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THE FRAME OF FIBREWISE CLOSED NUCLEI

by M. JIBLADZE and P.T. JOHNSTONE *

RÉSUMÉ: Dans cet article, on montre que les noyaux correspondant aux sous-locales fermés par B-fibres d'un locale donné A sur une base B, forment un sous-cadre du cadre de tous les noyaux sur A; et ce cadre peut-être caractérisé par un certain 'pushout' dans la catégorie des cadres. On caractérise ainsi les fonctions $B \rightarrow A$ qui se présentent comme restrictions à B de noyaux (fermés par fibres) sur A.

INTRODUCTION.

In [2], a notion of B-fibrewise denseness, and a corresponding notion of B-fibrewise closedness, were introduced for sublocales of locales over a base locale B (equivalently, frames under B). Although an explicit description was given of the nuclei on a frame A under B corresponding to B-fibrewise dense sublocales of A, the problem of describing the B-fibrewise closed nuclei was left open. An explicit, though rather complicated, answer to this problem was given in [3]. In this paper, we give a simpler but less explicit answer, and use it to show that the B-fibrewise closed nuclei on A form a subframe $N_B(A)$ of the frame $N(A)$ of all nuclei on A. We also show that there is a pushout diagram

$$\begin{array}{ccc}
 B & \longrightarrow & N(B) \\
 \downarrow & & \downarrow \\
 A & \longrightarrow & N_B(A)
 \end{array}$$

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in the category of frames; equivalently, that $N_B(A)$ is freely generated from A by adjoining complements for the elements in the image of the frame homomorphism $B \rightarrow A$. Finally, by constructing the above pushout another way, we arrive at a characterization of those functions $B \rightarrow A$ which are expressible as the composite of the given frame homomorphism with a nucleus on A .

The arguments which we shall employ are basically lattice-theoretic, and so we shall work throughout in the category of frames rather than that of locales. In other respects, however, our notation and terminology agree with those of [1]. In particular, a *nucleus* on a frame A is a mapping $j : A \rightarrow A$ satisfying

$$a \leq j(a) = jj(a) \quad \text{and} \quad j(a \wedge b) = j(a) \wedge j(b)$$

for all $a, b \in A$; the *sublocale* of A corresponding to j is the set A_j of j -fixed elements of A . Given $a \in A$, we write $c(a)$ and $u(a)$ for the corresponding closed and open nuclei on A , defined by

$$c(a)(b) = a \vee b \quad \text{and} \quad u(a)(b) = a \rightarrow b.$$

We write $N(A)$ for the set of all nuclei on A , ordered pointwise (i.e. $j \leq k$ if and only if $j(a) \leq k(a)$ for all a); we recall ([1], II 2.6) that $N(A)$ is a frame and $c : A \rightarrow N(A)$ is a frame homomorphism.

When working in the category of frames under a given frame B , we shall find it convenient (in the first two sections, at least) to suppress the name of the given frame homomorphism from B to a frame A under B , and instead identify elements of B with their images in A . (However, in section 3, where we shall need to refer to the right adjoint of the frame homomorphism as well as the homomorphism itself, it will be convenient to give the latter a name.) This convention is not intended to imply that the frame homomorphism $B \rightarrow A$ is necessarily injective; but we note that the definition of B -fibrewise closure depends only on the image of $B \rightarrow A$, and not on B itself.

A few words should be said about the joint authorship of this paper. Broadly, the results of sections 1 and 2 are due to the second author, and those of section 3 to the first; they were obtained independently, and it was only through the opportunity to meet in Bangor at ICTM '89 that we became aware of each other's work, and of the close connections between what we had done. We are deeply grateful to the organi-

zers of the Bangor meeting for providing that opportunity. The first author is also grateful to the staff of the Cultural Section of the British Embassy in Moscow for their essential help in obtaining a British visa.

1. The Frame $N_B(A)$

We begin by recalling the proof ([1], II 2.7) of the result that $N(A)$ is generated as a frame by the elements $c(a)$ and $u(a)$, $a \in A$: if j is any nucleus on A , we have

$$j = \bigvee \{c(j(a)) \wedge u(a) \mid a \in A\} .$$

Indeed, writing k_a for $c(j(a)) \wedge u(a)$, we have

$$k_a(a') \leq j(a')$$

for all a' , whence the join on the right is less than or equal to j , and $k_a(a) = j(a)$. Note also that the join in the above formula is computed pointwise, although this is generally not the case for joins in $N(A)$.

