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BOOLEAN ALGEBRAS AND ULTRACOMPACTNESS

by Ian PASEKA

Résumé. Dans la théorie des treillis locaux, quelques propriétés treillis-théoriques de certaines classes des treillis locaux (de Lindelöf, paracompact) se comportent mieux que dans la théorie des espaces topologiques. Dans ce contexte, on montre qu'un treillis local est un sous-objet fermé d'un produit des algèbres booléennes complètes dans la catégorie des treillis locaux si, et seulement si, il est ultraparacompact et de dimension 0. Cette proposition est une généralisation du résultat qu'un treillis local est '$\aleph$-compact' dans la catégorie des treillis locaux si, et seulement si, il est de Lindelöf et de dimension 0.

Introduction

This paper deals with the category of ultraparacompact 0-dimensional locales. In [9], we have shown that a 0-dimensional locale is localic '$\aleph$-compact' if and only if it is Lindelöf. We shall show that any ultraparacompact 0-dimensional locale can be described by means of a closed sublocale of a product of copies of a complete (atomic) Boolean algebra.

1 Preliminaries

The basic reference for the present text is the classic book by Johnstone [7], where the interested reader can find unexplained terms and notation concerning the subject. Our terminology and notation agree with the book [7] of Johnstone and with the papers [3], [8], [9]. The aspects of paracompactness and related properties are taken up in Dowker and Strauss [2], Engelking [4], Isbell [6] and Pultr [11],[12], [13]. The results obtained here are closely related to [10].

We now review some terminology from elementary topology and locales.

Recall that a locale is ultraparacompact if any open cover has a discrete open refinement. Clearly, any complete Boolean algebra is ultraparacompact and ultraparacompactness is inherited by arbitrary products and closed sublocales (see [3]).

Let K be a class of locales. We shall say that a locale L is K-regular (resp. K-compact) (see [5], [9]) if it is a sublocale (resp. closed sublocale) of a product of elements of K.

For a set S we put D(S) to be the discrete space on S, $D(S) = \Omega(\mathrm{D}(S))$.
We define elements $b(s) \in \tau(D(S))$ as follows:

$$b(s) = \{s\}$$

for each $s \in S$.

Evidently, $\{b(s) : s \in S\}$ is a disjoint (discrete) cover of $D(S)$. We put *ACBool* to be the class of all complete atomic Boolean algebras.

As in [7], Theorem IV.1.7 we can prove that a locale L is 0-dimensional if and only if it is $D(S)$-regular, S has at least two elements. Note that compact 0-dimensional locales are $D(\mathbf{2})$-compact locales and Lindelöf 0-dimensional locales are $D(\mathbb{N})$-compact locales. Consider 'evaluation' map $u_S(L) : L \longrightarrow D(S)^{C(L,D(S))}$. For each 0-dimensional locale L we shall denote by $\rho_S L$ the closure of the image of $u_S(L)$, $\rho_S(L) : L \longrightarrow \rho_S L$. Clearly, $\rho_S L$ is an ultraparacompact 0-dimensional locale and $\rho_S(L)$ is dense.

For a locale L, we shall denote $Clo(L)$ the set of all clopen i.e. complemented elements of L. Recall that an element $c \in L$ is clopen if and only if $c = \tau(\Phi)(b(s))$, $\Phi : L \longrightarrow D(S)$ being continuous, $s \in S$, S has at least two elements. We shall say that an element $a \in L$ is disjoint if it is a disjoint supremum of clopen elements of L. The corresponding open sublocale will be called disjoint as well.

Consider the evaluation map $u_2(L) : L \longrightarrow D(2)^{C(L,D(2))}$. Clearly, the Banaschewski compactification $B(L)$ of a 0-dimensional locale L is isomorphic to the closure of the image of $u_2(L)$. We put ρL to be the intersection of all disjoint sublocales of $B(L)$ containing L, $\rho(L) : L \longrightarrow \rho L$ will be the corresponding embedding.

Proposition 1.1. *Let L be a locale, $H \subseteq Clo(L)$, H be discrete. Then $\bigvee H$ is a clopen element of L as well.*

Proof. Clearly, H gives us a closed locally finite cover $\{L_{c(h^*)} : h \in H\} \cup \{L_{c(\bigvee H)}\}$ of L, h^* being the complement of h, a system of coinciding continuous maps $f_h : L_{c(h^*)} \longrightarrow D(\mathbf{2})$ for all $h \in H$ and uniquely determined $f_H : L_{c(\bigvee H)} \longrightarrow \Omega(\{0\}) \longrightarrow D(\mathbf{2})$ satisfying assumptions of [10], 1.1 such that $\tau(f_h)(\{1\}) = h \vee h^* = 1$ and $\tau(f_H)(\{1\}) = \bigvee H$.

