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MONADS AND COCOMPLETENESS OF CATEGORIES

by *Manuela SOBRAL* *

RÉSUMÉ: Soit \mathcal{E} une catégorie complète et bien-potentiée. La théorie des monades est utilisée pour démontrer que, pour chaque \mathcal{E} -objet A , l'enveloppe réflexive \mathcal{A} de $\{A\}$ dans \mathcal{E} est une catégorie cocomplète, lorsque A vérifie une condition d'injectivité. Ensuite, nous démontrons que ce résultat est une conséquence d'un théorème qui établit une relation entre \mathcal{A} et des monades induites dans \mathcal{E} et $\text{Set}^{\mathbb{I}}$ par l'adjonction associée au foncteur comparaison $\Phi : \mathcal{E}^{\text{op}} \rightarrow \text{Set}^{\mathbb{I}}$, τ est la monade induite dans Set par l'adjonction $\langle A^-, \text{Hom}(-, A) \rangle : \text{Set} \rightarrow \mathcal{E}^{\text{op}}$.

1. INTRODUCTION.

Let \mathcal{E} be a complete and well-powered category. Then, the reflective hull $\text{REF}(A)$ of $\{A\}$ in \mathcal{E} is its limit-closure and A is a strong cogenerator in $\text{REF}(A)$, for each single \mathcal{E} -object A (see [9] 6.1 where credit is given to Ringel [7]).

It is well-known that \mathcal{E} has (epi, strong mono) - factorizations and so that extremal monomorphisms are strong. If, furthermore, \mathcal{E} is co-well-powered then it has (epi, strong mono) - factorization of sources and this implies the existence of certain colimits in \mathcal{E} ([5] 1.1) and, consequently, in its reflective full subcategories.

Throughout this paper \mathcal{E} will denote a complete and well-powered category. We are going to prove that if a \mathcal{E} -object A is D -injective, D being a class of morphisms we shall define shortly, then $\text{REF}(A)$ is a cocomplete category. For that we

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show that $\text{REF}(A)$ is dually equivalent to a monadic category over a complete category.

Our general reference for monad theory is [1].

From now on, we consider a \mathcal{E} -object A and denote by \mathbb{T} , $\Phi : \mathcal{E}^{\text{op}} \rightarrow \text{Set}$, and \mathfrak{A} , the monad induced in Set by the adjunction $A \dashv \text{Hom}(-, A) : \mathcal{E}^{\text{op}} \rightarrow \text{Set}$, the comparison functor and $\text{REF}(A)$, respectively. We will often take $F = A$ and $H = \text{Hom}(-, A)$.

2. THE \mathfrak{A} -CLOSURE OPERATOR.

Let \mathcal{M} be the full subcategory of \mathcal{E}^2 (where $2 = \{0 \rightarrow 1\}$) with objects the external monomorphisms. The \mathfrak{A} -closure operator ([2], [3]), originally defined in [8] for $\mathcal{E} = \text{Top}$, is the functor

$$[]_{\mathfrak{A}} : \mathcal{M} \rightarrow \mathcal{M}$$

which assigns to each extremal monomorphism m the generalized pullback of

$$\{e = \text{eq}(f, g) \mid f.m = g.m \text{ and } \text{cod}f = \text{cod}g \in \mathfrak{A}\}.$$

We consider the extension of $[]_{\mathfrak{A}}$ to \mathcal{E}^2 , also denoted by $[]_{\mathfrak{A}}$, or simply $[]$, assigning to $f = m.e$ in \mathcal{E} the \mathfrak{A} -closure $[m]_{\mathfrak{A}}$ of m , $m.e$ being the (epi, extremal mono)-factorization of f .

Let $\varepsilon_X : X \rightarrow A^{\text{Hom}(X, A)}$ be the unique \mathcal{E} -morphism such that $p_f \cdot \varepsilon_X = f$ for all $f \in \text{Hom}(X, A)$, (p_f) being the canonical projections. Then ε^{op} is the counit of the adjunction $F = A \dashv H = \text{Hom}(-, A) : \mathcal{E}^{\text{op}} \rightarrow \text{Set}$.

The following is essentially Lemma 1 in [6], whose proof we include for convenience of the reader.