Lemma 1.1. *For any j , the B -fibrewise closure of j is given by the formula*

$$\bar{j} = \bigvee \{c(j(b)) \wedge u(b) \mid b \in B\} .$$

Proof. By definition, the B -fibrewise closure of j is the smallest nucleus on A which agrees with j on elements of B . By the remarks above, the definition of \bar{j} given in the statement satisfies $\bar{j} \leq j$, and $\bar{j}(b) = j(b)$ for all $b \in B$; also, if k is any nucleus agreeing with j on elements of B , then $\bar{j} = \bar{k}$ and so $\bar{j} \leq k$. So \bar{j} is the B -fibrewise closure of j . \square

Remark 1.2. As mentioned in [3], G.C. Wraith suggested the formula

$$\hat{j}(a) = \bigvee \{j(b) \wedge (b \rightarrow a) \mid b \in B\}$$

for the B -fibrewise closure of j . Clearly, we have

$$\begin{aligned} j(b) \wedge (b \rightarrow a) &= (j(b) \vee a) \wedge (b \rightarrow a) \\ &= c(j(b))(a) \wedge u(b)(a) , \end{aligned}$$

so Wraith's suggestion is equivalent to taking the pointwise join of the nuclei k_b , $b \in B$. Unfortunately, this pointwise join is not in general a nucleus, as was shown in [3]; so we must replace it by the join in the frame $N(A)$.

Proposition 1.3. *The B-fibrewise closure operation on $N(A)$ preserves binary meets.*

Proof. It is clear that $j \mapsto \bar{j}$ is order-preserving, so we have only to verify that $\bar{j} \wedge \bar{k} \leq \overline{j \wedge k}$. By the infinite distributive law in $N(A)$, the left-hand side is the join of all nuclei of the form

$$c(j(b_1)) \wedge c(k(b_2)) \wedge u(b_1) \wedge u(b_2)$$

with $b_1, b_2 \in B$. But $u(b_1) \wedge u(b_2) = u(b_1 \vee b_2)$, so this is less than or equal to

$$\begin{aligned} c(j(b_1 \vee b_2)) \wedge c(k(b_1 \vee b_2)) \wedge u(b_1 \vee b_2) \\ = c((j \wedge k)(b_1 \vee b_2)) \wedge u(b_1 \vee b_2) \end{aligned}$$

(recalling that meets in $N(A)$ are computed pointwise, and that $a \mapsto c(a)$ preserves finite meets), and this is one of the terms in the join defining $\overline{j \wedge k}$. \square

Corollary 1.4. *The set $N_B(A)$ of B-fibrewise closed nuclei on A is a subframe of $N(A)$; in particular, it is a frame.*

Proof. $N_B(A)$ is closed under arbitrary joins in $N(A)$, since $j \mapsto \bar{j}$ is right adjoint to the inclusion. Proposition 1.3 implies that it is closed under binary meets; and the top element of $N(A)$ (corresponding to the degenerate sublocale of A) is closed and hence B-fibrewise closed ([2], 2.3). \square

Corollary 1.5. *A nucleus on A is B-fibrewise closed if and only if it can be expressed as a join of nuclei of the form $c(a) \wedge u(b)$, where $a \in A$ and $b \in B$.*

Proof. One direction is immediate from Lemma 1.1. The converse will follow from Corollary 1.4, provided we can show that all nuclei of the form $c(a)$ or $u(b)$ are in $N_B(A)$. But, as we have just remarked, the closed nuclei on A are B-fibrewise closed for any B. Also, $u(b)$ is the smallest nucleus on A sending b to 1; it is therefore the smallest

nucleus agreeing with $u(b)$ at elements of B . □

2. The Universal Property of $N_B(A)$

Lemma 2.1. *The assignment $A \mapsto N_B(A)$ is functorial on the category B/Frm of frames under B , and the maps $c : A \rightarrow N_B(A)$ form a natural transformation from the identity to this functor.*

