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PRESENTATIONS OF ETENDUES

by Anders KOCK and Ieke MOERDIJK

RÉSUMÉ: Dans cet article, on prouve que toute étendue peut être présentée par un site dont les applications sont moniques, et l'on donne des conditions sur un groupoïde localique pour que son topos classifiant soit une étendue. Ces résultats reposent sur la notion d'application localement monique dans un topos.

We prove that any étendue can be presented by a site whose maps are monic; and also, we give conditions on a localic groupoid that its classifying topos be an étendue. Both these results hinge on the notion of *locally monic* map in a topos.

1. LOCALLY MONIC MAPS AND TORSION FREE OBJECTS.

Let $\gamma : \mathcal{E} \to \mathcal{S}$ be a geometric morphism between toposes (elementary toposes will suffice, for present §).

Definition 1.1. A map $f : A \to B$ in $\mathcal{E}$ is called *locally monic* relative to γ if there exists an $I \in \mathcal{S}$ and a commutative square

$$\begin{tikzcd} A' \arrow[r, tail, "f'"] \arrow[d, two heads, "q"'] & \gamma^* I \times B \arrow[d, "\mathrm{proj}"] \\ A \arrow[r, "f"'] & B \end{tikzcd} \tag{1.1}$$

with q epic and f′ monic.

This is equivalent to saying that f , as an object in

$\mathcal{E}/B$, is generated by subobjects of 1 , relative to $\mathcal{S}$, in the sense introduced by Johnstone [J2], cf. Proposition 1.8 below.

Note that f' may be factored $A' \to \gamma^* I \times A \to \gamma^* I \times B$ through $\gamma^* I \times f$; this provides the heuristics: if $\mathcal{S} = \mathit{Sets}$, A' is an I-indexed family A'_i of subobjects of A , covering A , and such that for each $i \in I$, the restriction of f to A'_i is monic.

We derive some elementary properties of this notion. For the moment, the geometric morphism γ is fixed, and we say 'locally monic' instead of 'locally monic relative to γ' .

Proposition 1.2 *Monic maps are locally monic.*

Proof. Take $I = 1$, $A' = A$, and $f' = f$ in (1.1).

The class of locally monic maps share some of the properties of the class of monic maps:

Proposition 1.3. *Consider maps in* $\mathcal{E}$

$$A \xrightarrow{f} B \xrightarrow{g} C .$$

1) *If* $g \circ f$ *is locally monic, then so is* f .
2) *If* f *and* g *are locally monic, then so is* $g \circ f$.

Proof.
1) We have a square witnessing that $g \circ f$ is locally monic

$$\begin{tikzcd} A' \arrow[r, "f'"] \arrow[d] & \gamma^* I \times C \arrow[d] \\ A \arrow[r, "f"'] & B \arrow[r, "g"'] & C \end{tikzcd} \quad ;$$

it may be factored into two squares by inserting $\gamma^* I \times B$ in the middle of the top row; the left hand square then witnesses local monicity of f .
2) This is a matter of contemplating the diagram

$$\begin{tikzcd}
A'' \arrow[r, tail] \arrow[d, two heads] \arrow[dr, phantom, "\mathrm{pb}"] & B' \times \gamma^*I \arrow[r, tail] \arrow[d, two heads] & (C \times \gamma^*J) \times \gamma^*I \arrow[d] & = C \times \gamma^*(J \times I) \\
A' \arrow[r, tail] \arrow[d, two heads] & B \times \gamma^*I \arrow[r] \arrow[d] & C \times \gamma^*I \arrow[d] & \\
A \arrow[r, "f"'] & B \arrow[r, "g"'] & C &
\end{tikzcd}$$

where the lower left hand square witnesses f locally monic, and the upper right hand square arises by applying $- \times \gamma^*I$ to a witness that g is locally monic. The square 'pb' is formed as a pullback and the fourth square is obvious. Now the total square is a witness that $g \circ f$ is locally monic.

We leave it to the reader to prove

Proposition 1.4 *The pullback of a locally monic map along any map is locally monic.*

It is not true that pulling back along an epic reflects the property of being locally monic. In fact, in an étendue (cf. below), every map can be pulled back along some epic so as to become locally monic. But unless the étendue is localic, not every map is itself locally monic.

We now consider variation of the toposes $\mathcal{E}$ and $\mathcal{S}$ involved. Propositions 1.5 and 1.6 are only recorded here for the sake of completeness, whereas Proposition 1.7 is essential for what follows.

Proposition 1.5. *If* $\sigma : \mathcal{S} \to \mathcal{S}'$ *is a geometric morphism, and if* f *in* $\mathcal{E}$ *is locally monic relative to* γ *(where* $\gamma : \mathcal{E} \to \mathcal{S}$*), then* f *is locally monic relative to* $\sigma \circ \gamma$. *Every map in* $\mathcal{E}$ *is locally monic relative to* $\mathrm{id}_{\mathcal{E}} : \mathcal{E} \to \mathcal{E}$.

Proof. If $I' \in \mathcal{S}'$ is the witnessing index object for local monicity relative to $\sigma \circ \gamma$, then $\sigma^* I \in \mathcal{S}$ witnesses local monicity relative to γ , using identically the same witnessing square in $\mathcal{E}$. For the second assertion, if $f : A \to B$, take $I = A$, $q = \mathrm{id}_A$, and take f' to be the graph $\langle \mathrm{id}_A, f \rangle$ of f .