2.1 Lemma. *For each \mathcal{E} -object X , $[\varepsilon_X]_{\mathfrak{A}} = e_X$ where $e_X = \text{eq}(F H \varepsilon_X, \varepsilon_{F H X})$.*

Proof. The functors $[]_{(A)}$ and $[]_{\mathfrak{A}}$ coincide ([4] 2.1) and so it is enough to prove that $[\varepsilon_X]_{(A)} = e_X$ for every $X \in \mathcal{E}$. We have that $p_h \cdot \varepsilon_{F H X} = h$ and $p_h \cdot F H \varepsilon_X = p_h \cdot \varepsilon_X$ for

all \mathcal{E} -morphisms $h : A^{\text{Hom}(X,A)} \rightarrow A$. For \mathcal{E} -morphisms $f, g : A^{\text{Hom}(X,A)} \rightarrow A$, if $f \cdot \varepsilon_X = g \cdot \varepsilon_X$ then $f \cdot e_X = g \cdot e_X$.
Indeed

$$\begin{aligned} f \cdot e_X &= p_f \cdot \varepsilon_{FHX} \cdot e_X = p_f \cdot FH\varepsilon_X \cdot e_X = p_{f \cdot \varepsilon_X} \cdot e_X \\ &= p_{g \cdot \varepsilon_X} \cdot e_X = p_g \cdot FH\varepsilon_X \cdot e_X = p_g \cdot \varepsilon_{FHX} \cdot e_X = g \cdot e_X. \end{aligned}$$

Since the converse is trivial, then $f \cdot \varepsilon_X = g \cdot \varepsilon_X$ if and only if $f \cdot e_X = g \cdot e_X$.

Furthermore, $\varepsilon_{FHX} \cdot \varepsilon_X = FH\varepsilon_X \cdot \varepsilon_X$. Hence

$$p_h \cdot \varepsilon_{FHX} \cdot [\varepsilon_X]_{(A)} = p_h \cdot FH\varepsilon_X \cdot [\varepsilon_X]_{(A)}$$

for all $h \in \text{Hom}(A^{\text{Hom}(X,A)}, A)$, and so

$$\varepsilon_{FHX} \cdot [\varepsilon_X]_{(A)} = FH\varepsilon_X \cdot [\varepsilon_X]_{(A)}.$$

It is now straightforward to prove that $[\varepsilon_X]_{(A)} = e_X$. □

Let $D = \{[\varepsilon_X]_{\mathcal{A}} \mid X \in \mathcal{E}\}$, α be the unit and β^{op} the co-unit of the adjunction $M \dashv \Phi : \mathcal{E}^{\text{op}} \rightarrow \text{Set}^{\mathbb{T}}$.

2.2 Lemma. *If A is D -injective then β_X is an A -epi for every \mathcal{E} -object X .*

Proof. We recall that $M(Y, \theta)$ is the equalizer object of the \mathcal{A} -morphisms $F\theta, \varepsilon_{FY} : A^Y \rightarrow A^{\text{Hom}(AY,A)}$ and so it is an \mathcal{A} -object, for every \mathbb{T} -algebra (Y, θ) . For $X \in \mathcal{E}$ we have the diagram

$$\begin{array}{ccc} & M\Phi X & \\ \beta_X \nearrow & & \searrow e_X \\ X & \xrightarrow{\varepsilon_X} & FHX \end{array} \quad \begin{array}{c} \xrightarrow{\varepsilon_{FHX}} \\ \xrightarrow{FH\varepsilon_X} \end{array} \quad FHFHX$$

where the triangle commutes. Let f and g be two parallel \mathcal{A} -morphisms such that $f \cdot \beta_X = g \cdot \beta_X$. Without loss of generality, since A is a cogenerator in \mathcal{A} , we assume that the codomain of f and g is A . The injectivity condition of

A with respect of $[\varepsilon_X] = e_X$ implies that there exist morphisms f' and g' such that $f' \cdot [\varepsilon_X] = f$ and $g' \cdot [\varepsilon_X] = g$. Hence $f' \cdot \varepsilon_X = g' \cdot \varepsilon_X$. By definition of \mathcal{A} -closure of ε_X $f' \cdot [\varepsilon_X] = g' \cdot [\varepsilon_X]$ and so $f = g$. \square

3. COCOMPLETENESS OF \mathcal{A} .

Let $S = (S, \alpha, \delta)$ be a monad in \mathcal{K} and $\text{Fix}(S, \alpha)$ be the full subcategory with objects all \mathcal{K} -objects X for which α_X is an isomorphism.

The equivalence of the following assertions is known.

3.1 Proposition. *For a monad $S = (S, \alpha, \delta)$ in \mathcal{K} the following are equivalent:*

- (i) δ is an isomorphism.
- (ii) $\alpha_S = S_\alpha$.
- (iii) α_S is an isomorphism.
- (iv) \mathcal{K}^S is concretely isomorphic to $\text{Fix}(S, \alpha)$.

A monad S is called **idempotent** if it satisfies one, and so all, of the conditions (i) - (iv) in 3.1.