Proof. We recall how $A \mapsto N(A)$ is made into a functor ([1], II 2.8): given a frame homomorphism $f : A \rightarrow A'$ (equivalently, a locale map $A' \rightarrow A$) and a nucleus j on A , we form the pullback locale

$$\begin{array}{ccc}
 A'' & \longrightarrow & A_j \\
 \downarrow & & \downarrow \\
 A' & \longrightarrow & A
 \end{array}$$

and define $N(f)(j)$ to be the nucleus on A' corresponding to the sublocale A'' . But pullbacks of B -fibrewise closed inclusions are B -fibrewise closed ([2], 1.8), so $N(f)$ maps $N_B(A)$ into $N_B(A')$. The remaining details are straightforward. □

Recalling in particular that every sublocale of B is B -fibrewise closed ([2], 2.5), we obtain

Corollary 2.2. *For any frame A under B , there is a commutative square*

$$\begin{array}{ccc}
 B & \xrightarrow{c} & N(B) \\
 \downarrow & & \downarrow \\
 A & \xrightarrow{c} & N_B(A)
 \end{array}$$

in Frm . □

Lemma 2.3. *Suppose that the image of the frame homomorphism $B \rightarrow A$ consists of complemented elements of A . Then $c : A \rightarrow N_B(A)$ is an isomorphism.*

Proof. If b has a complement a in A , then $u(b) = c(a)$ in $N(A)$. But finite meets and arbitrary joins of closed nuclei are closed, so it follows from 1.5 that every B -fibrewise closed nucleus on A is closed.

Proposition 2.4. *The commutative diagram of Corollary 2.2 is a pushout in Frm.*

Proof. Suppose given a commutative square

$$\begin{array}{ccc} B & \xrightarrow{c} & N(B) \\ \downarrow & & \downarrow \\ A & \xrightarrow{f} & C \end{array} .$$

Since every element of B becomes complemented in $N(B)$, it remains complemented in C ; so by Lemma 2.3 the homomorphism $N_B(f) : N_B(A) \rightarrow N_B(C)$ can be regarded as taking values in C . Thus we have a factorization of f through $c : A \rightarrow N_B(A)$. Since (in view of 1.5) $A \rightarrow N_B(A)$ is epic as well as monic, this factorization is unique; and since $B \rightarrow N(B)$ is also epic, it makes the triangle

$$\begin{array}{ccc} N(B) & & \\ \downarrow & \searrow & \\ N_B(A) & \xrightarrow{\quad} & C \end{array}$$

commute. □

Remark 2.5. The proof of 2.4 shows that $N_B(A)$ is obtained from A by freely adjoining complements for the elements of B ; that is, $A \rightarrow N_B(A)$ is universal among frame homomorphisms $A \rightarrow C$ sending all the elements of B to complemented elements of C . As a particular case, this includes the well-known fact that $N(A) = N_A(A)$ is obtained from A by freely adjoining complements for all its elements.

3. The Frame of B -nuclei.

By definition, a B -fibrewise closed nucleus on A is uniquely specified by its restriction to the image of the frame homomorphism $B \rightarrow A$. It is therefore possible to

identify such nuclei with certain functions $B \rightarrow A$, namely those which occur as the composites of the frame homomorphism with (arbitrary!) nuclei on A . But how can we characterize the functions which arise in this way?

In this section, unlike the previous two, we shall find it convenient to give a name to the frame homomorphism $B \rightarrow A$; we shall denote it by g , and write $g_* : A \rightarrow B$ for its right adjoint.

We now define a function $k : B \rightarrow A$ to be a *B-nucleus on A* if it satisfies the condition

$$(g(b) \rightarrow k(b')) = (k(b) \rightarrow k(b'))$$

for all $b, b' \in B$. For the moment, let us write $BN(A)$ for the set of B-nuclei on A , ordered pointwise; our aim is to show that this is (up to isomorphism) just another name for $N_B(A)$.

The definition of a B-nucleus just given is about the shortest possible, but perhaps not the easiest to work with. The following characterization will generally be more useful in practice.

