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BUSISO P. CHISALA
MBILA-MAMBU MAWANDA

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COUNTING MEASURE FOR KURATOWSKI FINITE PARTS AND DECIDABILITY

by Busiso P. CHISALA and MAWANDA Mbila-Mambu

RÉSUMÉ. Le but de l'article est de montrer qu'il existe une "counting measure" sur les parties Kuratowski finies d'un objet d'un topos **E** si et seulement si l'égalité sur cet objet est presque décidable. La décidabilité de l'objet équivaut à l'existence d'une "counting measure" forte. Quelques propriétés supplémentaires équivalentes à la loi de De Morgan sont aussi établies.

0. INTRODUCTION.

The goal of this paper is to establish a necessary and sufficient condition for the existence of a counting measure with values in the natural number object, on Kuratowski finite parts of an object X in a topos **E**. The starting point is a question posed to the second author by F.E.J. Linton: how to compute counting measures on the object K(X) of K-finite parts of an object X of **E**?

In Section 1, we show that X is almost decidable if a counting measure exists on K(X). When a strong condition is required on the measure, then X must be decidable. These observations lead to connections between logical properties of **E** and the existence of a counting measure (resp. strong counting measure) on K(X) for every object X of **E**, using a slight extension of 2.6 in [1] in the case of almost decidability.

In Section 2, we show that the sufficient conditions of Section 1 are also necessary. There is a counting (resp. strong counting) measure on K(X) if and only if X is almost decidable (resp. decidable). A corollary is that **E** satisfies De Morgan's law (resp. is a Boolean topos) iff there is a counting (resp. strong counting) measure for every X in **E**. This, together with Proposition 1.5 adds a further characterization to the list initated by P.T. Johnstone [2].

The last section emphasizes the fact that a counting measure is monotone and the natural number object is not well suited for counting measure. This raises the question of finding a suitable object.

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1. NECESSARY CONDITIONS.

Let **E** be a topos. As in [4], by a part of an object X of **E** we mean, strictly speaking, a term of type PX in the language of **E**. We will write $x \in X$ and $A \in PX$ for x a term of type X and A of type PX. The object of K-finite parts of X will be denoted by K(X).

DEFINITION 1.1. Suppose **E** has a natural number object N. A *counting measure* (with values in N) on K(X) is a morphism $\mu: K(X) \to N$ satisfying:

(1) $A \cap B = \emptyset \Rightarrow \mu(A \cup B) = \mu(A) + \mu(B)$.

(2) $\mu(\{x\}) = 1$.

An immediate consequence of (1) is that $\mu(\emptyset) = 0$.

EXAMPLE 1.2. Recall that an object X of a topos **E** is antidecidable if the following holds: $\neg\neg(x = y)$. Using 1.9 (i) of [4], we define $\mu: K(X) \in N$ by:

$$\mu(A) \begin{cases} = 0 & \text{if } A = \emptyset \\ = 1 & \text{otherwise.} \end{cases}$$

When X is antidecidable, μ is a counting measure on K(X). Indeed, by 1.9 (i) of [4], antidecidability of X implies

$$A \cap B = \emptyset \Rightarrow (A = \emptyset \vee B = \emptyset).$$

Now, let X be an arbitrary set and 1 a singleton. In Sierpinski topos $\mathbf{S}^2$ the object $Y = X \to 1$ is always antidecidable. When X has at least two elements, the measure of 1.2 on K(Y) does not satisfy:

$$\mu(A) = 1 \Rightarrow A \text{ is a singleton.}$$

This motivates:

DEFINITION 1.3. A *strong counting measure* on K(X) is a morphism $\mu: K(X) \to N$ satisfying (1) and

(2') $\mu(A) = 1 \Leftrightarrow$ A is a singleton.

Observe that condition (2) in 1.1 says that the square

$$\begin{array}{ccc} X & \longrightarrow & 1 \\ \big\downarrow \{\,\} & & \big\downarrow 1 \\ K(X) & \xrightarrow{\mu} & N \end{array}$$

commutes and (2') that it is a pullback.

By an *almost decidable* formula of the language of **E**, we mean a formula φ such that the following holds

$$\neg\varphi \vee \neg\neg\varphi .$$

When $x = y$ is almost decidable for $x \in X$, we say that X is almost decidable. A formula φ is decidable if $\varphi \vee \neg\varphi$ is valid and X is decidable when $x = y$ is decidable. The *object of $\in$-almost decidable parts of* X is defined by

$$\forall x \; (\neg(x \in A) \vee \neg\neg(x \in A))$$

and that of $\in$-decidable parts is defined by

$$\forall x \; (x \in A \vee \neg(x \in A))$$

(i.e., complemented parts of X). We say that X is $\in$-almost decidable if every part of X is $\in$-almost decidable.

