y).$$

1.5. Closed homomorfisms: A homomorfism $\phi : A \longrightarrow B$ is said to be *closed* if the image of a closed sublocale is always closed. That is, if for each $b \in B$ there is an $a = \psi(b)$ such that

$$x \vee \psi(b) = y \vee \psi(b) \text{ iff } \phi(x) \vee b = \phi(y) \vee b.$$

This condition can be easily rewritten to

$$\text{(CLOSED)} \qquad \phi(x) \leq y \vee \phi(z) \;\Rightarrow\; x \leq \phi_+(y) \vee z$$

where ϕ_+ is the right Galois adjoint to ϕ. In the regular case it can be reduced (using (CR) above) to

$$\phi(x) \vee y = 1 \;\Rightarrow\; x \vee \phi_+(y) = 1.$$

1.6. Coproducts (details see, e.g. [2] or [5]) :
The coproducts of frames A, B will be denoted by

$$A \xrightarrow{i_A} A \oplus B \xleftarrow{i_B} B.$$

One uses the symbol $a \oplus b$ for $i_A(a) \wedge i_B(b)$. We will need the following two facts:

$$(1) \qquad A \oplus B \text{ is join-generated by the elements } a \oplus b,$$

$$(2) \qquad \text{if } a \oplus b \leq a \oplus c \text{ and } a \neq 0 \text{ then } b \leq c$$

If $\phi_i : A_i \longrightarrow B_i$ $(i = 1, 2)$ are homomorphisms, we write

$$\phi_1 \oplus \phi_2 : A_1 \oplus A_2 \longrightarrow B_1 \oplus B_2$$

for the homomorphism given by $(\phi_1 \oplus \phi_2) \circ i_{A_i} = i_{B_i} \circ \phi_i$. Obviously,

$$(\phi_1 \oplus \phi_2)(a_1 \oplus a_2) = \phi_1(a_1) \oplus \phi(a_2).$$

The coproduct $A \oplus \mathbf{2}$ can be identified with A (then, $a \oplus 1$ is a, and $a \oplus 0 = 0$).

The functor Ω does not generally preserve products (in frame point of view, does not send products to coproducts). The natural homomorphism

$$\mu : \Omega(X_1) \oplus \Omega(X_2) \longrightarrow \Omega(X_1 \times X_2),$$

determined by $\mu \circ \iota_i = \Omega(p_i)$, obviously satisfies the formula

$$\mu(\bigvee_{j\in J} a_j \oplus b_j) = \bigcup_{j\in J} a_j \times b_j .$$

Thus, it is always onto.

1.7. Closed injections: Recall 1.5. We easily see that the natural injection $\iota : B \longrightarrow A \oplus B$ is closed iff

for each $u \in A \oplus B$ there is a $b \in B$ such that
$1 \oplus b \leq u$, and $(1 \oplus v \leq u \vee (1 \oplus w) \Rightarrow v \leq b \vee w)$.

For regular B this reduces to

$$1 \oplus b \leq u \quad \text{and} \quad (u \vee (1 \oplus w) = 1 \Rightarrow b \vee w = 1).$$

2. A characterization of α-compact frames

2.1. In this and the following sections, α is a *regular* cardinal. Recall that a frame A is *α-compact* if each cover of A has a subcover of cardinality $< \alpha$. A space is *α-discrete* if any intersection of $< \alpha$ open sets is an open set.

2.2. Construction: Let A be a frame which is not α-compact. Fix a cover $\mathcal{U}$ such that there is no subcover of cardinality $< \alpha$. Define a space X on A as the underlying set with $\Omega(X) = B$ consisting of the $M \subseteq A$ such that

$$\text{if } 1 \in M \text{ then } \uparrow \bigvee K \subseteq M \text{ for some } K \subseteq \mathcal{U} , \ |K| < \alpha .$$

2.3. Lemma. *In $A \oplus B$ define $c = \bigvee\{u \oplus \uparrow u | u \in \mathcal{U}\}$. Then $(1_A \oplus (A \setminus \{1\})) \vee c = 1_A \oplus 1_B$, and $1 \oplus M \leq c$ only for $M = \emptyset$.*

Proof: Put $z = (1 \oplus (A \setminus \{1\})) \vee c$. For $u \in \mathcal{U}$ we have $u \oplus 1 = u \oplus (A \setminus \{1\}) \vee u \oplus \uparrow u \leq z$. As $\mathcal{U}$ is a cover, $1 = \bigvee \mathcal{U} \oplus 1 \leq z$.