Lemma 3.1. *A function $k : B \rightarrow A$ is a B-nucleus on A if and only if it satisfies*

- (i) $g(b) \leq k(b)$,
- (ii) $k(b \wedge b') = k(b) \wedge k(b')$, and
- (iii) $kg_*(a \rightarrow k(b)) \leq (a \rightarrow k(b))$

for all $b, b' \in B$ and all $a \in A$.

Proof. First suppose k is a B-nucleus. Putting $b = b'$ in the definition, we obtain $(g(b) \rightarrow k(b)) = 1_A$, so (i) holds. Now if $b \leq b'$ we have $g(b) \leq g(b') \leq k(b')$ and so

$$(k(b) \rightarrow k(b')) = (g(b) \rightarrow k(b')) = 1,$$

i.e. k is order-preserving, and hence

$$k(b \wedge b') \leq k(b) \wedge k(b')$$

for any b and b' . For the reverse inequality, we have

$$\begin{aligned}
 ((k(b) \wedge k(b')) \rightarrow k(b \wedge b')) &= (k(b) \rightarrow (k(b') \rightarrow k(b \wedge b'))) \\
 &= (k(b) \rightarrow (g(b') \rightarrow k(b \wedge b'))) \\
 &= (g(b') \rightarrow (k(b) \rightarrow k(b \wedge b'))) \\
 &= (g(b') \rightarrow (g(b) \rightarrow k(b \wedge b'))) \\
 &= ((g(b) \wedge g(b')) \rightarrow k(b \wedge b')) \\
 &= (g(b \wedge b') \rightarrow k(b \wedge b')) \\
 &= 1 ;
 \end{aligned}$$

so (ii) holds. Finally, for (iii) we have

$$\begin{aligned}
 a \leq ((a \rightarrow k(b)) \rightarrow k(b)) &\leq (gg_*(a \rightarrow k(b)) \rightarrow k(b)) \\
 &= (kg_*(a \rightarrow k(b)) \rightarrow k(b)) ,
 \end{aligned}$$

so $(a \wedge kg_*(a \rightarrow k(b))) \leq k(b)$, whence the result.

Conversely, suppose the given conditions hold, and let $b, b' \in B$. Write a for $(g(b) \rightarrow k(b'))$; we have $a \geq (k(b) \rightarrow k(b'))$ by (i), so it suffices to show the reverse inequality. But we have $g(b) \leq (a \rightarrow k(b'))$, or equivalently $b \leq g_*(a \rightarrow k(b'))$, whence (ii) and (iii) imply

$$k(b) \leq kg_*(a \rightarrow k(b')) \leq (a \rightarrow k(b')) ,$$

so $a \leq (k(b) \rightarrow k(b'))$. □

From the particular case of 3.1 when g is the identity, we readily deduce that a function $j : A \rightarrow A$ is a nucleus if and only if it satisfies

$$(a \rightarrow j(a')) = (j(a) \rightarrow j(a'))$$

for all $a, a' \in A$. (This result is probably well-known, but we have been unable to find it in the literature.) Using it, we immediately obtain

Corollary 3.2. *If j is a nucleus on A , then the composite jk is a B -nucleus.* □

In the converse direction, we first describe a means of producing nuclei on B from B -nuclei on A .

Lemma 3.3. *Let k be a B -nucleus on A , and let $a \in A$.*

Then the mapping $b \rightarrow g_*(a \rightarrow k(b))$ is a nucleus on B .

Proof. It is clear that

$$b \leq g_*g(b) \leq g_*k(b) \leq g_*(a \rightarrow k(b)) ;$$

and the mapping preserves finite meets, since g_* , k and $(a \rightarrow (-))$ all do so. So we have only to verify idempotency: for this, we need to show that

$$g_*(a \rightarrow kg_*(a \rightarrow k(b))) \leq g_*(a \rightarrow k(b)) .$$

But this is immediate from 3.1 (iii), since

$$(a \rightarrow (a \rightarrow k(b))) = ((a \wedge a) \rightarrow k(b)) = (a \rightarrow k(b)) . \quad \square$$

Proposition 3.4. $BN(A)$ is a frame, and there is a pushout diagram

$$\begin{array}{ccc} B & \xrightarrow{c} & N(B) \\ \downarrow g & & \downarrow \\ A & \xrightarrow{\quad} & BN(A) \end{array}$$

in \mathbf{Frm} .