PROPOSITION 1.4. *Let* X *be an object of a topos* **E** *with natural number object. Then:*

(a) X *is almost decidable if there is a counting measure on* K(X).

(b) X *is decidable if there is a strong counting measure on* K(X).

PROOF. (a) Suppose that there is a counting measure μ on K(X). Let $\{x,y\}$ be the term

$$\{t \mid t = x \vee t = y\}.$$

In view of (1) and (2), $\mu(\{x,y\}) = 1$ implies $\neg\neg(x = y)$. On the other hand $\neg(\mu(\{x,y\}) = 1)$ gives $\neg(x = y)$ using (2). The result follows from the decidability of N.

(b) With a strong counting measure , $\mu(\{x,y\}) = 1$ implies $x = y$. ∎

We define two axioms related to toposes with natural number object:

(CM) For every object X of **E** there is a counting measure on K(X).

(SCM) For every object X of **E** there is a strong counting measure on K(X)

From 2.6 of [1] and 1.4 above, **E** is boolean if it satisfies (SCM). Furthermore, we claim that **E** satisfies De Morgan's law when it satisfies (CM). We need an analogue of 2.6 (iii) where decidability is replaced by almost decidability. We will do more, the following proposition includes a general version of 1.5 of [4].

Recall that 2 is linearly ordered and the trichotomy is satisfied. Any part of 2 is bounded above and below. Furthermore 2 is defined as the extension of $\alpha = 0 \vee \alpha = 1$ where $\alpha \in \Omega$.

PROPOSITION 1.5. *The following properties are equivalent for a topos* **E**:

(0) **E** *satisfies De Morgan's law.*
(i) Every object of **E** *is almost decidable.*
(ii) Ω *is almost decidable.*
(iii) Every object of **E** *is* $\in$*-almost decidable.*
(iv) 2 is $\in$*-almost decidable.*
(v) Every part of 2 has an infimum.
(vi) Every part of 2 has a supremum.

PROOF. First observe that the following implications are trivial:

$$(0) \Rightarrow (i), \ (iii) \Rightarrow (i) \Rightarrow (ii) \text{ and } (iii) \Rightarrow (iv).$$

Let $[\![\cdot \in \cdot]\!]: X \times PX \to \Omega$ be the characteristic morphism of membership. If (ii) holds then for $x \in A$ and $A \in PX$, either

$$\neg([\![x \in A]\!] = 1) \text{ or } \neg\neg([\![x \in A]\!] = 1).$$

Since $[\![x \in A]\!] = 1$ iff $x \in A$, (iii) follows. Suppose (iv) holds. For $A \in P2$ either $\neg(0 \in A)$ or $\neg\neg(0 \in A)$. In the first case, 1 is the infimum, being a lower bound of A. In the other case 0 is the infimum. Both facts use trichotomy and the definition of 2, and yield (v). Suppose (v) is true. For any formula φ in the language of **E** the term

$$\{\alpha \mid \alpha = 1 \vee (\alpha = 0 \wedge \varphi),$$

has an infimum α_0. From decidability of 2, a comparison of α_0 and 1 gives $\neg\varphi$ or $\neg\neg\varphi$, whence (0). By symmetry the equivalence with (vi) follows. ■

Notice that Boolean versions of (i) to (vi) follow on replacing almost decidable, part, infimum and supremum by, respectively. decidable, inhabited part, minimum and maximum. From 1.4, 2.6 of [1] and 1.5, we infer:

COROLLARY 1.6. *(a) A topos* **E** *satisfies De Morgan's law if it satisfies* (CM).

(b) A topos **E** *is Boolean if it satisfies* (SCM).

2. SUFFICIENT CONDITIONS.

Let X be an object of a topos **E**. Recall that K(X) is defined by

$$K(X) = \bigcap\{P \in PPX \mid \emptyset \in P \wedge \forall P \in P\ \forall x\ (P \cup \{x\}) \in P)\}.$$

Notice that this asserts an induction principle for K-finite parts of X. To show that almost decidability and decidability in 1.4 are sufficient conditions, we begin with two lemmas.