Now consider, for $x \in A$, the homomorphisms $\xi_x : B \longrightarrow \mathbf{2}$ defined by $\xi_x(M) = 1$ iff $x \in M$. We have

$$(id \oplus \xi_x)(u \oplus \uparrow u) = u \oplus \xi_x(\uparrow u) = \begin{cases} u & \text{if } u \leq x , \\ 0 & \text{otherwise} . \end{cases}$$

Thus,

$$(id \oplus \xi_x)(c) = \bigvee \{u | u \leq x\} \leq x \ .$$

Each non-void open M contains an $x \neq 1$. Then

$$(id \oplus \xi_x)(1 \oplus M) = 1 \not\leq x$$

and hence $(1 \oplus M) \not\leq c$. ◇

2.4. Corollary. *Let A not be α-compact. Then there is a space X such that*

(1) X *has only one non-isolated point ,*

(2) X *is α-discrete, and*

(3) *the natural injection* $\Omega(X) \longrightarrow A \oplus \Omega(X)$ *is not closed .*

2.5. Theorem. *A frame A is α-compact iff for each α-discrete space X the natural injection $\Omega(X) \longrightarrow A \oplus \Omega(X)$ is closed.*

Proof: Let X be α-discrete. For $x \in X$ consider the $\xi_x : \Omega(X) \longrightarrow \mathbf{2}$ with $\xi_x(u) = 1$ iff $x \in u$ .Take $y = \bigvee_J a_i \oplus b_i$ in $A \oplus \Omega(X)$ and put

$$M = \{x \in X | (id \oplus \xi_x)(y) = 1\} \ .$$

Thus, for $x \in M$ we have $1 = (id \oplus \xi_x)(y) = \bigvee \{a_i | x \in b_i\}$ and, by α-compactness, there is a $K(x) \subseteq J$, $|K(x)| < \alpha$, such that

$$\bigvee_{K(x)} a_i = 1 \text{ and } b(x) = \bigwedge_{K(x)} b_i = \bigcap_{K(x)} b_i \ni x \ .$$

Put $b = \bigvee_{x \in M} b(x)$. Since obviously for $a_i \in K(x)$ we have $a_i \oplus b(x) \leq y$, we infer $1 \oplus b(x) \leq y$ and consequently $1 \oplus b \leq y$. Now let $1 \oplus v \leq y \vee (1 \oplus w)$. If $x \in v$ we have $1 = (id \oplus \xi_x)(1 \oplus w) = (id \oplus \xi_x)(y) \vee \xi_x(w)$ .Hence either $x \in M \subseteq b$ or $x \in w$. Thus, $v \leq b \vee w$.

If A is not α-compact, consider the X from 2.4. ◇

2.6 Remark. By 2.4.(1), of course, the "testing class" for the α-compactness in 2.5 can be reduced to the α-discrete spaces with at most one non-isolated point.

3. Pointfree Kuratowski Theorem

3.1. We say that a frame A satisfies the **unit decomposition property** (briefly, UD) if for each frame B and each decomposition of the unit

$$1_{A\oplus B} = \bigvee\{a_i \oplus b_i | i \in J\}$$

the system $\{\bigwedge\{b_i | i \in K\} \mid K \subseteq J$ such that $\bigvee\{a_i | i \in K\} = 1\}$ is a cover of B. If this is required only for the frames B from a class $\mathcal{C}$ we speak on the property UD$\mathcal{C}$.

3.2. Lemma. *Let* A *satisfy* UD *and let*

$$1 \oplus v \leq \bigvee\{a_i \oplus b_i \,| i \in J\}.$$

then $\bigvee_K\{\bigwedge_{i\in K} b_i \,|K \subseteq J$ *such that* $\bigvee_J a_i = 1\} \geq v$.

