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GEORGE JANELIDZE

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WHAT IS A DOUBLE CENTRAL EXTENSION? (the question was asked by Ronald Brown)

by George JANELIDZE

RÉSUMÉ. Cet article développe une notion d'extension centrale double comme cas particulier d'un objet normal dans une catégorie. Ceci répond à une question de R. Brown relative à une "théorie de Galois dans les catégories".

The theory of central extensions of groups can be considered as a particular case of the "Galois theory in categories" given in [2].

In a similar manner we obtain here the new notion of "double central extension" (as a particular case of a normal object in a category in the sense of [2]). This answers a question R. Brown asked the author in Tbilisi in 1987.

The idea comes from the generalization of the Hopf formula given by R. Brown and G. Ellis [1], but the method and results are independent of [1].

The results of this paper were reported at the International Category Theory Meeting, 1989, in Bangor.

The paper contains two sections. The first section recalls with some improvements a part of [2] and continues that work. The second section considers the *double central extensions of groups* of our title.

1. ON GALOIS THEORY IN CATEGORIES

Let $\mathcal{C}$ be a category with pullbacks and $\mathcal{E}$ a class of morphisms in $\mathcal{C}$ containing all isomorphisms, closed under composition and such that for a pullback diagram

$$\begin{array}{ccc} C_1 \times_{C_0} C_2 & \xrightarrow{\pi_2} & C_2 \\ {\scriptstyle \pi_1}\downarrow & & \downarrow{\scriptstyle \sigma_2} \\ C_1 & \xrightarrow[\sigma_1]{} & C_2 \end{array}$$

$\sigma_1,\sigma_2 \in \mathcal{E}$ implies $\pi_1 \in \mathcal{E}$ (and so $\pi_2 \in \mathcal{E}$) .

For a fixed $(\mathcal{C},\mathcal{E})$ we say that an object (C,σ) of $(\mathcal{C} \downarrow C_0)$ is an *extension* of C_0 if $\sigma \in \mathcal{E}$; the full subcategory of $(\mathcal{C} \downarrow C_0)$ with objects all extensions of C_0 is denoted by $\mathcal{E}(C_0)$.

For an extension (C,σ) of C_0 there is the composition functor $U^\sigma : \mathcal{E}(C) \longrightarrow \mathcal{E}(C_0)$ defined by $(A,\alpha) \mapsto (A,\sigma\alpha)$ and its right adjoint: the pullback functor $P^\sigma : \mathcal{E}(C_0) \longrightarrow \mathcal{E}(C)$, defined by

$$(A,\alpha) \mapsto (C \times_{C_0} A,\ proj_1) .$$

A "Galois structure" (Definition 3.1 of [2]) consists of an adjunction

$$(I,H,\eta,\varepsilon) : \mathcal{C} \longrightarrow \mathcal{X}$$

and classes $\mathcal{E}$ and $\mathcal{Z}$ of morphisms of $\mathcal{C}$ and $\mathcal{X}$ respectively such that: $(\mathcal{C},\mathcal{E})$ satisfies the conditions above; $(\mathcal{X},\mathcal{Z})$ satisfies the same conditions; $I(\mathcal{E}) \subset \mathcal{Z}$; $H(\mathcal{Z}) \subset \mathcal{E}$; ε is an isomorphism[1]; $\eta_C \in \mathcal{E}$ for each $C \in \mathrm{Ob}\mathcal{C}$.

We consider a fixed Galois structure Γ and so assume that $\mathcal{C},\mathcal{X},I,H,\eta,\varepsilon,\mathcal{E}$ and $\mathcal{Z}$ are fixed.