Proof. We use the result (essentially due to Wigner [5], but see also [4], II 2.1(v)) that the pushout of a diagram

$$\begin{array}{ccc} B & \xrightarrow{c} & C \\ \downarrow g & & \\ A & & \end{array}$$

in \mathbf{Frm} may be identified with the poset of those Galois connections $(\hat{p} : C \rightarrow A, \hat{h} : A \rightarrow C)$ such that

$$a \wedge g(b) \leq \hat{p}(c) \text{ if and only if } c \wedge h(b) \leq \hat{h}(a) \quad (*)$$

for all $a \in A, b \in B, c \in C$. Given such a Galois connec-

tion in the case $C = N(B)$, we note that, since \check{p} carries joins to meets and

$$\check{p}(j \wedge c(b)) = (g(b) \rightarrow \check{p}(j))$$

for any $j \in N(B)$, by (*), it follows that \check{p} is determined by its effect on nuclei of the form $u(b)$, $b \in B$. Now the composite $\check{p}u : B \rightarrow A$ is order-preserving, since both \check{p} and u are order-reversing; we claim that $\check{p}u$ is a B-nucleus.

Certainly we have $\check{p}u(b) \geq g(b)$, since

$$u(b) \wedge c(b) = 0_{N(B)} \leq \hat{p}(1_A);$$

and $\check{p}u$ preserves finite meets, since u sends them to finite joins and \check{p} converts them back to meets. To verify 3.1 (iii), we note first that by (*) we have

$$\begin{aligned} \check{p}g(b) &= \hat{p}(1 \wedge g(b)) \\ &= (c(b) \rightarrow \hat{p}(1)) \\ &= (u(b) \vee \hat{p}(1)) \end{aligned}$$

since $c(b)$ and $u(b)$ are complementary elements of $N(B)$. So

$$\begin{aligned} \check{p}u(b) &= \check{p}u(b) \wedge \check{p}\hat{p}(1) \\ &= p(u(b) \vee \hat{p}(1)) \\ &= \check{p}\hat{p}g(b). \end{aligned}$$

More generally,

$$\begin{aligned} \check{p}\hat{p}(a \wedge g(b)) &= \check{p}(c(b) \rightarrow \hat{p}(a)) \\ &= \check{p}(u(b) \vee \hat{p}(a)) \\ &= \check{p}\hat{p}(a) \wedge \check{p}u(b); \end{aligned}$$

thus the closure operator $\check{p}\hat{p}$ preserves binary meets, provided one of the factors is in the image of g . Now we have

$$\begin{aligned}
 \check{y}u_{g_*}(a \rightarrow \check{y}u(b)) \wedge a &= \check{y}\hat{p}g_{g_*}(a \rightarrow \check{y}\hat{p}g(b)) \wedge a \\
 &\leq \check{y}\hat{p}g_{g_*}(a \rightarrow \check{y}\hat{p}g(b)) \wedge \check{y}\hat{p}(a) \\
 &= \check{y}\hat{p}(g_{g_*}(a \rightarrow \check{y}\hat{p}g(b)) \wedge a) \\
 &\leq \check{y}\hat{p}((a \rightarrow \check{y}\hat{p}g(b)) \wedge a) \\
 &\leq \check{y}\hat{p}\check{y}\hat{p}g(b) = \check{y}\hat{p}g(b) = \check{y}u(b) ,
 \end{aligned}$$

so $\check{y}\hat{u}g(a \rightarrow \check{y}u(b)) \leq (a \rightarrow \check{y}u(b))$, as required.