Proof: Consider the sublocale $q : B \longrightarrow [0, v]$ (the interval between 0 and v) given by $q(x) = x \wedge v$. Then

$$1 \leq (id \oplus q)(\bigvee_J a_i \oplus b_i) = \bigvee_J a_i \oplus q(b_i)$$

so that by UD

$$\bigvee\{\bigwedge_K b_i \wedge v \,|K \subseteq J \text{ such that } \bigvee\{a_i \,|i \in K\} = 1\} = v(= 1_{[0,v]}) \ ,$$

that is, $v \wedge \bigvee\{\bigwedge_K b_i \,|K \subseteq J$ such that $\bigvee\{a_i \,|i \in K\} = 1\} = v$. ◇

3.3. Lemma. *Let* A *be compact,* B *arbitrary non trivial (that is,* $1_B \neq 0_B$ *) . Let*

$$1_{A\oplus B} = \bigvee\{a_i \oplus b_i \,|i \in J\} \ .$$

Then there exists a finite $K \subseteq J$ *such that* $\bigvee_K a_i = 1$ *and* $\bigwedge_K b_i \neq 0$.

Proof: In [6] (part of the proof of theorem 3.9, pp. 39-40) . ◇

3.4. Proposition. *Each compact frame satisfies* UD .

Proof: Let A be compact, $1_{A\oplus B} = \bigvee\{a_i \oplus b_i \,|i \in J\}$. Put $U = \{\bigwedge_K b_i \,|K \subseteq J,\ K$ finite such that $\bigvee_K a_i = 1\}$, $c = \bigvee U$. Consider the congruence

$$x \sim y \text{ iff } x \vee c = y \vee c$$

Suppose U is not a cover. Then $\overline{B} = B/\sim$ is not trivial. Let $q : B \longrightarrow \overline{B}$ be the sublocale homomorphism. We have

$$1_{A\oplus\overline{B}} = (id \oplus q)(\bigvee a_i \oplus b_i) = \bigvee a_i \oplus q(b_i) \ .$$

Thus, by 3.3 , there is a finite $K \subseteq J$ such that $\bigvee_K a_i = 1$ and $q(\bigwedge_K b_i) = \bigwedge_K q(b_i) \neq 0$. By the definition of q , $(\bigwedge b_i) \vee c \neq c$ which is a contradition since $\bigwedge_K b_i \in U$. $\diamond$

3.5. Theorem: *The following statements are equivalent:*

(a) A is compact,

(b) A satisfies UD,

(c) For each frame B the natural injection $B \longrightarrow A \oplus B$ is closed ,

(d) for each space X with at most one non-isolated point the natural injection $\Omega(X) \longrightarrow A \oplus \Omega(X)$ is closed.

Proof: (a)$\Rightarrow$(b) is proved in 3.4. (b)$\Rightarrow$(c) : Recall 1.7. Consider $u = \bigvee\{a_i \oplus b_i \,|i \in J\}$ in $A \oplus B$. Put $b = \bigvee\{\bigwedge_K b_i \,|K \subseteq J$ such that $\bigvee_K a_i = 1\}$. Since in each individual case $1 \oplus \bigwedge b_i \leq u$, we have $1 \oplus b \leq u$. Let $1 \oplus v \leq u \vee (I \oplus w)$. By 3.2 , $b \vee w \geq v$.

(c)$\Rightarrow$(d) is trivial and (d)$\Rightarrow$(a) by 2.5. $\diamond$

4. α-Discrete-frames

4.1. Recall the following simple charachteristics of α-discrete spaces (see, e.g., [10] cor.2.3; the proof however, can be left to the reader as an easy exercise) :

Proposition: *A space Y is α-discrete iff for each discrete X with $|X| < \alpha$ the natural projection $X \times Y \longrightarrow Y$ is closed.*

4.2. We introduce the following condition

$$\mathcal{D}(\alpha) \ : \ \text{If } |J| < \alpha \text{ then } (\bigwedge_J a_i) \vee b = \bigwedge_J (a_i \vee b) \ .$$