Proposition 1.6. *If* f *is locally monic relative to* $\gamma : \mathcal{E} \to \mathcal{S}$, *and* $\varepsilon : \mathcal{E}' \to \mathcal{E}$ *is a geometric morphism, then* $\varepsilon^*(f)$ *is locally monic relative to* $\gamma \circ \varepsilon$.

Proof. Just apply ε^* to a witnessing square for f .

Proposition 1.7. *Consider a commutative triangle in* $\mathcal{E}$

$$\begin{tikzcd} A \arrow[rr, "f"] \arrow[dr, "a"'] & & B \arrow[dl, "b"] \\ & X & \end{tikzcd}$$

Then f *is locally monic relative to* $\gamma : \mathcal{E} \to \mathcal{S}$ *if and only if* f , *viewed as a map in* $\mathcal{E}/X$, *is locally monic relative to* $\gamma \circ \pi$, *where* $\pi : \mathcal{E}/X \to \mathcal{E}$ *is the natural geometric morphism.*

Proof. For $I \in \mathcal{S}$,

$$(\gamma \circ \pi)^*(I) = (\gamma^* I \times X \xrightarrow{\text{proj}} X) ,$$

so

$$(\gamma \circ \pi)^*(I) \times_X B = \gamma^* I \times B ,$$

so if (1.1) is a witnessing square for local monicity of f relative to γ , we may read it as

$$\begin{tikzcd} A' \arrow[r, "f'"] \arrow[d, "q"'] & (\gamma \circ \pi)^*(I) \times_X B \arrow[d] \\ A \arrow[r] \arrow[dr] & B \arrow[d] \\ & X \end{tikzcd}$$

and vice versa, and it is well known that maps (q and f′ , say) are epi (resp. mono) in $\mathcal{E}/X$ if and only if they are so in $\mathcal{E}$.

Proposition 1.8 *A geometric morphism* $\gamma : \mathcal{E} \to \mathcal{S}$ *is localic if and only if every map in* $\mathcal{E}$ *is locally monic relative to* γ .

Proof. The Proposition refers to the notion of 'localic' given by Johnstone [J1]; according to this definition, γ is localic if and only if for every $A \in \mathcal{E}$, there is a diagram

$$\begin{tikzcd} A' \arrow[r, tail, "i"] \arrow[d, two heads, "q"'] & \gamma^* I \\ A & \end{tikzcd}$$

with q epic and i monic; equivalently, inserting 1 in the lower right hand corner, if and only if for every $A \in \mathcal{E}$, the map $A \to 1$ is locally monic according to Definition 1.1; but it follows from Proposition 1.3 1) that if every map to 1 is locally monic, then every map is locally monic.

The rationale for the terminology of *torsion free* object that we shall introduce now is the following. Consider the topos $\mathcal{E} = \mathcal{S}^G$ (G-sets), where G is a discrete group. If A and B are G-sets, clearly a map $f : A \to B$ is locally monic if and only if its restriction to every G-orbit in A is monic. This will in particular be the case if all the G-orbits in B are isomorphic to G itself, i.e. if B is a (torsion-) free G-set.

Let $\gamma : \mathcal{E} \to \mathcal{S}$ be a geometric morphism.

Definition 1.9. An object $B \in \mathcal{E}$ is called *torsion-free* relative to γ if every map with codomain B is locally monic relative to γ.

Combining Propositions 1.7 and 1.8, we see that this may equivalently be expressed by saying: $\mathcal{E}/B$ is localic relative to $\gamma \circ \pi$.

The following observations refer to a fixed $\gamma : \mathcal{E} \to \mathcal{S}$, so we again omit the phrases 'relative to γ'. From Proposition 1.3 immediately follows

Proposition 1.10. *If* C *is torsion free in* $\mathcal{E}$, *and* $B \to C$ *is any map, then* B *is torsion free.*

Recall that if $\gamma : \mathcal{E} \to \mathcal{S}$ is a geometric morphism, then a *bound* for $\mathcal{E}$ relative to γ is an object $G \in \mathcal{E}$ such that for every $A \in \mathcal{E}$ there exists $I \in \mathcal{S}$ and a diagram of form

$$\begin{tikzcd} A' \arrow[r, tail, "i"] \arrow[d, two heads, "q"'] & \gamma^*I \times G \\ A & \end{tikzcd}$$

with q epi and i mono, cf [J1] 4.42-4.43 (where such G is called an *object of generators*). If G is a bound and $G' \to G$ is epi, then G' is a bound. If a bound exists, $\mathcal{E}$ is called *bounded* over $\mathcal{S}$.

Proposition 1.11. *Let $\gamma : \mathcal{E} \to \mathcal{S}$ be a topos bounded over $\mathcal{S}$. Then the following conditions are equivalent:*
1) *there exists a torsion free object* B *with full support (i.e.* $B \to 1$ *is epi);*
2) *there exists a torsion free bound.*

Proof. Assume G is a bound and C is a torsion free object with full support (both relative to γ) . Then $\mathrm{proj}_1 : G \times C \to G$ is epi, so $G \times C$ is a bound; and since C is torsion free, the existence of $\mathrm{proj}_2 : G \times C \to C$ implies that $G \times C$ is torsion free, by Proposition 1.10. This shows that 1) implies 2). The converse is clear since every bound has full support.