Certainly there is a continuous map $f : L \longrightarrow D(2)$ such that f coincides with each f_h on $L_{c(h^*)}$ and $\bigvee H = \tau(f)(\{1\})$.
□

Proposition 1.2. *Let L be a locale, S be a set, $a \notin S, H \subseteq Clo(L)$. Then*

(i) *$H = \{h(s) : s \in S\}$ is discrete if and only if there is a continuous map $g : L \longrightarrow D(S \cup \{a\})$ such that $\tau(g)(b(s)) = h(s)$ for each $s \in S$.*

(ii) *$H = \{h(s) : s \in S\}$ is discrete, $\bigvee H = 1$ if and only if there is a continuous map $g : L \longrightarrow D(S)$ such that $\tau(g)(b(s)) = h(s)$ for each $s \in S$.*

Proof. "(i):" Clearly as in 1.1, for each $s \in S$ there is a continuous map $g_s : L_{Cl(u(h(s)))} \longrightarrow D(S \cup \{a\})$ such that $\tau(g_s)(b(s)) = h(s) \vee h(s)^* = 1$. Applying the same argument as in 1.1, we have a continuous map $g_H : L_{c(\bigvee H)} \to D(S \cup \{a\})$ satisfying $\tau(g_H)(b(a)) = \bigvee H$. Of course, g_s, $s \in S$ and g_H are coinciding and we have a continuous map $g : L \to D(S)$ such that $\tau(g)(b(s)) = h(s)$ for all $s \in S$.

The other implication is evident.

"(ii):"Factorizing through the open sublocale $D(S)$ we obtain the required result. □

According to 1.2, we shall say that a set S is ultragood for a locale L, if $H \subseteq Clo(L)$, $\bigvee H = 1$, H is discrete implies there is $G \subseteq Clo(L)$, $\bigvee G = 1$, G discrete, G refines H $(G < H)$ such that

$$G = \{g(s) : s \in S\}, \quad g(s) \wedge g(t) = 0 \quad \text{for } s, t \in S, \ s \neq t.$$

2 Boolean algebras and ultraparacompactness

This section contains the central result of the paper.

Lemma 2.1. *Let L_j be a dense ultraparacompact sublocale of a 0-dimensional locale L and let S be an ultragood set for L_j. If every continuous map $f : L_j \to D(S)$ has a unique extension to a continuous map $g : L \to D(S)$, then $L = L_j$.*

Proof. First, let us check that j is codense i.e. $a \in L$, $j(a) = 1$ implies $a = 1$. Clearly, $a = \bigvee F, F = \{x \in L : x \lhd a\}$. Since L_j is ultraparacompact we have $1 = j(\bigvee H)$, $H < F$, $H \subseteq Clo(L_j)$, H is discrete in L_j. Then there is $G \subseteq Clo(L_j)$, $1 = j(\bigvee G)$, G discrete, $G < H < F$ such that $G = \{g(s) : s \in S\}, g(s) \leq a, g(s) \wedge g(t) = 0$ for $s, t \in S, s \neq t$. By 1.2 (ii) there is a continuous map $f : L \to D(S)$ such that $\tau(f)(b(s)) = g(s)$ for each $s \in S$. We have an extension $h : L \to D(S)$ of f such that $j \circ \tau(h) = \tau(f)$. Clearly, for each $s \in S$

$$\tau(h)(b(s)) \leq j(\tau(h)(b(s)) = \tau(f)(b(s)) = g(s) \leq a.$$

Then $1 = \tau(g)(\bigvee\{b(s) : s \in S\}) \leq a$. Evidently, any codense dense embedding of 0-dimensional locales is an isomorphism.

□

Recall that the Lemma works for the topological case as well.

Theorem 2.2. *Let L be a 0-dimensional locale and let S be an ultragood set for L. Then the following are equivalent:*

(i) *L is ultraparacompact.*
(ii) *If L is dense and $D(S)$−embedded in a 0-dimensional locale L_1 then $L = L_1$.*
(iii) *L is a closed sublocale of a (localic) product of copies of $D(S)$.*
(iv) *L is an intersection of clopen sublocales of $D(S)^{C(L,D(S))}$ containing L.*
(v) *L is an intersection of disjoint sublocales of $B(L)$ containing L.*

Proof. "(i) $\Longrightarrow$ (ii):" Lemma 2.1.

"(ii) $\Longrightarrow$ (iii):" Let us consider the embedding $u_S(L) : L \to D(S)^{C(L,D(S))}$. Clearly, we have a factorization of $u_S(L)$ through $\rho_S(L) : L \to \rho_S L$. By (ii), $\rho_S(L)$ is an isomorphism, so we get L is closed in $D(S)^{C(L,D(S))}$.

"(iii) $\Longrightarrow$ (i):" This is immediate since ultraparacompactness is preserved by arbitrary products and closed sublocales.

"(i) $\Longrightarrow$ (iv):" Consider the embedding $u_S : L \to D(S)^{C(L,D(S))}$, we put $K = D(S)^{C(L,D(S))}$ and let $j : \tau(K) \to \tau(K)$ be the nucleus associated with L. It suffices to show that for each $v \in \tau(K)$ such that $v \neq j(v)$ there is a clopen element $c \in \tau(K)$ such that $j(c) = 1$ and $(c \to v) \neq v$. Let $v \neq j(v)$. By 0-dimensionality, there is $w \in Clo(K)$ such that

(1) $w \vee v > v$, and

(2) $w \leq j(v)$.