We have

Theorem: *A frame B satisfies $\mathcal{D}(\alpha)$ iff for each discrete X with $|X| < \alpha$ the natural injection $B \longrightarrow \Omega(X) \oplus B$ is closed .*

Proof: Let the injection be closed. Consider $\{a_i\}_J \subseteq B$ with $|J| < \alpha$, and endow J with discrete topology. Let b be in B. Put $y = \bigvee\{\{i\} \oplus a_i | i \in J\}$. By 1.7 we have an $a \in B$ such that

$$1 \oplus a \leq y \text{ and } (1 \oplus v \leq y \vee (1 \oplus b) \Rightarrow v \leq a \vee b) .$$

Thus in particular $\{i\} \oplus a = (1 \oplus a) \wedge (\{i\} \oplus 1) \leq y \wedge (\{i\} \oplus 1) = \{i\} \oplus a_i$, hence (recall 1.6(2)) $a \leq a_i$ and finally

$$a \leq \bigwedge a_i .$$

Consider $v = \bigwedge_J (a_i \vee b)$. Since $\{i\} \oplus v \leq \{i\} \oplus a_i \vee \{i\} \oplus b$, we have $1 \oplus v \leq y \vee (1 \oplus b)$ and hence $\bigwedge (a_i \vee b) \leq a \vee b \leq (\bigwedge a_i) \vee b$. As the opposite inequality is trivial, $\mathcal{D}(\alpha)$ holds.

On the other hand let B satisfy $\mathcal{D}(\alpha)$ and let X be discrete, $|X| < \alpha$. Consider an element $y \in \Omega(X) \oplus B$. Put $a_x = \bigvee\{c | \{x\} \oplus c \leq y\}$, $a = \bigwedge_{x \in X} a_x$. As $\{x\} \oplus a \leq y$ for all x, we have $1 \oplus a \leq y$. Now let $1 \oplus v \leq y \vee (1 \oplus w)$. That is,

$$1 \oplus v \leq \bigvee_X \{x\} \oplus a_x \vee \bigvee_X \{x\} \oplus w = \bigvee \{x\} \oplus (a_x \vee w)$$

and meeting both sides with $\{x\} \oplus 1$ we infer that $v \leq \bigwedge (a_x \vee w)$, by $\mathcal{D}(\alpha)$, $v \leq a \vee w$. $\Diamond$

4.3. Proposition: *Let Y be a space. Then Y is α-discrete iff $\Omega(Y)$ satisfies $\mathcal{D}(\alpha)$.*

Proof: As soon as we have realized that, for a discrete X, $|X| < \alpha$, $\Omega(X \times Y) = \Omega(X) \oplus \Omega(Y)$, the statement will follow from 4.1 and 4.2.

Consider the μ from 1.6. We have to show that it is one-one, that is, that $\bigcup_i a_i \times b_i = \bigcup_i a'_i \times b'_i$ implies $\bigvee a_i \oplus b_i = \bigvee a'_i \oplus b'_i$. For $x \in X$ put $b(x) = \bigcup\{b_i | x \in a_i\}$. If $\bigcup a_i \times b_i = \bigcup a'_i \times b'_i$, we have also $b(x) = \bigcup\{b'_i | x \in a'_i\}$ and obtain

$$\bigvee a_i \oplus b_i = \bigvee_i \bigvee_{x \in a_i} \{x\} \oplus b_i = \bigvee \{x\} \oplus b(x) = \bigvee a'_i \oplus b'_i . \Diamond$$

4.4. Proposition 4.3 justifies proclaiming a *frame α-discrete* if it satisfies $\mathcal{D}(\alpha)$.

It should be noted that for T_0-spaces the statement of 4.3 is immediate : if Y is not α-discrete, we have an instance of open $u_i \subseteq Y$, $i \in J, |J| < \alpha$ such that $\bigwedge_J u_i \subsetneqq \bigcap_J u_i$. Consider $x \in \bigcap u_i - \bigwedge v_i$. We have $\bigwedge(u_i \vee (Y - \{x\})) = Y$ while $x \notin (\bigwedge u_i) \vee (Y - \{X\})$.