Let $I^C : (\mathcal{C} \downarrow C) \longrightarrow (\mathcal{X} \downarrow I(C))$ be the functor induced by I . It has the right adjoint H^C constructed as follows: for $(X,\varphi) \in \mathrm{Ob}(\mathcal{X} \downarrow I(C))$ consider the pullback diagram:

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[1] this is not necessary, but it holds in the case considered in section 2.

$$\begin{array}{ccc} C \times_{HI(C)} H(X) & \xrightarrow{\mathrm{proj}_2} & H(X) \\ {\scriptstyle \mathrm{proj}_1}\downarrow & & \downarrow{\scriptstyle H(\varphi)} \\ C & \xrightarrow[\eta_C]{} & HI(C) \end{array}$$

and

$$H^C : (X,\varphi) \mapsto (C \times_{HI(C)} H(X), \mathrm{proj}_1) \ .$$

Moreover I^C itself induces a functor $\mathcal{E}(C) \longrightarrow \mathcal{Z}(I(C))$ and H^C induces a functor $\mathcal{Z}(I(C)) \longrightarrow \mathcal{E}(C)$; denote them by $I^{C,\Gamma}$ and $H^{C,\Gamma}$ respectively.

Let (C,σ) and (A,α) be extensions of C_0 ; (A,α) is *split* over (C,σ) with respect to Γ if the canonical morphism

$$P^{\sigma}((A,\alpha)) \longrightarrow H^{C,\Gamma}\, I^{C,\Gamma}\, P^{\sigma}((A,\alpha))$$

is an isomorphism; the full subcategory of $\mathcal{E}(C_0)$ with objects all extensions split over (C,σ) with respect to Γ is denoted by $\mathrm{Spl}_{\Gamma}((C,\sigma))$ (3.2 of [2]).

Clearly we have

PROPOSITION 1.1. *Let (C,σ) and (A,α) be extensions of C_0 . If the counit $I^{C,\Gamma} H^{C,\Gamma} \longrightarrow 1_{\mathcal{Z}(I(C))}$ is an isomorphism, then the following conditions are equivalent:*

(a) *(A,α) is split over (C,σ) with respect to Γ ;*

(b) *the morphism*

$$\langle \mathrm{proj}_1, \eta_{C \times_{C_0} A} \rangle : C \times_{C_0} A \longrightarrow C \times_{HI(C)} HI(C \times_{C_0} A)$$

is an ismorphism;

(c) *there exists $(X,\varphi) \in \mathrm{Ob}\mathcal{Z}(I(C))$ such that the objects $(C \times_{C_0} A, \mathrm{proj}_1)$ and $(C \times_{HI(C)} H(X), \mathrm{proj}_1)$ of $\mathcal{E}(C)$ are isomorphic.* □

After that we can prove

PROPOSITION 1.2. *If (C,σ) is split over (C,σ) with respect to Γ and the counit above is an isomorphism, then*

$U^{\sigma} H^{C,\Gamma}((X,\varphi))$ *is split over* (C,σ) *with respect to* Γ *for each* $(X,\varphi) \in \mathrm{Ob}Z((I(C))$.

PROOF. The diagram

$$\begin{array}{ccc}
C \times_{C_0} C & \xrightarrow{\langle \mathrm{proj}_1, \eta_{C \times_{C_0} C} \rangle} & C \times_{HI(C)} HI(C \times_{C_0} C) \\
\mathrm{proj}_2 \downarrow & & \downarrow \mathrm{proj}_2 \\
C & & HI(C \times_{C_0} C) \\
\eta_C \downarrow & & \downarrow HI(\mathrm{proj}_2) \\
HI(C) & = & HI(C)
\end{array}$$

commutes and so we can write

$$(C \times_{C_0} C) \times_{HI(C)} H(X) \cong C \times_{HI(C)} HI(C \times_{C_0} C) \times_{HI(C)} H(X)$$

and this isomorphism commutes with the first projection. The left side is isomorphic to $C \times_{C_0} (C \times_{HI(C)} H(X))$ and the right side to

$$C \times_{HI(C)} H(I(C \times_{C_0} C) \times_{I(C)} X) ,$$

and these isomorphisms commute with the first projection too. Now the implication (c) $\Rightarrow$ (a) of 1.1 yields that

$$U^{\sigma}H^{C,\Gamma}((X,\varphi)) = (C \times_{HI(C)} H(X), \sigma.\mathrm{proj}_1)$$

is split over (C,σ) with respect to Γ . □

From this proposition it follows that Definition 3.3 of [2] is equivalent to the following one:

