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ALMOST SMOOTH ALGEBRAS

by Alfredo R. GRANDJEAN and Maria J. VALE

RÉSUMÉ. Le but de cet article est d' introduire et de caractériser les algèbres "presque lisses". Une A-algèbre B est presque lisse si et seulement si, pour tout homomorphisme surjectif de A-algèbres commutatives de but B la seconde suite exacte fondamentale associée est une suite exacte courte.

INTRODUCTION.

We introduce and characterize "almost smooth algebras", which generalize the formally smooth algebras of Grothendieck [1]. There are "almost smooth algebras" that are not formally smooth.

It is well known that for every epimorphism of groups there is a short exact sequence of modules [2]. Also, for every surjective homomorphism of commutative algebras there is a right exact sequence (the second fundamental exact sequence [5]), whose lack of exactness is measured by the cotangent functors [3], which play an important role in algebraic deformation theory. "Almost smooth algebras" are characterized by the short exactness property of the second fundamental exact sequence.

DEFINITION.

An A-algebra B is *almost smooth* if for any A-algebra C , any ideal I of C satisfying $I^2 = 0$, and any A-algebra homomorphism $g : B \to C/I$ such that I is an injective B-module via g , there exists a lifting $f : B \to C$ of g that is an A-algebra homomorphism.

$$\begin{array}{ccccccccc} & & & & B & & & & \\ & & & {}^{f}\swarrow & & \searrow^{g} & & & \\ I^2 = 0 , \quad 0 & \longrightarrow & I & \longrightarrow & C & \longrightarrow & C/I & \longrightarrow & 0 . \end{array}$$

THEOREM. *Let* B *be an* A-algebra. *Then the following conditions are equivalent:*

(1) B *is an almost smooth* A-*algebra;*

(2) $T^1(B\backslash A,I) = 0$ *for every injective* B-*module* I , *where* T^1 *is Lichtenbaum and Schlessinger's first upper cotangent functor* [3];

(3) $T_1(B\backslash A,B) = 0$, *where* T_1 *is the first lower cotangent functor* [3];

(4) *for every surjective homomorphism of* A-*algebras* $g : D \twoheadrightarrow B$, *the imperfection module of the* D-*algebra* B *relative to* A , $\gamma_{B\backslash D\backslash A}$ ([1], p.136), *is isomorphic to* J/J^2 , *where* $J = \mathrm{Ker} g$;

(5) *for every surjective homomorphism of* A-*algebras* $g : D \twoheadrightarrow B$, *the second fundamental exact sequence of* B-*modules*

$$0 \longrightarrow J/J^2 \longrightarrow \Omega_{D\backslash A} \otimes_D B \longrightarrow \Omega_{B\backslash A} \longrightarrow 0$$

is short exact, where $J = \mathrm{Ker} g$;

(6) *there is a polynomial ring* R *over* A *and a surjective homomorphism of* A-*algebras* $R \twoheadrightarrow B$ *such that*

$$0 \longrightarrow J/J^2 \longrightarrow \Omega_{R\backslash A} \otimes_R B \longrightarrow \Omega_{B\backslash A} \longrightarrow 0$$

is short exact, where $J = \mathrm{Ker} g$;

(7) *there is an isomorphism* $\mathrm{Ext}^1_B(\Omega_{B\backslash A},M) \simeq T^1(B\backslash A,M)$, *for every* B-*module* M .

(8) *a singular* A-*extension of* B *by an injective* B-*module* I *is* A-*trivial if its image by a surjective homomorphism of* A-*algebras* $u : D \twoheadrightarrow B$ *is* A-*trivial;*

(9) *if* $0 \to M \to E \to B \to 0$ *is a singular* A-*extension of* B *by the* B-*module* M , *then the second fundamental exact sequence of* B-*modules*

$$0 \longrightarrow M \longrightarrow \Omega_{E\backslash A} \otimes_E B \longrightarrow \Omega_{B\backslash A} \longrightarrow 0$$

is short exact;

(10) *every singular* A-*extension of* B ,

$$0 \to M \to E \to B \to 0 ,$$

allows a split embedding, i.e. there is a split singular A-*extension*

$$0 \to N \to H \rightleftarrows B \to 0$$

of B *with* $M \subset N$, $E \subset H$, *such that the diagram*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M & \longrightarrow & E & \longrightarrow & B & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \| & & \\ 0 & \longrightarrow & N & \longrightarrow & H & \rightleftarrows & B & \longrightarrow & 0 \end{array}$$

is commutative.