We thus rephrase a classical definition [SGA4], [JT],

Definition 1.12. A geometric morphism $\gamma : \mathcal{E} \to \mathcal{S}$ is *étendue-like* (or *makes $\mathcal{E}$ étendu* over $\mathcal{S}$) if $\mathcal{E}$ is bounded over $\mathcal{S}$ and the equivalent conditions of Propositions 1.11 are satisfied.

2. THE CANONICAL MONIC MAP SITE.

In this section, we consider the case where $\mathcal{S} = \mathit{Sets}$, and $\gamma : \mathcal{E} \to \mathcal{S}$ is the global sections functor of a Grothendieck topos $\mathcal{E}$ (hence bounded). Let $L(\mathcal{E})$ be the full subcategory of $\mathcal{E}$ consisting of all the torsion free objects. Thus all maps in $L(\mathcal{E})$ are locally monic in $\mathcal{E}$. Let $M(\mathcal{E})$ be the (non-full) subcategory of $L(\mathcal{E})$ with the same objects as $L(\mathcal{E})$ but with maps only those maps which are monic in $\mathcal{E}$.

For every $A \in L(\mathcal{E})$ (or $M(\mathcal{E})$) , $M(\mathcal{E})/A$ is equivalent as a category to the partially ordered set (frame) of subobjects of A . Thus $M(\mathcal{E})$ is a pre-frame in the sense of the following

Definition 2.1. A category $\mathbf{A}$ is a *preframe* if for every $A \in \mathbf{A}$, the category $\mathbf{A}/A$ is equivalent to a frame.

Every preframe carries a natural Grothendieck topology: a family $\{A_i \to A \mid i \in I\}$ covers A if $\vee(A_i \to A) = 1_A$ in the frame $\mathbf{A}/A$. (This topology need not be subcanonical, see below.) A preframe has the property that every map in it is a monic map. For, if $h \circ f = h \circ g$ (= k, say), then f and g are maps from k to h in the appropriate slice category $\mathbf{A}/A$, which however is a preordered set (being equivalent to a partial order), so $f = g$.

If $\mathbf{A}$ is a small preframe (or essentially small, meaning that it is a U-site in the sense of [SGA4] II.3.0.2), it therefore follows from the Theorem of Rosenthal ([R], Theorem 1.5 and Proposition 1.3) that the topos $\mathrm{sh}(\mathbf{A})$ of sheaves on it is an étendue (relative to *Sets*).

The following Theorem implies that, conversely, every étendue $\mathcal{E}$ arises as the topos of sheaves on an essentially small preframe, in particular as sheaves on a site with monic maps. This answers in the positive a conjecture by Lawvere.

Theorem 2.2 *Let* $\mathcal{E}$ *be an étendue. Then* $\mathcal{E} \simeq \mathrm{sh}(M(\mathcal{E}))$.

Proof. $L(\mathcal{E})$ is a full subcategory of $\mathcal{E}$, and since $\mathcal{E}$ is an étendue, $L(\mathcal{E})$ contains a bound for $\mathcal{E}$, by Proposition 1.11. It follows from [SGA4], IV.1.2.1 that $\mathcal{E} \simeq \mathrm{sh}(L(\mathcal{E}))$, where the topology on $L(\mathcal{E}) \subseteq \mathcal{E}$ is the one induced from $\mathcal{E}$, i.e. consists of families which are jointly epi in $\mathcal{E}$. Therefore it suffices to prove $\mathrm{sh}(L(\mathcal{E})) \simeq \mathrm{sh}(M(\mathcal{E}))$, induced by the (non-full) inclusion $j : M(\mathcal{E}) \hookrightarrow L(\mathcal{E})$. Note that the covering families in $M(\mathcal{E})$ are precisely those covering families in $L(\mathcal{E})$ whose maps are monic in $\mathcal{E}$.

This is a consequence of the basic Comparison Lemma in the theory of sites, as in [SGA4] III, except that the formulation given to it there deals only with full and faithful functors. The functor j at hand is only *locally* full (cf. 2) below), so we need a somewhat more general Comparison Lemma. We state it in more generality than we need, namely what we think is the natural generality. We need the following conditions for a functor $u : \mathbf{C} \to \mathbf{C}'$ between essentially small sites (= U-sites, [SGA4] II.3.0.2); it is clear that the inclusion $j : M(\mathcal{E}) \hookrightarrow L(\mathcal{E})$ satisfies these:

1) u is *cover preserving*.
2) u is *locally full*, meaning: if $g : u(C) \to u(D)$ is a map in $\mathbf{C}'$, there exists a cover $(\xi_i : C_i \to C)_{i \in I}$ in $\mathbf{C}$

and maps $(f_i : C_i \to D)_{i\in I}$ such that

$$g\circ u(\xi_i) = u(f_i) \qquad \forall i \in I .$$

3) u is *locally faithful*, meaning: if $f,f' : C \to D$ in $\mathbf{C}$ have $u(f) = u(f')$, then there exists a cover $(\xi_i)_{i\in I}$ of C with $f\circ\xi_i = f'\circ\xi_i \quad \forall i \in I$.

4) u is *locally surjective* on objects, meaning that to every $C' \in \mathbf{C}'$, there exists a covering family of the form $(u(C_i) \to C')_{i\in I}$.