DEFINITION 1.3. An extension (C,σ) of C_0 is Γ-*normal* if the following conditions hold:

(a) the counit $I^{C,\Gamma} H^{C,\Gamma} \to 1_{Z(I(C))}$ is an isomorphism;
(b) the functor P^{σ} is monadic;
(c) (C,σ) is split over (C,σ) with respect to Γ . □

We shall write $(A,\alpha) \leq_\Gamma (C,\sigma)$ if (A,α) is split over (C,σ) with respect to Γ and (C,σ) satisfies the conditions (a) and (b) of 1.3.

DEFINITION 1.4.
(a) An extension (A,α) of C_0 is a *Γ-covering* if there exists an extension (C,σ) of C_0 such that $(A,\alpha) \leq_\Gamma (C,\sigma)$;

(b) An extension (C,σ) of C_0 is a *weak universal Γ-covering* if it is a Γ-covering and for each extension (A,α) of C_0 which is a Γ-covering one has $(A,\alpha) \leq_\Gamma (C,\sigma)$. □

The full subcategory of $\mathcal{E}(C_0)$ with objects all Γ-coverings is denoted by $\mathrm{Spl}(\Gamma,C_0)$. If there exists a weak universal Γ-covering (C,σ) then we can write

$$\mathrm{Spl}(\Gamma,C_0) = \mathrm{Spl}_\Gamma((C,\sigma)) \ .$$

Moreover, clearly (C,σ) is a Γ-normal extension of C_0 and so using the Theorem 3.6 of [2] we obtain the following description of this category:

THEOREM 1.5. *Let (C,σ) be an extension of C_0 which is a weak universal covering and let $G = \mathrm{Gal}_I((C,\sigma))$ be the Galois groupoid of (C,σ) in the sense of* [2]. *Then the category $\mathrm{Spl}(\Gamma,C_0)$ is equivalent to the full subcategory of $\mathcal{X}^G$ with objects all internal functors $F = (F_0,\pi,\xi) : G \longrightarrow \mathcal{X}$ such that (F_0,π) is an extension of $I(C)$.* □

2. DOUBLE CENTRAL EXTENSIONS

Let Γ be the following Galois structure: $\mathcal{E}$ is the full subcategory of the category Ar(Groups) with objects all surjective homomorphisms of groups; if A is an object of $\mathcal{E}$ then we write $A = (A^0,\zeta_A,A^1)$, where $\zeta_A : A^1 \longrightarrow A^0$ is a group epimorphism, and a morphism $\alpha : A \longrightarrow B$ in $\mathcal{E}$ is a commutative square

$$\begin{array}{ccc} A^1 & \xrightarrow{\alpha^1} & B^1 \\ {\scriptstyle\zeta_A}\downarrow & & \downarrow{\scriptstyle\zeta_B} \\ A^0 & \xrightarrow[\alpha^0]{} & B^0 \end{array}$$

in the category of groups:

$\mathcal{E}$ consists of all morphisms $\alpha : A \longrightarrow B$ such that for each $b^1 \in B^1$ and $a^0 \in A^0$ with $\alpha^0(a^0) = \zeta_B(b^1)$ there exists $a^1 \in A^1$ with $\zeta_A(a^1) = a^0$ and $\alpha^1(a^1) = b^1$, i.e. the homomorphism

$$<\zeta_A,\alpha^1> : A^1 \longrightarrow A^0 \times_{B^0} B^1$$

is surjective;

$\mathcal{X}$ is the full subcategory of $\mathcal{E}$ with objects all central extensions;

$\mathcal{Z} = \mathcal{X} \cap \mathcal{E}$;

I is the "centralization", i.e.

$$I(A) = \left[\begin{array}{c} A^1/[\mathrm{Ker}\ \zeta_A,A^1] \\ \downarrow \bar{\zeta}_A \\ A^0 \end{array}\right],$$

where $\bar{\zeta}_A$ is the homomorphism induced by ζ_A, i.e. $\bar{\zeta}_A(cls(a^1)) = \zeta_A(a^1)$ for each $a^1 \in A^1$;