PROOF. (1) $\Longrightarrow$ (2). Let $0 \to I \to E \xrightarrow{p} B \to 0$ be a singular A-extension of B by an injective B-module I . Since B is an almost smooth A-algebra, there is an A-algebra section s of p . Thus $0 \to I \to E \to B \to 0$ is split.
(2) $\Longrightarrow$ (1). Let C be an A-algebra and I an ideal of C satisfying $I^2 = 0$. Let $p : C \to C/I$ be the projection and $g : B \to C/I$ an A-algebra homomorphism such that I is an injective B-module via g . Consider the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & I & \longrightarrow & C \times_{C/I} B & \xrightarrow{q} & B & \longrightarrow & 0 \\ & & \| & & \downarrow h & & \downarrow g & & \\ 0 & \longrightarrow & I & \longrightarrow & C & \xrightarrow{p} & C/I & \longrightarrow & 0 \end{array}$$

where

$$C \times_{C/I} B = \{(c,b) \in C \times B \mid g\, b = p\, c\} .$$

Since $T^1(B\backslash A,I) = 0$,

$$0 \to I \to C \times_{C/I} B \to B \to 0$$

is a split A-extension. For the section s of q , hs is a lifting of g .
(2) $\Longleftrightarrow$ (3). Let P be a polynomial ring over A mapping onto B with kernel J . The sequence

$$0 \longrightarrow T_1(B\backslash A,B) \longrightarrow J/J^2 \longrightarrow \Omega_{P\backslash A} \otimes_P B \longrightarrow \Omega_{B\backslash A} \longrightarrow 0$$

is exact. If I is an injective B-module, the sequence

$$0 \longrightarrow \mathrm{Hom}_B(\Omega_{B\backslash A},I) \longrightarrow \mathrm{Hom}_B(\Omega_{P\backslash A} \otimes_P B,I) \longrightarrow \mathrm{Hom}_B(J/J^2,I) \longrightarrow$$
$$\longrightarrow \mathrm{Hom}_B(T_1(B\backslash A,B),I) \longrightarrow 0$$

is also exact. Thus

$$T^1(B\backslash A,I) \cong \mathrm{Hom}_B(T_1(B\backslash A,B),I) .$$

if

$$T_1(B\backslash A,B) = 0 \quad \text{then} \quad T^1(B\backslash A,I) = 0 .$$

Conversely, let $T^1(B\backslash A,I) = 0$ for every injective B-module I . Choose I to contain $T_1(B\backslash A,B)$; it follows that $T_1(B\backslash A,B) = 0$.

(3) $\Longrightarrow$ (4) Consider the surjective homomorphism of A-algebras $D \twoheadrightarrow B$ and the associated exact sequences

$$\begin{array}{ccccccccc}
 & & \gamma_{B\backslash D\backslash A} & \longrightarrow & \Omega_{D\backslash A} \otimes_D B & \longrightarrow & \Omega_{B\backslash A} & \longrightarrow & 0 \\
 & & \downarrow & & \| & & \| & & \\
0 = T_1(B\backslash A,B) & \longrightarrow & J/J^2 & \longrightarrow & \Omega_{D\backslash A} \otimes_D B & \longrightarrow & \Omega_{B\backslash A} & \longrightarrow & 0
\end{array}$$

Since $T_1(B\backslash A,B) = 0$, $\gamma_{B\backslash D\backslash A} \cong J/J^2$.

(4) $\Longrightarrow$ (5). Trivial.
(5) $\Longrightarrow$ (6). Trivial.
(6) $\Longrightarrow$ (7). Let $0 \to U/U_0 \to F/U_0 \to R \to B \to 0$ be a free extension of B over A [3] and let J be the kernel of $R \twoheadrightarrow B$. From the exact sequence

$$0 \longrightarrow J/J^2 \longrightarrow \Omega_{R\backslash A} \otimes_R B \longrightarrow \Omega_{B\backslash A} \longrightarrow 0$$

we obtain the exact sequence

$$\begin{array}{ccccccccc}
U/U_0 \otimes_R B & \to & F/U_0 \otimes_R B & \longrightarrow & \Omega_{R\backslash A} \otimes_R B & \to & \Omega_{B\backslash A} & \to & 0 \\
\downarrow \cong & & & \searrow \quad \nearrow & & & & & \\
U/U_0 & & & J \otimes_R B \cong J/J^2 & & & & &
\end{array}$$

Since $\Omega_{R\backslash A} \otimes_R B$ and $F/U_0 \otimes_R B$ are free B-modules,

$$T^1(B\backslash A,M) \cong \mathrm{Ext}^1_B(\Omega_{B\backslash A},M) .$$

(7) $\Longrightarrow$ (8). Let $u^* : T^1(B\backslash A,I) \to T^1(D\backslash A,I)$ be the homomorphism induced by $u : D \twoheadrightarrow B$ and I an injective B-

module. Since $T^1(B\backslash A,I) \simeq \mathrm{Ext}^1_B(\Omega_{B\backslash A},I) = 0$, u^* is injective.