Also, u is *co-continuous* ([SGA4] III.2.1), meaning

5) if $(\xi_i : C'_i \to u(C))_{i\in I}$ is a cover in $\mathbf{C}'$, then the set of arrows $f : D \to C$ in $\mathbf{C}$, such that $u(f)$ factors through some ξ_i, covers C in $\mathbf{C}$.

The comparison Lemma that will give that j induces an equivalence $sh(M(\mathcal{E})) \simeq sh(L(\mathcal{E}))$ is then the following:

Comparison Lemma. *Let* $u : \mathbf{C} \to \mathbf{C}'$ *be a functor between essentially small sites. If it satisfies* 1) - 4), *then if* F *is a sheaf on* $\mathbf{C}'$, $F\circ u$ *is a sheaf on* $\mathbf{C}$ (i.e. *"u is continuous"*, [SGA4] III.1.1), *and the functor* $u^* : sh(\mathbf{C}') \to sh(\mathbf{C})$ *thus defined is full and faithful. If further u satisfies* 5), u^* *is an equivalence.*

The proof of the first two assertions is essentially routine, as in [SGA4] III. For the last assertion, we get easily (cf. [SGA4] III Proposition 2.6) that u^* preserves colimits, and also that $\varepsilon(C) = u^*(\varepsilon'(u(C)))$, where $\varepsilon(C)$ is the sheaf associated to the presheaf represented by $C \in \mathbf{C}$, and similarly for ε'. Since now every object X in $sh(\mathbf{C})$ is of the form $\varinjlim \varepsilon(C_i)$,

$$X = \varinjlim \varepsilon(C_i) = \varinjlim u^*\, \varepsilon' u(c_i) = u^* \varinjlim \varepsilon' u C_i$$

so u^* is essentially surjective on objects.

3. WHEN IS BG AN ÉTENDUE?

Recall from [JT] or [M3] that if $\mathbf{G}$ is a groupoid in the category of locales (with $d_0,d_1 : G_1 \to G_0$ open surjections - this will be assumed throughout, sometimes tacitly), then one defines the category $\mathbf{BG}$ of sheaves $E \to G_0$ (local homeomor-

phisms) with a right action α by $\mathbf{G}$. It is a topos, and the functor $\pi^* : BG \longrightarrow sh(G_0)$ which forgets the action is the inverse image of a geometric morphism π .

All the following considerations are stated for toposes over *Sets*, but they readily generalize to arbitrary base topos $\mathcal{S}$.

There is a diagram of toposes (where we do not distinguish between a locale X and the topos sh(X) to which it gives rise),

(3.1)
$$\begin{tikzcd} G_1 \arrow[r, "d_1"] \arrow[d, "d_0"'] \arrow[dr, phantom, "\overset{\cong}{\swarrow}\alpha"] & G_0 \arrow[d, "\pi"] \\ G_0 \arrow[r, "\pi"'] & BG \end{tikzcd}$$

Following [M1], we call $\mathbf{G}$ *étale-complete* if this square is a 2-pullback (up to equivalence, as usual, not up to isomorphism).

For localic groups, étale-complete means something like prodiscrete, cf. [M3] §3, and for localic groupoids in general, étale-completeness is sometimes a difficult condition to verify. We do, however, have

Proposition 3.1. *If* $\mathbf{G}$ *is an étale localic groupoid (i.e.* d_0 *and* d_1 *are étale (= local homeomorphisms)), then it is étale-complete.*

Proof. Consider the nerve G. of $\mathbf{G}$; so G_n is the "locale of composable n-tuples", with $\partial_i : G_n \longrightarrow G_{n-1}$ being 'omit i'th vertex' (i = 0,...,n) . Note that $\partial_0 = d_1 : G_1 \longrightarrow G_0$, and vice versa. With $\partial_1 \circ \ldots \partial_n : G_n \longrightarrow G_0$ as structural map, G_n carries an action by $\mathbf{G}$, by precomposition on the first arrow in the n-tuple, thus we get an object $G_n \in BG$. All maps $\partial_i : G_n \longrightarrow G_{n-1}$ for $i \geq 1$ are then maps in BG ; and pulling back along $\partial_0 : G_n \longrightarrow G_{n-1}$ induces an equivalence $sh(G_{n-1}) \simeq BG/G_n$ (with quasi-inverse: pulling back along the degeneracy map $\sigma_0 : G_{n-1} \longrightarrow G_n$) .

Under these equivalences, the square (3.1) becomes equivalent to the square

$$\begin{tikzcd} BG/G_2 \arrow[r, "\partial_1"] \arrow[d, "\partial_2"'] & BG/G_1 \arrow[d, "\partial_1"] \\ BG/G_1 \arrow[r, "\partial_1"'] & BG/G_0 \simeq BG \end{tikzcd} \tag{3.2}$$

(in the comparison box, two of the comparison faces are commutative up to isomorphism, just by simplicial identities among the ∂_i's , and the remaining two are of the form

$$\begin{tikzcd} BG/G_1 \arrow[r, "\partial_1"] \arrow[d, "\partial_0"', "\simeq"] & BG/G_0 \arrow[d, "\simeq"] \\ sh(G_0) \arrow[r, "\pi"'] & BG \end{tikzcd}$$

which commutes up to an isomorphism which is essentially the α of (3.1)). Now since **G** is a groupoid, the square

$$\begin{tikzcd} G_2 \arrow[r, "\partial_1"] \arrow[d, "\partial_2"'] & G_1 \arrow[d, "\partial_1"] \\ G_1 \arrow[r, "\partial_1"'] & G_0 \end{tikzcd}$$

is a pullback; since the maps are étale and G-equivariant, it may therefore be seen as a product diagram in BG . But it is well known that if $C = A \times B$ in a topos $\mathcal{E}$, then one obtains $\mathcal{E}/C$ as the 2-pullback $\mathcal{E}/A \times_{\mathcal{E}} \mathcal{E}/B$. Therefore (3.2) is a 2-pullback, and by the equivalence of (3.1) and (3.2), we see that (3.1) is a 2-pullback as well, proving that **G** is étale-complete.