MAIN LEMMA 2.1. *The following properties hold for a topos* **E**:

(a) X *is almost decidable iff every* K*-finite part of* X *is* $\in$*-almost decidable.*

(b) X *is decidable iff any* K*-finite part of* X *is* $\in$*-decidable.*

PROOF. We will prove only (a), the other statement was established as a definition of decidability (see 2.2 (iv) of [1]). The sufficient condition follows immediately from the fact that singletons are K-finite parts. For necessity, we use induction on K(X). It is clear that $\emptyset$ is $\in$-almost decidable. For $x, y \in X$, either $\neg(y \in A)$ or $\neg\neg(y \in A)$ and either $\neg(x = y)$ or $\neg\neg(x = y)$. It is easy to infer the following:

$$\neg(y \in A) \text{ and } \begin{cases} \neg(x = y) & \text{implies } \neg(y \in A \cup \{x\}) \\ \neg\neg(x = y) & \text{implies } \neg\neg(y \in A \cup \{x\}) \end{cases}$$

$$\neg\neg(y \in A) \text{ implies } \neg\neg(y \in A \cup \{x\}).$$

Thus, if $A \in K(X)$ is $\in$-almost decidable then $A \cup \{x\}$ is $\in$-almost decidable. ∎

LEMMA 2.2. *If* μ *is a counting measure on* K(X) *then the following holds:*

$$\mu(A \cup \{x\}) = \begin{cases} \mu(A) + 1 & \textit{if } \neg(x \in A) \\ \mu(A) & \textit{otherwise.} \end{cases}$$

PROOF. By 1.4 we have assumed that X is almost decidable. By 2.1 all its K-finite parts are $\in$-almost decidable. In particular μ is well defined. Applying Axioms (1) and (2) of a counting measure,

$$\mu(A \cup \{x\}) = \mu(A) + 1 \text{ when } \neg(x \in A).$$

Now suppose that $\neg\neg(x \in A)$. As N is a decidable object and

$$\neg(\mu(A \cup \{x\})) = \mu(A) \text{ implies } \mu(x \in A)$$

then $\neg\neg(x \in A)$ gives

$$\mu(A \cup \{x\}) = \mu(A). \quad \blacksquare$$

THEOREM 2.3. *The following properties hold for an object* X *of a topos* **E***:*

(a) X *is almost decidable iff there is a counting measure on* K(X).

(b) X *is decidable iff there is a strong counting measure on* K(X).

PROOF. From 1.4. it suffices to prove sufficient conditions Let μ: $K(X) \rightarrow N$ be defined (inductively) by:

$$\mu(\varnothing) = 0,$$

$$\mu(A \cup \{x\}) = \begin{cases} \mu(A) + 1 & \text{if } \neg(x \in A) \\ \mu(A) & \text{otherwise.} \end{cases}$$

Here, we have used the Main Lemma. We will verify

$$\forall A \in K(X) [\forall B \, (A \cap B = \varnothing \Rightarrow \mu(A \cup B) = \mu(A) + \mu(B)]$$

and

$$\mu(A) = 1 \Rightarrow A \text{ is a singleton.}$$

when X is decidable. For the first verification, we make an induction on A, the case A empty being obvious. Suppose the assertion is valid for A, and let $x \in X$. First note that

$$\mu((A \cup \{x\}) \cup B) = \begin{cases} \mu(A \cup B) + 1 & \text{if } \neg(x \in A \cup B) \\ \mu(A \cup B) & \text{otherwise.} \end{cases}$$

Suppose that $(A \cup \{x\}) \cap B = \varnothing$. It follows that $\neg(x \in B)$ and $A \cap B = \varnothing$. By the Main Lemma

$$\neg(x \in A \cup B) \text{ or } \neg\neg(x \in A \cup B).$$

Here $\neg(x \in A \cup B)$ is equivalent to $\neg(x \in A)$, so

$$\mu((A \cup \{x\}) \cup B) = \mu(A \cup B) + 1 = \mu(A) + \mu(B) + 1 = \mu(A \cup \{x\}) + \mu(B)$$

as desired. When $\neg\neg(x \in A \cup B)$, we have

$$\mu(A) = \mu(A \cup \{x\}) \quad \text{as } \neg\neg(x \in A).$$

Now

$$\mu((A\cup\{x\})\cup B) = \mu(A\cup B) = \mu(A) + \mu(B) = \mu(A\cup\{x\}) + \mu(B).$$

To verify

$$\mu(A) = 1 \Rightarrow A \text{ is a singleton}$$

we make an induction on A. supposing X is decidable. The case A empty is obvious. Suppose that the assertion is valid for A and that $x \in X$. By 2.1, either $x \in A$ or $\neg(x \in A)$. If $x \in A$ then by the induction hypothesis $A\cup\{x\} = A$ is a singleton when $\mu(A\cup\{x\}) = 1$. Now let $\neg(x \in A)$ and

$$\mu(A\cup\{x\}) = 1 = \mu(A) + 1.$$

Then $\mu(A) = 0$ implies that $A = \emptyset$. This last assertion follows by induction using the definition of μ and axiom (2). ■

COROLLARY 2.4. *For a topos* **E** *with a natural number object. the following assertions hold:*

(a) **E** *satisfies De Morgan's law iff it satisfies* (CM).

