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THE EXACT SEQUENCE IN THE HOMOLOGY OF GROUPS WITH INTEGRAL COEFFICIENTS MODULO q ASSOCIATED TO TWO NORMAL SUBGROUPS

by C. RODRÍGUEZ-FERNÁNDEZ and E. G. RODEJA FERNÁNDEZ

RÉSUMÉ. Dans cet article, la suite à 8 termes de Brown et Loday associée à deux sous-groupes normaux d'un groupe est généralisée au cas où les coefficients sont dans $\mathbb{Z}_q = \mathbb{Z}/_q\mathbb{Z}$, où q est un entier non-negatif.

1. INTRODUCTION.

In this paper we generalize to the case of coefficients in $\mathbb{Z}_q = \mathbb{Z}/q\mathbb{Z}$, q non-negative integer, the eight-term sequence of Brown and Loday associated to two normal subgroups of a group [B-L]. For this we extend the definition of $N\Delta^qG$, introduced in [E-R] to the case of two normal subgroups of G and then if M and N are two normal subgroups of a group G such that $MN = G$, there exists an exact sequence

$$\ldots \to H_3(G,\mathbb{Z}_q) \to H_3(G/N,\mathbb{Z}_q) \oplus H_3(G/M,\mathbb{Z}_q) \to V \to$$

$$\to H_2(G,\mathbb{Z}_q) \to H_2(G/N,\mathbb{Z}_q) \oplus H_2(G/M,\mathbb{Z}_q) \to \frac{M\cap N}{M \#_q N} \to$$

$$\to (G)^q_{ab} \to (G/N)^q_{ab} \oplus (G/M)^q_{ab} \to 0$$

where $V = \mathrm{Ker}\,(M\Delta^qN \xrightarrow{[\ ,\ ]} M \cap N$ and $(G)^q_{ab} = G/([G,G].G^q)$.

The proof is a combination of Proposition 1, Remark 2 and Theorem 19.

2. THE HOMOLOGY SEQUENCE WITH COEFFICIENTS IN $\mathbb{Z}_q$.

Let $\mathcal{V}$ be a variety of groups. We denote by V(G) the verbal subgroup of a group G with respect to $\mathcal{V}$ and consider the functor $V : \mathbf{Gr} \to \mathbf{Gr}$ taking G to V(G) and

$\mathcal{V} : \mathbf{Gr} \longrightarrow \mathbf{Gr}$ taking G to $G/V(G)$.

With these notations, the derived functors L_nV_m and $L_n\mathcal{V}_m$ are defined for $n,m \geq 0$ [B-R].

Proposition 1. *Let* M *and* N *be two normal subgroups of a group* G *such that* $MN = G$. *Let* (α,γ) *be the object of* $\mathbf{Gr}_2$ *given by*

$$\begin{array}{ccc} G & \xrightarrow{\ \alpha\ } & G/N \\ \beta\Big\downarrow & & \Big\downarrow\delta \\ G/M & \xrightarrow{\ \gamma\ } & 0 \end{array}$$

and let $H_n(G , \mathbb{Z}_q)$ *be the n-th homology group of* G *with coefficients in* $\mathbb{Z}_q$. *Then, if* $\mathcal{V}$ *is the variety of abelian groups of exponent* q , *there exists a long exact sequence*

$$\ldots\longrightarrow H_3(G, \mathbb{Z}_q) \longrightarrow H_3(G/N, \mathbb{Z}_q) \oplus H_3(G/M, \mathbb{Z}_q) \longrightarrow L_1\mathcal{V}_2(\alpha,\gamma) \longrightarrow$$

$$\longrightarrow H_2(G, \mathbb{Z}_q) \longrightarrow H_2(G/N, \mathbb{Z}_q) \oplus H_2(G/M, \mathbb{Z}_q) \longrightarrow L_0\mathcal{V}_2(\alpha,\gamma) \longrightarrow$$

$$\longrightarrow H_1(G, \mathbb{Z}_q) \longrightarrow H_1(G/N, \mathbb{Z}_q) \oplus H_1(G/M, \mathbb{Z}_q) \longrightarrow 0 \ .$$

Proof. [B-R, Prop. 4.4].

Remark 2. In [B-R] it is shown that

$$L_1\mathcal{V}_2(\alpha,\gamma) = \mathrm{Ker}(L_0V_2^q(\alpha,\gamma) \longrightarrow M \cap N) \ ,$$

$$L_0\mathcal{V}_2(\alpha,\gamma) = \mathrm{Coker}(L_0V_2^q(\alpha,\gamma) \longrightarrow M \cap N)$$

where V^q denotes the verbal subgroup functor of the variety of abelian groups of exponent q .

Proposition 3. *Let* Q , R *and* S *be groups and take* $X = Q*R*S$ *as the free product (coproduct). If we write* $A = Q*S$ *and* $B = Q*R$, *viewed as subgroups of* X , *then we have the following*

(i) $B^X \cap A^X \cap [X,X] = [A,B]$,

(ii) $B^X \cap A^X \cap (X \#_q X) = A \#_q B$

where B^X *denotes the normal closure of* B *in* X *and*

$A \#_q B$ *is the subgroup of* X *generated by the elements of the form* $[a,b]c^q$, *for* $a \in A$, $b \in B$ *and* $c \in A \cap B$.

Proof. (i) follows from (ii) for $q = 0$.
(ii) Clearly, the left hand side contains the right hand side.