A basic idea of Grothendieck *et al* ([SGA4] IV.9.8.2) is

localic groupoid, and conversely, such BG's are always étendues (except that [SGA4] deals with *topological* groupoids and étendues only, but cf. [JT] for the general case).

We would like to emphasize here that this correspondence between étendues and étale groupoids should not be interpreted as a reduction of the study of étendues to that of étale groupoids. There are many examples of étendues which are more naturally represented by a localic groupoid which is not étale. For example, if M is a manifold equipped with a foliation F, the holonomy groupoid $\mathrm{Hol}(M,F)$ of Ehresmann [E2], [P1] is most naturally defined as the groupoid whose space $\mathrm{Hol}(M,F)_0$ of objects is M itself, and whose space of morphisms $\mathrm{Hol}(M,F)_1$ is a space of equivalence classes of paths within leaves, see [W]. For this groupoid, the domain and codomain maps are submersions with connected fibres. $\mathrm{BHol}(M,F)$ is an étendue, because $\mathrm{Hol}(M,F)$ is equivalent to an étale groupoid, obtained by pulling back along a complete transversal section.

The notion of equivalence used here is the obvious one: a homomorphism $f : \mathbf{G} \to \mathbf{H}$ of localic groupoids is an equivalence if 1) f is essentially surjective on objects, and 2) f is fully faithful, meaning that

$$\begin{tikzcd} G_1 \arrow[r, "f_1"] \arrow[d, "(d_0,d_1)"'] & H_1 \arrow[d, "(d_0,d_1)"] \\ G_0 \times G_0 \arrow[r, "f_0 \times f_0"'] & H_0 \times H_0 \end{tikzcd}$$

is a pullback. Such an equivalence induces an equivalence of classifying toposes $Bf : \mathbf{BG} \to \mathbf{BH}$ (cf. [M1]).

It follows from Proposition 3.1 and the following Proposition that $\mathrm{Hol}(M,F)$, being equivalent to an étale groupoid, is étale-complete.

Proposition 3.2 *If* $f : \mathbf{G} \to \mathbf{H}$ *is an equivalence, as above, then* $\mathbf{G}$ *is étale-complete if and only if* $\mathbf{H}$ *is.*

Proof. If $f : \mathbf{G} \to \mathbf{H}$ is an equivalence, one can construct in a standard way a commutative (up to natural isomorphism) diagram

$$\begin{tikzcd} & \mathbf{K} \arrow[dl] \arrow[dr] \arrow[d, phantom, "\cong"] & \\ \mathbf{G} \arrow[rr, "f"'] & & \mathbf{H} \end{tikzcd}$$

where $\mathbf{K} \to \mathbf{G}$ and $\mathbf{K} \to \mathbf{H}$ are also equivalences, and $K_0 \to G_0$ and $K_0 \to H_0$ are open surjections (take $K_0 = G_0 \times_{H_0} H_1$, etc.). Therefore, it is enough to prove the Proposition under the assumption that $f_0 : G_0 \to H_0$ is an open surjection. Consider the diagram

$$\begin{tikzcd} G_1 \arrow[r, "f_1"] \arrow[d] & H_1 \arrow[r] \arrow[d] & \mathbf{BG} \arrow[r, equal] \arrow[d] & \mathbf{B} \arrow[d] \\ G_0 \times G_0 \arrow[r, "f_0 \times f_0"'] & H_0 \times H_0 \arrow[r] & \mathbf{BG} \times \mathbf{BG} \arrow[r, equal] & \mathbf{BH} \times \mathbf{BH} . \end{tikzcd}$$

The left hand square is a pullback since f is full and faithful. If $\mathbf{H}$ is étale-complete, the middle square is a 2-pullback, and hence so is the rectangle composed out of the two left hand squares, i.e. $\mathbf{G}$ is étale-complete. Conversely, if $\mathbf{G}$ is étale-complete, the composed rectangle is a 2-pullback. Since $f_0 \times f_0$ is an open surjection, it follows that the middle square must also be a 2-pullback, so $\mathbf{H}$ is étale-complete.

As a start to an answer to the question of the present paragraph, we have

Proposition 3.3 *Let* $\mathbf{G}$ *be a localic groupoid. Then the following conditions are equivalent:*
1) $\mathbf{BG}$ *is localic, and* $\mathbf{G}$ *is étale-complete;*
2) $\mathbf{G}$ *is an effective equivalence relation (= kernel pair of its co-equaliser).*

Proof. Assume $\mathbf{BG}$ is (sheaves on) a locale and that $\mathbf{G}$ is étale-complete. Then the 2-pullback (3.1) in the category of toposes is *a fortiori* a pullback in the full subcategory *Loc* of locales (in which of course invertible 2-cells are equal-

ities). Thus d_0,d_1 is the kernel pair in *Loc* of π. Conversely, if $G_1 \rightrightarrows G_0$ is the kernel pair of its coequaliser $q : G_0 \to B$ in *Loc*, q is an open surjection, by the assumption that d_0,d_1 are open surjections, [M2] 1.2, and hence by [JT], q is a descent map, this means that $BG \simeq sh(B)$, since actions of the kernel pair for a map q is the same thing as descent data for descent along q (and pulling back along q preserves and reflects the property of being étale). The functor sh from locales to toposes preserves pullbacks, so the pullback defining G_1 goes by sh to the pullback (3.1), proving étale-completeness.