(b) **E** *is Boolean iff it satisfies* (SCM).

3. COMMENTS.

OBSERVATION 3.1. As $\mu(\emptyset)$ must be equal to 0. from the induction principle on K(X) Lemma 2.2 asserts that when a counting measure exists on K(X). it is unique and given as in the proof of Theorem 2.3.

In order to prove monotonicity of a counting measure. we need a few more observations.

OBSERVATION 3.2. Let X be an object in a topos **E**. For $A \in PX$ and $x \in X$. we define

$$\langle x_A \rangle = \{y \in A \mid \neg\neg(x = y)\}.$$

If X is almost decidable then for all $A \in K(X)$ and $x \in A$. $A \setminus \langle x_A \rangle$ and $\langle x_A \rangle$ are K-finite. Proofs are by induction in K(X).

OBSERVATION 3.3. Let $A_d(B)$ be the predicate on K(X) defining antidecidable K-finite parts of an object X of **E** (i.e..

$$A_d(B) \equiv \forall x.y \in B(\neg\neg(x = y)).$$

Example 1.2 can be extended by: the measure of an antidecidable part is 0 or 1. In fact, the property trivially holds for $\emptyset$. Suppose that for $B \in K(X)$,

$$A_d(B) \Rightarrow \mu(B) = 0 \vee \mu(B) = 1.$$

Let $x \in X$ be such that $A_d(B \cup \{x\})$. By 1.9 (i) of [4], either B is empty or inhabited. If B is empty then $\mu(B \cup \{x\}) = 1$ by axiom (2). If B is inhabited, then $\neg\neg(x \in B)$ because $\neg(x \in B)$ contradicts $A_d(B \cup \{x\})$, B being inhabited. So $\mu(B \cup \{x\}) = \mu(B)$ is either 0 or 1 since $A_d(B \cup \{x\})$ implies $A_d(B)$. Note that if B is antidecidable and inhabited then $\mu(B) = 1$.

PROPOSITION 3.4. *A counting measure* μ *on* K(X) *is always a monotone morphism (i.e.,* $A \subset B$ *implies* $\mu(A) \leq \mu(B)$*).*

PROOF. We prove by induction the following:

$$\forall A \, [\forall B \, (A \subset B \Rightarrow \mu(A) \leq \mu(B)] \qquad A, B \in K(X).$$

The property trivially holds for A empty. Suppose that the property holds for A. Let $x \in X$ be such that $A \cup \{x\} \subset B$. As X is almost decidable, either $\neg\neg(x \in A)$ or $\neg(x \in A)$. In the former case,

$$\mu(A \cup \{x\}) = \mu(A) \leq \mu(B).$$

When $\neg(x \in A)$, $B = (B \backslash \langle x_B \rangle) \cup \langle x_B \rangle$, a disjoint union of K-finite parts of X. Here, $A \subset B \backslash \langle x_B \rangle$, so $\mu(A) \leq \mu(B \backslash \langle x_B \rangle)$ by our induction hypothesis. So

$$\mu(A \cup \{x\}) = \mu(A) + 1 \leq \mu(B \backslash \langle x_B \rangle) + 1 = \mu(B)$$

since $\langle x_B \rangle$ is antidecidable and inhabited. ■

OBSERVATION 3.5. We have shown that the existence of a strong counting measure on K(X) is equivalent to decidability of X. In some sense this condition on the measure explains the suitability of the natural number object in describing Kuratowski finiteness for a decidable object. In fact, as pointed out in [4], K-finiteness for decidable objects is precisely local cardinal finiteness as defined in 1.1 [4].

OBSERVATION 3.6. For arbitrary X however, N is evidently not well suited for counting measure. One problem is clear in 1.4 - existence imposes conditions on X. Another is that when it does exist, the measure does not reflect the complexity of K(X), as was demonstrated for antidecidable objects that are not decidable.

A natural question is to determine what a "Kuratowski natural number object" remedying N's deficiences would be in general. This requires a careful examination of K(X) in Grothendieck toposes, or possibly a general axiomatization. We intend to investigate this matter and relations between such an object and the object of natural numbers in the near future.

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UNIVERSITY OF ZIMBABWE
MATHEMATICS DEPARTMENT
P.O. BOX MP 167
MOUNT PLEASANT.
HARARE. ZIMBABWE