Conversely, suppose we are given a B-nucleus k on A . We define

$$\check{y}(j) = \wedge\{gj(b) \rightarrow k(b) \mid b \in B\}$$

for $j \in N(B)$, and

$$\hat{p}(a)(b) = g_*(a \rightarrow k(b))$$

for $a \in A$, $b \in B$. By Lemma 3.3, $\hat{p}(a) \in N(B)$ for all a . We must show that the pair (\check{y}, \hat{p}) is a Galois connection satisfying (*). Clearly, both \check{y} and \hat{p} are order-reversing maps. Suppose we have

$$a \wedge g(b) \leq \check{y}(j) = \wedge\{gj(b') \rightarrow k(b') \mid b' \in B\}$$

for some $a \in A$, $b \in B$, $j \in N(B)$. Then for each $b' \in B$ we have

$$a \wedge g(b \wedge j(b')) \leq k(b') ,$$

whence

$$b \wedge j(b') \leq g_*(a \rightarrow k(b')) = \hat{p}(a)(b') .$$

But

$$\begin{aligned}
 (j \wedge c(b))(b') &= j(b') \wedge (b \vee b') \\
 &= (j(b') \wedge b) \vee b'
 \end{aligned}$$

and we also have $b' \leq \hat{p}(a)(b')$, so $j \wedge c(b) \leq \hat{p}(a)$. Conversely, if $j \wedge c(b) \leq \hat{p}(a)$, then for each b' we have

$$j(b') \wedge b \leq j(b') \wedge c(b)(b') \leq \hat{p}(a)(b') ,$$

so reversing the above argument we get

$$a \wedge g(b) \leq gj(b') \rightarrow k(b'), \text{ whence } a \wedge g(b) \leq \bigvee j.$$

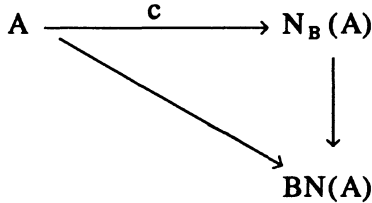
Finally, we note that

$$\begin{aligned} \bigvee u(b) &= \bigwedge \{g(b \rightarrow b') \rightarrow k(b') \mid b' \in B\} \\ &= \bigwedge \{k(b \rightarrow b') \rightarrow k(b') \mid b' \in B\} \\ &= k(b), \end{aligned}$$

since $k(b) \leq k(b \rightarrow b') \rightarrow k(b')$, with equality if $b = b'$. Since we have already remarked that $\bigvee j$ (and hence also \hat{p}) is uniquely determined by $\bigvee u$, this means that we have set up a bijection (which is clearly order-preserving) between $BN(A)$ and the set of Galois connections satisfying (*). So the result follows from the quoted result of Wigner. \square

Corollary 3.5. *The mapping $j \mapsto jg$ is a frame isomorphism $N_B(A) \rightarrow BN(A)$. In particular, every B -nucleus on A is the composite of g with a nucleus on A .*

Proof. On comparing Propositions 2.4 and 3.4, it is immediate that the frames $N_B(A)$ and $BN(A)$ are isomorphic; we have to verify that the stated map yields the isomorphism. For this, it suffices (since $B \rightarrow N(B)$ is epimorphic) to verify that it makes the triangle



commute, where the diagonal map is that appearing as the bottom edge of the square in the statement of 3.4. But this map sends $a \in A$ to the B -nucleus corresponding to the smallest Galois connection (\hat{p}, \hat{p}) satisfying (*) and $\hat{p}(1) \geq a$, i.e. to the (pointwise) meet of all such connections. But for any such (\hat{p}, \hat{p}) and any b , we have

$$\bigvee u(b) = \bigvee \hat{p}g(b) \geq g(b) \vee \hat{p}(1) \geq g(b) \vee a = c(a)(g(b));$$

to show that the smallest (\hat{p}, \hat{p}) yields equality here, it suf-

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faces to show that there is some (\check{p}, \hat{p}) satisfying (*) which does so, since it will then satisfy

$$\check{p}(1) = \check{p}u(0) = c(a)(0) = a .$$

However, we already know that $c(a)g$ is a B-nucleus by 3.2, so this follows from what we established in the proof of 3.4. \square

Remark 3.6. The inverse of the isomorphism $j \mapsto jg$ is now easy to describe: using 1.1, we see that it sends a B-nucleus k to the join (in $\tilde{N}(A)$) of the nuclei $(c(k(b)) \wedge u(g(b)))$, $b \in B$. However, because of the fact that joins in $\tilde{N}(A)$ are not in general pointwise, it does not seem possible to give a direct proof of 3.5 using this description.

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