Proposition 3.4. *Let* $\mathbf{G}$ *be an étale-complete localic groupoid, and assume* $\mathbf{H} \to \mathbf{G}$ *is a discrete fibration of localic groupoids (meaning that a certain diagram, cf. the proof below, is a pullback). Assume further that* $H_0 \to G_0$ *is étale. Then* $\mathbf{H}$ *is étale-complete.*

Proof. We have a box of geometric morphisms among toposes

(3.3)
$$\begin{tikzcd}
 & H_1 \arrow[rr] \arrow[dl] \arrow[dd] & & H_0 \arrow[dl] \arrow[dd] \\
G_1 \arrow[rr] \arrow[dd, "d_1"'] & & G_0 \arrow[dd] & \\
 & H_0 \arrow[rr] \arrow[dl] & & \mathbf{BH} \arrow[dl] \arrow[ul, phantom, "\Nearrow"] \\
G_0 \arrow[rr] & & \mathbf{BG} \arrow[ul, phantom, "\Nearrow"] &
\end{tikzcd}$$

The front is a 2-pullback by assumption on $\mathbf{G}$, and the left hand square is a pullback since $\mathbf{H} \to \mathbf{G}$ is a discrete fibration. The right hand square is a 2-pullback since it is equivalent to the square

$$\begin{tikzcd}
sh(H_0) \simeq sh(G_0)/\pi^*H \arrow[r] \arrow[d] & sh(G_0) \arrow[d, "\pi"] \\
\mathbf{BH} \simeq \mathbf{BG}/\mathbf{H} \arrow[r] & \mathbf{BG}
\end{tikzcd}$$

where $\mathbf{H}$ is the sheaf $H_0 \to G_0$ with $\mathbf{G}$-action, corresponding to

the discrete fibration, cf. [M1] 5.11 for

(3.4) $$\mathbf{BG/H} \simeq \mathbf{BH} .$$

But 2-pullbacks share with ordinary pullbacks the property that, in two consecutive squares with the right hand square a (2-)pullback, the left hand square is a (2-) pullback if and only if the total square is, cf [B] 3.1.6. From this it follows that the back square in (3.3) is a 2-pullback, so **H** is étale-complete.

Proposition 3.5 *Let* **G** *be a localic groupoid, and let* $H \to G_0$ *be a sheaf with a right* **G**-*action. If the total groupoid* **H** *of the action* $(H_0 = H\ ,\ H_1 = H_0 \times_{G_0} G_1)$ *is an effective equivalence relation,* H *is torsion free in* **BG** ; *and conversely, provided* **G** *is étale-complete.*

Proof. Since $\mathbf{BG/H} \simeq \mathbf{BH}$ (cf. (3.4)), **BG/H** is localic if **H** is an effective equivalence relation, by 2) $\Rightarrow$ 1) in Proposition 3.3. On the other hand, if **G** is étale-complete, then so is **H** , by Proposition 3.4, so if **BH** ($\simeq$**BG/H**) is localic, **H** is an effective equivalence relation, by 1) $\Rightarrow$ 2) in Proposition 3.3.

Recall that a *principal* **G**-bundle for a localic groupoid **G** is a locale $\alpha : E \to G_0$ over G_0 , equipped with a right **G**-action, such that the total groupoid **E** of the action (cf. **H** in Proposition 3.5) is an effective equivalence relation; we denote its coequaliser $\beta : E \to E/\mathbf{G}$ (careful - the symbol / is used in two ways by now!). It is an open surjection, and E/**G** is called the *basespace/locale* of the principal **G**-bundle E . (The classical case is when **G** is a (topological) group, and $\beta : E \to E/\mathbf{G}$ is locally trivial; α carries no information.) Unlike β , $\alpha : E \to G_0$ need not be an open surjection, but if it is, E is called a *fully supported* principal **G**-bundle, and if α is étale, E is called an *étale* principal **G**-bundle.

We can now give some answers to the question raised in the headline of the paragraph:

Theorem 3.6. *Let* **G** *be a localic groupoid (with* $d_0, d_1 : G_1 \to G_0$ open). *Then the following two conditions are equivalent:*

1) **BG** *is an étendue, and* **G** *is étale-complete;*
2) *There exists an étale, fully supported principal*

G-bundle.

Proof. Assuming 1), let $\alpha : E \to G_0$ be a torsion free object in **BG** with full support. Then α is an open surjection. By Proposition 3.5, **E** is an effective equivalence relation, so E is an étale and fully supported principal **G**-bundle.