H is the inclusion functor;

$\eta_A : A \longrightarrow HI(A)$ is

$$\begin{array}{ccc} A^1 & \longrightarrow & A^1/[\mathrm{Ker}\ \zeta_A,A^1] \\ \zeta_A \downarrow & & \downarrow \bar{\zeta}_A \\ A^0 & = & A^0 \end{array},$$

where the upper arrow is the canonical epimorphism;

$\varepsilon_X : IH(X) \longrightarrow X$ is

$$\begin{array}{ccc} X^1/\{1\} & \xrightarrow{\cong} & X^1 \\ \bar{\zeta}_X \downarrow & & \downarrow \zeta_X \\ X^0 & = & X^0 \end{array}.$$

LEMMA 2.1. *Let* (C,σ) *and* (A,α) *be extensions of* C_0 . *Then the following conditions are equivalent:*

(a) (A,α) *is split over* (C,σ) *with respect to* Γ ;
(b) *the diagram*

$$\begin{array}{ccc} C^1 \times_{C_0^1} A^1 & \longrightarrow & C^1 \times_{C_0^1} A^1/[(\mathrm{Ker}\ \zeta_C) \times_{C_0^1} (\mathrm{Ker}\ \zeta_A), C^1 \times_{C_0^1} A^1] \\ \downarrow & & \downarrow \\ C^1 & \longrightarrow & C^1/[\mathrm{Ker}\ \zeta_C, C^1] \end{array}$$

is a pullback;

(c) *the canonical homorphism*

$$[(\mathrm{Ker}\ \zeta_C) \times_{C_0^1} (\mathrm{Ker}\ \zeta_A)\ ,\ C^1 \times_{C_0^1} A^1] \to [\mathrm{Ker}\ \zeta_C, C^1]$$

is an isomorphism.

PROOF. (a) $\Leftrightarrow$ (b) follows easily from (a) $\Leftrightarrow$ (b) of 1,1 and (b) $\Leftrightarrow$ (c) follows from a well known property of group extensions. □

DEFINITION 2.2. An extension (A,α) of C_0 is a *double central extension* if the commutants

$$[\mathrm{Ker}\ \zeta_A \cap \mathrm{Ker}\ \alpha^1, A^1]\ ,\ [\mathrm{Ker}\ \zeta_A, \mathrm{Ker}\ \alpha^1]$$

are trivial groups, i.e.

$$ka_1 = a_1k,\ k_{\zeta_A}k_{\alpha^1} = k_{\alpha^1}k_{\zeta_A}$$

for each $k \in \mathrm{Ker}\ \zeta_A \cap \mathrm{Ker}\ \alpha^1$, $a_1 \in A^1$, $k_{\zeta_A} \in \mathrm{Ker}\ \zeta_A$, $k_{\alpha^1} \in \mathrm{Ker}\ \alpha^1$. □

LEMMA 2.3.

(a) *If* (A,α) *is split over* (C,σ) *with respect to* Γ , *then* (A,α) *is a double central extension;*

(b) *If* (A,α) *is a central double extension and for an extension* (C,σ) *there exists a morphism* $\gamma : C \to A$ *with* $\alpha\gamma = \sigma$, *then* (A,α) *is split over* (C,σ) *with respect to* Γ .

PROOF.

(a) Let $k, a_1, k_{\zeta_A}, k_{\alpha^1}$ be as above. First consider k and a_1.

Choose $c_1 \in C^1$ with $\sigma^1(c_1) = \alpha^1(a_1)$ and consider the element

$$t = [(1,k),(c_1,a_1)] \in [(\mathrm{Ker}\ \zeta_C) \times_{C^1_0} (\mathrm{Ker}\ \zeta_A)\ ,\ C^1 \times_{C^1_0} A^1]\ ;$$

its image in $[\mathrm{Ker}\ \zeta_C, C^1]$ is $[1,C^1] = 1$ and so $t = 1$ by the condition (c) of 2.1. Hence $[k,a_1] = 1$, i.e. $ka_1 = a_1k$.