(8) $\Longrightarrow$ (9). Let $0 \to M \to E \to B \to 0$ be a singular A-extension and $M \to \Omega_{E\backslash A} \otimes_E B \to \Omega_{B\backslash A} \to 0$ the second fundamental sequence associated with the A-algebra homomorphism $E \twoheadrightarrow B$. Since $p : T^2(B\backslash A,I) \to T^1(E\backslash A,I)$ is injective for every injective B-module I , the sequence

$$0 \longrightarrow \mathrm{Der}_A(B,I) \longrightarrow \mathrm{Der}_A(E,I) \longrightarrow \mathrm{Hom}_B(M,I) \longrightarrow 0$$

is exact for any injective B-module I . Thus $M \to \Omega_{E\backslash A} \otimes_E B$ is injective.

(9) $\Longrightarrow$ (3). Let P be a polynomial ring over A mapping onto B and with kernel J . Consider the diagram

$$\begin{array}{ccccccccc}
 & & J^2 & & & & & & \\
 & & \downarrow & \searrow & & & & & \\
0 & \longrightarrow & J & \longrightarrow & P & \longrightarrow & B & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & J/J^2 & \longrightarrow & E = P/J^2 & \longrightarrow & B & \longrightarrow & 0
\end{array}$$

If $J/J^2 \to \Omega_{E\backslash A} \otimes_E B \to \Omega_{B\backslash A} \to 0$ is the second fundamental exact sequence associated with the A-algebra homomorphism $E \twoheadrightarrow B$, then by the naturality of the Jacobi-Zariski sequence, the following diagram is commutative:

$$\begin{array}{ccccccccccc}
 & & 0 & \longrightarrow & J/J^2 & \longrightarrow & \Omega_{E\backslash A} \otimes_E B & \longrightarrow & \Omega_{B\backslash A} & \longrightarrow & 0 \\
 & & & & \| & & \uparrow & & \| & & \\
0 & \longrightarrow & T_1(C\backslash A,C) & \longrightarrow & J/J^2 & \longrightarrow & \Omega_{P\backslash A} \otimes_P B & \longrightarrow & \Omega_{B\backslash A} & \longrightarrow & 0
\end{array}$$

Since $\Omega_{E\backslash A} \otimes_E B \cong \Omega_{P\backslash A} \otimes_P B$, $T_1(C\backslash A,C) = 0$.

(2) $\Longrightarrow$ (10). Let $0 \to M \to E \to B \to 0$ be a singular A-extension of B , and choose an injective B-module I containing M . Consider the diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & M & \xrightarrow{i} & E & \xrightarrow{p} & B & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & I & \longrightarrow & E' & \longrightarrow & B & \longrightarrow & 0
\end{array}$$

where $E' = (I \oplus E)/H$, $I \oplus E$ is an A-algebra with multiplication

$$(x,e)\cdot(x',e') = (pe\cdot x' + pe'\cdot x, ee') ,$$

and $H = \{(m,-im) \mid m \in M\}$. The singular A-extension $0 \to I \to E' \to B \to 0$ is split, because $T^1(B \backslash A, I) = 0$.

(10) $\Longrightarrow$ (2). Let $0 \to I \to E \to B \to 0$ be a singular A-extension of B by an injective B-module I , and $0 \to N \to D \rightleftarrows B \to 0$ a split embedding. Let $j : I \to N$ be inclusion and $\psi : N \to I$ a B-module homomorphism such that $\psi j = 1_I$. Consider the commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & I & \longrightarrow & E & \longrightarrow & B & \longrightarrow & 0 \\
 & & \downarrow j & & \downarrow & & \| & & \\
0 & \longrightarrow & N & \xrightarrow{i} & D & \underset{s}{\rightleftarrows} & B & \longrightarrow & 0 \\
 & & \downarrow \psi & & \downarrow h & & \| & & \\
0 & \longrightarrow & I & \longrightarrow & D_\psi & \rightleftarrows & B & \longrightarrow & 0
\end{array}$$

where $D_\psi = (I \oplus D)/\{(\psi n,-in) \mid n \in N\}$. Since the singular extension $0 \to I \to D_\psi \to B \to 0$ splits, so too does $0 \to I \to E \to B \to 0$.

NOTE. Every formally smooth algebra is almost smooth, but there are almost smooth algebras that are not formally smooth. As an example consider a perfect field A and $B = S/I$, where

$$S = A[X_1,\ldots,X_r,Y_1,\ldots,Y_r]_{(X_1,\ldots,X_r,Y_1,\ldots,Y_r)} , \quad r > 1 ,$$

and I the ideal of S generated by

$$F_1 = X_1^2 - X_2 Y_1, \dots , F_{r-1} = X_{r-1}^2 - X_r Y_{r-1} ,$$

$$F_r = X_r^2 - X_1 Y_r , F_{r+1} = X_1 \dots X_r - Y_1 \dots Y_r .$$

Then B is an almost smooth A-algebra but is not formally smooth, because the homological dimension of $\Omega_{B \setminus A}$ is infinite [4].

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