Conversely, assume 2). Let $\alpha : E \to G_0$ be an étale and fully supported principal **G**-bundle. We shall out of E and the action explicitly construct an étale localic groupoid $\mathcal{E}\mathcal{E}^{-1}$, with locale of objects E/**G** and with $B(\mathcal{E}\mathcal{E}^{-1}) \simeq \mathbf{BG}$. Thus **BG** will be proved an étendue, but since in fact we shall have $\mathcal{E}\mathcal{E}^{-1} \simeq \mathbf{G}$ as localic groupoids, it follows from Propositions 3.2 and 3.1 that **G** is étale-complete.

We shall describe the construction of the groupoid $\mathcal{E}\mathcal{E}^{-1}$ in two equivalent ways, one topos theoretic, the other fibre-bundle theoretic.

Since the total category **E** of the action is an effective equivalence relation, with quotient E/**G**, and since **BG**/E $\simeq$ **BE**, we have the middle equivalence in

$$
(3.5)\qquad
\begin{tikzcd}
\mathbf{BG}/(E \times E) \arrow[r, shift left] \arrow[r, shift right] \arrow[d, "\simeq"'] & \mathbf{BG}/E \arrow[r] \arrow[d, "\simeq"'] & \mathbf{BG} \arrow[d, equal] \\
sh((E \times E)/\mathbf{G}) \arrow[r, shift left] \arrow[r, shift right] & sh(E/\mathbf{G}) \arrow[r] & \mathbf{BG}
\end{tikzcd}
$$

Now $E \times E$ (formed in **BG**) is a torsion free object with full support, as well, so the same procedure that gave E/**G** out of E gives $(E \times E)/\mathbf{G}$ (diagonal action of **G** on $E \times_{G_0} E$ - the underlying locale of $E \times E$ in **BG**). The row on the top is 2-exact (2-kernel-pair/descent), hence so is the bottom row, and thus

$$(E \times E)/\mathbf{G} \rightrightarrows E/\mathbf{G}$$

is a localic groupoid with classifying topos **BG**; and it is étale, since the geometric morphisms in (3.5) are all slices. (In fact, this is the [JT] construction of an étale groupoid out of a localic slice of an étendue.) In the present case, we can, however, describe the étale localic groupoid $(E \times E)/\mathbf{G} \rightrightarrows E/\mathbf{G}$ in more explicit terms, and through this see

that this groupoid is equivalent (not just 'Morita-equivalent') to the given $\mathbf{G}$. We give this description in point set terms, and refer the reader to [K2] for a diagrammatic justification.

Thus the points of $(E \times E)/\mathbf{G}$ are represented by pairs $x,y \in E$ with $\alpha(x) = \alpha(y)$, and with (x,y) identified with $(x \circ g, y \circ g)$, (for those $g \in G_1$ for which it makes sense). This justifies a "fraction"-notation yx^{-1} for the equivalence class of (x,y) , and the notation EE^{-1} for $(E \times E)/\mathbf{G}$. The groupoid structure is given by

$$d_0(yx^{-1}) = \alpha(x) , \qquad d_1(yx^{-1}) = \alpha(y) ,$$

and

$$(zy^{-1}) \cdot (yx^{-1}) = zx^{-1} .$$

The localic groupoid $(E \times E)/\mathbf{G} \rightrightarrows E/\mathbf{G}$ itself is therefore denoted $\mathcal{E}\mathcal{E}^{-1}$.

In the terminology of [K1], [K2], E is a pregroupoid on $A = G_0$, $B = E/\mathbf{G}$, with associated groupoids $E^* = \mathcal{E}\mathcal{E}^{-1}$ and $E_* = \mathbf{G}(" = \mathcal{E}^{-1}\mathcal{E}")$. The equivalence of $\mathcal{E}^{-1}\mathcal{E}$ and $\mathcal{E}\mathcal{E}^{-1}$ as localic groupoids is an immediate consequence of the Pradines butterfly diagram [P1], [P3] p. 536, or [K2] §2, in which we have canonical equivalences

$$\mathcal{E}\mathcal{E}^{-1} \longleftarrow \Lambda \longrightarrow \mathcal{E}^{-1}\mathcal{E}$$

for a certain groupoid Λ (with E as locale of objects). This proves the theorem.

Let us remark that the construction of the groupoid $\mathcal{E}\mathcal{E}^{-1}$ out of a principal $\mathbf{G}$-bundle E in essence goes back to Ehresmann [E1], and has been reconsidered on various levels of generality, in [P2], [P3], [K1], [K2]. Pradines says that $\mathcal{E}\mathcal{E}^{-1}$ is the *conjugate action* of the (principal) action of $\mathbf{G}$ on E .

Let us further remark that, from the construction in [K2] of $\mathcal{E}\mathcal{E}^{-1}$, one immediately gets that it is an étale groupoid if E is étale over G_0 .

We may summarise part of the discussion and theorem in

Theorem 3.6′ *If for a localic groupoid* $\mathbf{G}$ *, there exists an étale fully supported principal* $\mathbf{G}$*-bundle* E *, then* $\mathbf{BG}$ *is an étendue, and an étale localic groupoid for* $\mathbf{BG}$ *comes about as the conjugate action of* $\mathbf{G}$ *on* E .