Now consider k_{ζ_A} and k_{α^1}.

Choose $c_1 \in C^1$ with $\sigma^1(c_1) = \alpha^1(k_{\zeta_A})$ and $\zeta_C(c_1) = 1$ (this is possible because

$$\zeta_{C_0}(\alpha^1(k_{\zeta_A})) = \alpha^0(\zeta_A(k_{\zeta_A})) = 1 = \sigma^1(1)\ ).$$

Consider the element

$$t = [(c_1,k_{\zeta_A}),(1,k_{\alpha^1})] \in [(\mathrm{Ker}\ \zeta_C)\times_{C^1_0}(\mathrm{Ker}\ \zeta_A), C^1\times_{C^1_0} A^1]\ ;$$

its image in $[\mathrm{Ker}\ \zeta_C, C^1]$ is $[c_1,1] = 1$ and so $t = 1$ by the condition (c) of 2.1. Hence $[k_{\zeta_A}, k_{\alpha^1}] = 1$, i.e. $k_{\zeta_A}k_{\alpha^1} = k_{\alpha^1}k_{\zeta_A}$.

(b) We will prove that the condition (c) of 2.1 is satisfied. It is sufficient to prove that the composition

$$[(\mathrm{Ker}\ \zeta_C) \times_{C^1_0} (\mathrm{Ker}\ \zeta_A)\ ,\ C^1 \times_{C^1_0} A^1] \to [\mathrm{Ker}\ \zeta_C, C^1] \to$$
$$\to [(\mathrm{Ker}\ \zeta_C) \times_{C^1_0} (\mathrm{Ker}\ \zeta_A)\ ,\ C^1 \times_{C^1_0} A^1]\ ,$$

where the second arrow is defined by $c_1 \mapsto (c_1, \gamma^1(c_1))$ (clearly correct!), is the identity map. Thus it is sufficient to prove that for each $k_C \in \mathrm{Ker}\ \zeta_C, k_A \in \mathrm{Ker}\ \zeta_A, c_1 \in C^1$ and $a_1 \in A^1$ with $\sigma^1(k_C) = \alpha^1(k_A)$ and $\sigma^1(c_1) = \alpha^1(a_1)$ we have

$$[k_A,a_1] = [\gamma^1(k_C),\gamma^1(c_1)] \ .$$

To prove this, first observe that

$$\alpha^1(k_A) = \alpha^1(\gamma^1(k_C)),\ \alpha^1(a_1) = \alpha^1(\gamma^1(c_1))$$

and so there exist $k,k' \in \operatorname{Ker} \alpha^1$ with

$$\gamma^1(k_C) = k_A k\ ,\ \gamma^1(c_1) = a_1 k'$$

Moreover $k \in \operatorname{Ker} \zeta_A \cap \operatorname{Ker} \alpha^1$ because $\gamma^1(k_C)$ and k_A are in $\operatorname{Ker} \zeta_A$. After that we have

$$[\gamma^1(k_C),\ \gamma^1(c_1)] = [k_A k, a_1 k'] = k_A k a_1 k' k^{-1} k_A^{-1} k'^{-1} a_1^{-1} \ .$$

By the condition $[\operatorname{Ker} \zeta_A \cap \operatorname{Ker} \alpha^1, A^1] = \{1\}$ we can write $k a_1 k' k^{-1} = a_1 k'$ and by the condition $[\operatorname{Ker} \zeta_A, \operatorname{Ker} \alpha^1] = \{1\}$ we can write $k' k_A^{-1} k'^{-1} = k_A^{-1}$. Thus we have

$$k_A k a_1 k' k^{-1} k_A^{-1} k'^{-1} a_1^{-1} = k_A a_1 k' k_A^{-1} k'^{-1} a_1^{-1}$$
$$= k_A a_1 k_A^{-1} a_1^{-1} = [k_A,a_1] \ ,$$

which completes the proof. □

LEMMA 2.4. *The conditions* (a) *and* (b) *of* 1.3 *hold for any extension* (C,σ) .