4.5. We are so far unable to tell whether the positive part of 2.5 can be extended to general frames (as in Section 3 for $\alpha = \omega_0$) in case of general α. That is, we do not know whether, if A is α-compact, the injection $B \longrightarrow B \oplus A$ is closed for all α-discrete frames. In the following two paragraphs we will show that this cannot be decided by a simple modification of the techniques from 2.5 and Section 3. Note, in particular, that the proof of 4.6 will be virtually the same as that of 2.5.

4.6. Proposition: *Let $\mathcal{C}(\alpha)$ be the class of all spatial α-discrete frames. Then A is α-compact iff it satisfies $UDC(\alpha)$.*

Proof: If A is not α-compact, take the B from 2.2. By the proof of 2.3, the system corresponding to the decomposition $1 = (u \oplus (A - \{1\})) \vee \bigvee\{u \oplus \uparrow u \mid u \in \mathcal{U}\}$ does not cover the element 1.

On the other hand, let A be α-compact, B α-discrete and $1 = \vee a_i \oplus b_i$. In the notation of the proof of 2.5 we have $y = 1$, hence $M = X$ and $\{\bigwedge\{b_i | i \in K(x)\} | x \in X\}$ is a cover . ◊

4.7. The class $\mathcal{C}(\alpha)$ in 3.6 cannot be replaced by that of all α-discrete frames, not even for $\alpha = \omega_1$. Consider the real line $\mathbf{R}$, $A = \Omega(\mathbf{R})$ and B the Boolean algebra of regular open subset of $\mathbf{R}$. B is α-discrete for all α . The decomposition

$$1_{A \oplus B} = \bigvee_{r \in \mathbf{Z}} \bigvee_{s \in \mathbf{Z}} (r, r+2) \oplus \left(\frac{s}{|r|+1}, \frac{s+1}{|r|+1} \right)$$

($\mathbf{Z}$ the set of integers), if $\bigvee_K a_i = 1$ we always have $\bigwedge_K b_i = 0$, however.

5. Complete Boolean algebras

5.1. We say that a complete lattice is completely distributive if meets distribute over general joins (as in frames), and also joins distribute over general meets. Theorem 4.2 immediately yields

Corollary: *A frame is completely distributive iff for each discrete X the natural injection $B \longrightarrow \Omega(X) \oplus B$ is closed.*

5.2. The following is well known, but easier to prove than quote:

Proposition: *A regular frame is completely distributive iff it is a complete Boolean algebra.*

Proof: Obviously, complete Boolean algebras are completely distributive. On the other hand, let B be regular completely distributive. For $a \in B$ we have $a = \bigvee\{x | x^* \vee a = 1\}$, hence $a^* = \bigwedge\{x^* | x^* \vee a = 1\}$, and finally $a^* \vee a = \bigwedge\{x^* \vee a | x^* \vee a = 1\} = 1$. ◇

5.3. Proposition: Let B be a complete Boolean algebra and A an arbitrary frame. Then each homomorphism $\varphi : B \longrightarrow A$ is closed.

Proof: Recall 1.5; Let $\varphi(x) \vee y = 1$. Meeting both sides with $\varphi(x^*)$ we obtain $\varphi(x^*) \wedge y = \varphi(x^*)$ and hence $\varphi(x^*) \leq y$. Thus, $x^* \leq \varphi_+(y)$ and hence $x \vee \varphi_+(y) \geq x \vee x^* = 1$. ◇

5.4. Theorem.: *Let B be a regular frame. Then the following statements are equivalent:*

(a) B is a complete Boolean algebra,

(b) every homomorfism $\varphi : B \longrightarrow A$ is closed,

(c) for each A the natural injection $B \longrightarrow B \oplus A$ is closed,

(d) for each atomic complete Boolean algebra A the natural injection $B \longrightarrow B \oplus A$ is closed.

Proof: (a)⇒(b) by 5.3, (b)⇒(c)⇒(d) is trivial, and (d)⇒(a) follows from 5.1. ◇

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