Consider the collection $C = \{u \in Clo(K) : u \leq w^* \text{ or } u \leq v\}$. By the 0-dimensionality, $\bigvee C = v \vee w^*$ and $j(\bigvee C) = j(j(v) \vee j(w^*)) = 1$ by (2). By ultraparacompactness, there is $H = \{h(s) : s \in S\}, H < j(C), H \subseteq Clo(L)$, $j(\bigvee H) = 1$, H is discrete in L. By 1.2 (ii), there is a continuous map $g : L \to D(S)$ such that $\tau(g)(b(s)) = h(s) \leq j(c(s))$ for some $c(s) \in C$ for each $s \in S$. Put $c = \bigvee G, G = \{g(s) : s \in S\}$, $g(s) = \tau(\pi_g)(b(s)) \wedge c(s)$. Since G is discrete we have that $c \in Clo(K)$,

$$\begin{aligned} j(c) &= j(\bigvee G) \\ &= j(\bigvee\{g(s) : s \in S\}) \\ &= j(\bigvee\{h(s) : s \in S\}) \\ &= j(\bigvee H) = 1, \end{aligned}$$

$c \to v > v$. Namely, $(w \vee v) \wedge g(s) \leq v$ for each $s \in S$. Thus

$$\begin{aligned} (w \vee v) \wedge c &= (w \vee v) \wedge \bigvee G \\ &\leq (w \vee v) \wedge (\bigvee\{\tau(\pi_g)(b(s)) \wedge c(s) : s \in S\}) \\ &\leq v. \end{aligned}$$

Now, we have $v < w \vee v \leq c \to v$.

"(iv) $\Longrightarrow$ (i):" It follows immediately from the fact that every clopen sublocale of an ultraparacompact locale is ultraparacompact and that ultraparacompactness is inherited by intersections.

"(v) $\Longrightarrow$ (i):" It is immediate since any disjoint sublocale of an ultraparacompact locale is ultraparacompact as well.

"(i) $\Longrightarrow$ (v):" The proof is in fact the same as $(i) \Longrightarrow (iv)$. The only distinction is that we use the density of L in $B(L)$ to prove that G is disjoint in L, $G \subseteq Clo(L)$.

□

Corollary 2.3. *Let L be a 0-dimensional locale. Then the following are equivalent:*

(i) *L is ultraparacompact.*
(ii) *L is embeddable as a closed sublocale of a (localic) product of copies of a discrete space.*
(iii) *L is $ACBool$-compact.*

The following result is in fact Theorem 2.2 of [9].

Corollary 2.4. *A 0-dimensional locale is Lindelöf if and only if it is embeddable as a closed sublocale of a (localic) product of copies of $D(\mathbb{N})$.*

Recall that a locale L is said to be c-*compact*, c is an infinite cardinal, if any cover of L has a subcover of a cardinality less or equal c. Clearly, in the presence of the Tychonoff condition, any product of c-compact locales is c-compact (see [3]).

Corollary 2.5. *A 0-dimensional locale is ultraparacompact and* c*-compact if and only if it is embeddable as a closed sublocale of a (localic) product of copies of a complete atomic Boolean algebra which has at most* c *atoms.*

Proof. We put $S = \mathrm{c}$. Then $D(\mathrm{c})$ has exactly c atoms. The rest follows from 2.2. □

Now, we will compare the situation described in Theorem 2.2 with topology. Each of the properties (i) through (v) has its topological analogue, (i_T), (ii_T), etc., which we shall obtain by writing 'X' instead of 'L', 'D(S)' instead of '$D(S)$', etc. By the topological version of 2.2, we have $(i_T) \Longrightarrow (ii_T)$, evidently $(ii_T) \Longleftrightarrow (iii_T)$, $(i_T) \Longrightarrow (iv_T)$, $(i_T) \Longrightarrow (v_T)$. The difference between locales and topology is that, though any ultraparacompact 0-dimensional space is a closed subspace of a product of copies of a discrete space, a closed subspace of a product of copies of a discrete space need not be ultraparacompact while the $ACBool$-compactness is equivalent to ultraparacompactness.

Finally, we shall establish the (c-*compact) ultraparacompactification* for locales.

Theorem 2.6. *(* c*-compact ultraparacompactification for locales) The inclusion functor from the category of* c*-compact ultraparacompact 0-dimensional locales has a left adjoint ρ_{c}. Moreover, the unit $\rho_{\mathrm{c}}(L) : L \longrightarrow \rho_{\mathrm{c}}L$ of the adjunction in Loc is a regular monomorphism if and only if L is 0-dimensional.*

Theorem 2.7. *(ultraparacompactification for locales) The inclusion functor from the category of ultraparacompact 0-dimensional locales has a left adjoint ρ. Moreover, the unit $\rho(L) : L \longrightarrow \rho L$ of the adjunction in Loc is a regular monomorphism if and only if L is 0-dimensional.*

Proof. The same machinery is used as in [7]. □

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