PROOF. 1.3(a): For an extension (X,φ) of $I(C)$ (in $\mathfrak{X}$) consider the pullback

$$\begin{array}{ccc} D & \xrightarrow{\ \pi_2\ } & X^1 \\ {\scriptstyle \pi_1}\downarrow & & \downarrow{\scriptstyle \varphi^1} \\ C^1 & \xrightarrow[\ \eta_C^1\ ]{} & C^1/[\operatorname{Ker} \zeta_C, C^1] \ ; \end{array}$$

we need to prove that π_2 induces an isomorphism

$$D/[\operatorname{Ker} \zeta_X \pi_2, D] \cong X^1 \ ,$$

i.e. that $\pi_2(d) = 1$ implies $d \in [\operatorname{Ker} \zeta_X \pi_2, D]$ for each $d \in D$.

Let d be an element of D with $\pi_2(d) = 1$. Then $\pi_1(d)$ is in $[\mathrm{Ker}\ \zeta_C, C^1]$ and so we can write

$$d = ([k_1,c_1]...[k_n,c_n],1),$$

where $k_1,...,k_n$ are in $\mathrm{Ker}\ \zeta_C$. Choose $x_1,...,x_n \in X^1$ with $\varphi^1(x_i) = c_i[\mathrm{Ker}\ \zeta_C,C^1]$ and $y_1,...,y_n \in X^1$ with

$$\varphi^1(y_i) = k_i[\mathrm{Ker}\ \zeta_C,C^1],\ \zeta_X(y_i) = 1$$

for $i = 1, ..., n$; we have

$$d = [(k_1,y_1),(c_1,x_1)]...[(k_n,y_n),(c_n,x_n)]$$

and so d is in $[\mathrm{Ker}\ \zeta_X\pi_2, D]$.

1.3(b) easily follows from the fact that for each group epimorphism $\mathcal{B} \longrightarrow \mathcal{A}$ the pullback functor

$$(\mathrm{Groups} \downarrow A) \longrightarrow (\mathrm{Groups} \downarrow B)$$

is monadic. □

Now let (C,σ) be the extension of C_0 constructed by the following three steps:

1^0, $\sigma^0 : C^0 \longrightarrow C^0_0$ is an arbitrary epimorphism with a free C^0_0;

2^0, after that consider an arbitrary epimorphism from a free group F to the group $C^1_0 \times_{C^0_0} C^0$ and denote the compositions

$$F \longrightarrow C^1_0 \times_{C^0_0} C^0 \xrightarrow{\mathrm{proj}_1} C^1_0,$$

$$F \longrightarrow C^1_0 \times_{C^0_0} C^0 \xrightarrow{\mathrm{proj}_2} C^0$$

by π_1,π_2 respectively.

3^0, (C,σ) is the "double centralization": C^1 is the factor group of F by the commutants $[\mathrm{Ker}\ \pi_1 \cap \mathrm{Ker}\ \pi_2, F]$ and $[\mathrm{Ker}\ \pi_1, \mathrm{Ker}\ \pi_2]$, and σ^1,ζ_C are induced by π_1, π_2 respectively.

Clearly (C,σ) is a double central extension of C^0, and if (A,α) also is a double central extension of C_0, then there exists a morphism $\gamma : C \longrightarrow A$ with $\alpha\gamma = \sigma$. Using the lemmas above we obtain:

THEOREM 2.5. *The extension* (C,σ) *constructed above is a weak universal covering and for an extension* (A,α) *of* C_0 *the following conditions are equivalent:*

(a) (A,α) *is a double central extension;*
(b) (A,α) *is split over* (C,σ) *with respect to* Γ *;*
(c) (A,α) *is a* Γ*-covering;*
(d) (A,α) *is a* Γ*-normal extension.* □

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Dr G Janelidze
Mathematical Institute
Z Rukhadze Str 1
Tbilisi
Georgia
380093 U.S.S.R.