3.2 Theorem. *If \mathcal{A} is injective with respect to \mathcal{D} then \mathcal{A} is a cocomplete category.*

Proof: We consider the monad $S = (\Phi M, \alpha, \Phi \beta^{\text{op}} M)$ induced in Set^\top by the adjunction $M \dashv \Phi : \mathcal{E}^{\text{op}} \rightarrow \text{Set}^\top$. The restriction Φ_1 of Φ to \mathcal{A}^{op} is part of an adjunction which induces the same monad in Set^\top as the former adjunction does. Indeed, it is enough to remark that $M(Y, \Theta) \in \mathcal{A}$ for all \top -algebra (Y, Θ) (see proof of 2.2).

If $X \in \mathcal{A}$ there exists an \mathcal{A} -extremal monomorphism from X to some power of A and so ε_X is an extremal monomorphism in \mathcal{A} . Since $\varepsilon_X = [\varepsilon_X] \cdot \beta_X$ and β_X is an epimorphism in \mathcal{A} (2.2), then it is an isomorphism. Hence $\Phi \beta^{\text{op}} M$ is an isomorphism and so S is an idempotent monad. It is now clear that $\Phi_1 : \mathcal{A}^{\text{op}} \rightarrow \text{Fix}(\Phi M, \alpha) \cong (\text{Set}^\top)^S$ is an equivalence. We have therefore concluded that \mathcal{A} is dually equivalent to a monadic category $(\text{Set}^\top)^S$ over a complete category Set^\top and so that \mathcal{A} is cocomplete. \square

4. MONADS INDUCED BY THE COMPARISON ADJUNCTION

The adjunction associated with the comparison functor $\Phi : \mathcal{E}^{\text{op}} \rightarrow \text{Set}^\top$ induces a monad $S = (\Phi M, \alpha, \Phi \beta^{\text{op}} M)$ and a co-

monad in \mathcal{E}^{op} , i.e. a monad $S' = (M\phi, \beta, (M\alpha\phi)^{\text{op}})$ in \mathcal{E} . The monad S was the main tool for proving the cocompleteness of $\text{REF}(A) = \mathcal{A}$, whenever A is injective with respect to D . This is a consequence of the fact that the injectivity condition is equivalent to some close relations between \mathcal{A} , S and S' .

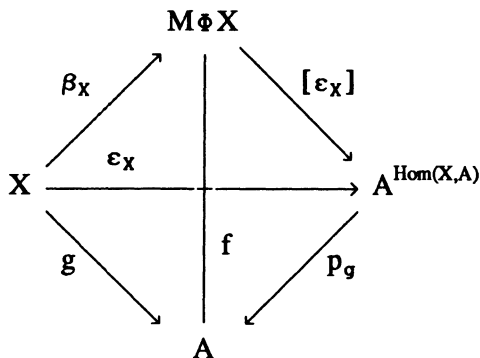
4.1 Theorem. *The following assertions are equivalent.*

- (i) A is D -injective.
- (ii) The comparison functor ϕ induces an equivalence $\phi_1 : \mathcal{A}^{\text{op}} \rightarrow (\text{Set}^{\mathbb{T}})^S$.
- (iii) A and $\mathcal{E}^{S'}$ are concretely isomorphic over \mathcal{E} .

Proof: (i) \Rightarrow (ii) See proof of 3.2.

(ii) \Rightarrow (iii) Since ϕ_1 is an equivalence, β_X is an isomorphism for every $X \in \mathcal{A}$. Then $\beta_{M\phi}$ is an isomorphism. This tells us that S' is an idempotent monad (3.1 (iii)). The functor $L : \mathcal{A} \rightarrow \mathcal{E}^{S'} \cong \text{Fix}(M\phi, \beta)$ defined on objects by $L(X) = (X, \beta_X^{-1})$ is a concrete isomorphism, i.e. $U^{S'}.L = E$ where $E : \mathcal{A} \rightarrow \mathcal{E}$ is the embedding and $U^{S'} : \mathcal{E}^{S'} \rightarrow \mathcal{E}$ is the forgetful functor from the Eilenberg-Moore category of algebras $\mathcal{E}^{S'}$ to \mathcal{E} .

(iii) \Rightarrow (i) Let $L : \mathcal{A} \rightarrow \mathcal{E}^{S'}$ be an isomorphism such that $U^{S'}.L = E$. Then β_X is the reflection of X in \mathcal{A} . Indeed, $M\phi X$ is an A -object and $\beta_X : X \rightarrow M\phi X \cong U^{S'} F^{S'} X$ where $F^{S'}$ is the left adjoint to $U^{S'}$, is universal from X to E . Let $f : M\phi X \rightarrow A$ be a \mathcal{E} -morphism. Then, by definition of ε_X , $p_g \cdot \varepsilon_X = g$ for $g = f \cdot \varepsilon_X$.



Since β_X is the reflection of X in \mathcal{A} , hence $p_G.[\varepsilon_X].\beta_X = f.\beta_X$ implies that $p_G.[\varepsilon_X] = f$ and so that A is D -injective. \square

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