4. CONSTRUCTION OF A SITE WITH MONIC MAPS OUT OF AN ÉTALE GROUPOID.

We consider a localic groupoid $\mathbf{G} = G_1 \rightrightarrows G_0$ with d_0, d_1 étale. For each open sublocale $U \hookrightarrow G_0$, we have a right action of $\mathbf{G}$ on the locale over G_0

$$(4.1) \qquad d_1^{-1}(U) \xrightarrow{\ d_0\ } G_0 ,$$

utilising the composition law of $\mathbf{G}$. Here, d_0 denotes the restriction of $d_0 : G_1 \to G_0$ to the open sublocale $d_1^{-1}(U)$. Since $d_0 : G_1 \to G_0$ is assumed étale, and $d_1^{-1}(U)$ is open in G_1, the d_0 in (4.1) is étale, and thus defines an object of $B\mathbf{G}$. This object we denote simply $d_1^{-1}(U)$.

The objects of form $d_1^{-1}(U)$ generate $B\mathbf{G}$; for, if $p : H \to G_0$ is a sheaf with a right $\mathbf{G}$-action, H is covered, as a locale, by sections $s : U \to H$ of p, defined over open sublocales U of G_0; but such a section s defines a $\mathbf{G}$-equivariant map

$$(4.2) \qquad \bar{s} : d_1^{-1}(U) \longrightarrow H ,$$

in point set notation: $\bar{s}(g) = s(d_1(g)) \bullet g$. Since s factors through $\bar{s}$ (using the formation $i : G_0 \to G_1$ of identity arrows), the maps (4.2) cover H in $sh(G_0)$, but $\pi^* : B\mathbf{G} \to sh(G_0)$ (preserves and) reflects covers. Every $\mathbf{G}$-equivariant map $d_1^{-1}(U) \to H$ arises as $\bar{s}$ for a unique section s in this way.

Furthermore, the objects $d_1^{-1}(U) \in B\mathbf{G}$ are torsion free in the sense introduced in §1; by Proposition 3.5, we just have to see that the action groupoid for the $\mathbf{G}$-action on $d_1^{-1}(U)$ is an effective equivalence relation. But since $\mathbf{G}$ is a groupoid (with d_1 an open surjection), the diagram

$$G_1 \times_{G_0} G_1 \underset{\mathrm{proj}_1}{\overset{\circ}{\rightrightarrows}} G_1 \xrightarrow{\ d_1\ } G_0$$

is exact (kernel pair/coequalizer), and the action groupoid for the $\mathbf{G}$-action on $d_1^{-1}(U)$ appears by pulling back this whole diagram along the inclusion $U \hookrightarrow G_0$, and thus is the kernel pair of $d_1 : d_1^{-1}(U) \to U$.

It follows that all maps in $B\mathbf{G}$ between objects of form $d_1^{-1}(U)$ are locally monic. We proceed to identify the monic ones.

Lemma 4.1. *Let* $\bar{s} : d_1^{-1}(U) \to d_1^{-1}(V)$ *be a* G*-equivariant map. Then* $\bar{s}$ *is monic in* BG *if and only if the composite*

$$U \xrightarrow{s} d_1^{-1}(V) \xrightarrow{d_1} V$$

is monic (where $s = \bar{s} \circ i$ *is the section corresponding to* $\bar{s}$ *, as above).*

Proof. By stability of everything involved under base change, it suffices to do a point set argument, and utilising that $\pi^* : BG \to sh(G_0)$ preserves and reflects monos. Assume $\bar{s}$ is monic, and let $x,y \in U$ have $d_1(s(x)) = d_1(s(y)) \in V$. Consider the arrow g from y to x such that

$$s(x) \circ g = s(y) \ .$$

Then

$$\bar{s}(g) = s(x) \circ g = s(y) = \bar{s}(i(y)) \ ,$$

and since $\bar{s}$ is monic, $g = i(y)$, so $x = y$. Conversely, assume that $d_1 \circ s : U \to V$ is monic and that $\bar{s}(g_1) = \bar{s}(g_2)$, where $g_1 : z \to x$ and $g_2 : z \to y$. Then

$$s(y) \circ g_2 = \bar{s}(g_2) = \bar{s}(g_1) = s(x) \circ g_1 \ ;$$

in particular, $d_1(s(y)) = d_1(s(x))$, so by assumption $x = y$. Thus the above equation reads $s(x) \circ g_2 = s(x) \circ g_1$, so $g_1 = g_2$; this proves the Lemma.

Since the $d_1^{-1}(U)$'s generate BG , and are torsion free, and since any subobject of $d_1^{-1}(U)$ is of form $d_1^{-1}(V)$ for a unique open sublocale V of U , the Comparison Lemma of §2 gives that the monic maps between objects of the form $d_1^{-1}(U)$ in BG form a site of definition for BG . Utilising Lemma 4.1, we may describe this site in an alternative way, arriving at a site analogous to the one utilised by Tapia [T] p. 43 in the context of smooth manifolds, namely

Theorem 4.2 *Let* G *be an étale groupoid, defining the étendue* BG . *Then a site of definition of* BG *may be described as follows:*

objects: *open sublocales* $U \subseteq G_0$;
arrows: $U \to V$: *sections* $s : U \to G_1$ *of* d_0 *such* that $d_1 \circ s$ *is monic and factors through* $V \subseteq G_0$;
composition: *if* $t : V \to W$ *is a further arrow, then*

$t \circ s$ *is the section over* U *given by (in point set terms)*

$$(t \circ s)(x) = t(d_1(s(x)) \circ s(x) \ ;$$

coverings: *given by open coverings* $\{U_i \subseteq U \mid i \in I\}$ of U. *This is a site with monic maps.*

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