Conversely, if

$$x = \prod_{i=1}^{n} k_i r_i s_i \in X = Q*R*S \ , \ k_i \in Q \ , \ r_i \in R \ , \ s_i \in S$$

we have

$$x = \left(\prod_{i=1}^{n-1} \left[\prod_{j=1}^{i} k_j r_j \ , \ \prod_{j=1}^{i} s_j \right] . \left[\prod_{j=1}^{i} s_j \ , \ \prod_{j=1}^{i+1} k_j r_j \right] \right) .$$

$$. \left(\prod_{i=1}^{n-1} \left[\prod_{j=1}^{i} k_j \ , \ \prod_{j=1}^{i} r_j \right] . \left[\prod_{j=1}^{i} r_j \ , \ \prod_{j=1}^{i+1} k_j \right] \right) .$$

$$. \left(\prod_{i=1}^{n} k_i \right) . \left(\prod_{i=1}^{n} r_i \right) . \left(\prod_{i=1}^{n} s_i \right) .$$

Furthermore, if

$$x \in B^X \cap A^X \cap (X \#_q X)$$

and we consider the homomorphisms

$$\begin{bmatrix}0\\1\\0\end{bmatrix} : Q*R*S \longrightarrow R \ , \ \begin{bmatrix}0\\0\\1\end{bmatrix} : Q*R*S \longrightarrow S \ , \ \begin{bmatrix}1\\0\\0\end{bmatrix} : Q*R*S \longrightarrow Q$$

and

$$\alpha = \mu . \begin{bmatrix}0\\1\\0\end{bmatrix} \ , \ \beta = \mu . \begin{bmatrix}0\\0\\1\end{bmatrix} \ , \ \gamma = \mu . \begin{bmatrix}1\\0\\0\end{bmatrix} ,$$

μ being the inclusion homomorphism in the free product, then

$$\text{Ker } \alpha = A^X \ , \ \text{Ker } \beta = B^X$$

and we have

$$1 = \alpha(x) = \prod_{i=1}^{n} r_i \; ; \; 1 = \beta(x) = \prod_{i=1}^{n} s_i$$

and

$$\prod_{i=1}^{n} k_i = \gamma(x) \in \mu(Q \#_q Q) \subset A \#_q B \, .$$

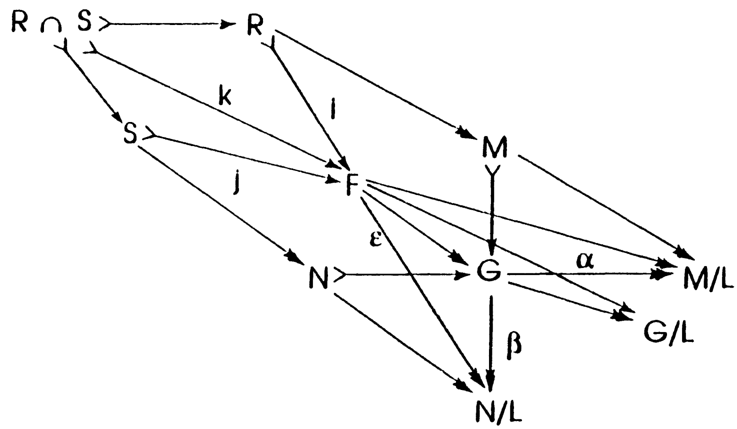
Consequently

$$x = \left(\prod_{i=1}^{n-1} \left[\prod_{j=1}^{i} k_j r_j \, , \prod_{j=1}^{i} s_j \right] \cdot \left[\prod_{j=1}^{i} s_j , \prod_{j=1}^{n+1} k_j r_j \right] \right) \cdot$$

$$\cdot \left(\prod_{i=1}^{n-1} \left[\prod_{j=1}^{i} k_j , \prod_{j=1}^{i} r_j \right] \cdot \left[\prod_{j=1}^{i} r_j , \prod_{j=1}^{n+1} k_j \right] \right) \cdot \left(\prod_{i=1}^{n} k_i \right) \in A \#_q B$$

From now on, M and N will be two normal subgroups of a given group G , such that MN = G . We write L = M ∩ N , and consider $\varepsilon : F \longrightarrow G$ a free presentation of G and let R be the kernel of the composite morphism $\beta\varepsilon : F \longrightarrow G \longrightarrow N/L$ and S , the kernel of $\alpha\varepsilon : F \longrightarrow G \longrightarrow M/L$.

Here is an illustration:



We will consider

$$X_0 = (R \cap S)'*R*S \; , \; A = (R \cap S)'*S \quad \text{and} \quad B = (R \cap S)'*R \; ,$$

where by $(R \cap S)'$ we denote an isomorphic copy of $R \cap S$.

d will be the morphism $\begin{pmatrix}\varepsilon k\\ \varepsilon i\\ \varepsilon j\end{pmatrix} : X_0 \longrightarrow G$ and T its kernel.

$\mu : G \to F$ will be any set theoretic section of ε (i.e., $\varepsilon\mu = 1$) and μ_1,μ_2 and μ_3 the set theoretic maps:

$$\mu_1 : M \xrightarrow{\mu} R \longrightarrow B^{X_0} \longrightarrow X_0 ,$$

$$\mu_2 : M \xrightarrow{\mu} S \longrightarrow A^{X_0} \longrightarrow X_0 ,$$

$$\mu_3 : M \cap N \xrightarrow{\mu} R \cap S \xrightarrow{\cong} (R \cap S)' \longrightarrow X_0 .$$

Finally, D will denote

$$((T \cap B^{X_0}) \#_q A^{X_0})\cdot(B^{X_0} \#_q (T \cap A^{X_0})) .$$

Lemma 4. *With the above notation we have:*

(i) $[T,A^{X_0} \cap B^{X_0}] \subset [T \cap A^{X_0},B^{X_0}]\cdot[T \cap B^{X_0},A^{X_0}]$.

(ii) $T \#_q (A^{X_0} \cap B^{X_0}) \subset ((T \cap A^{X_0}) \#_q B^{X_0})\cdot((T \cap B^{X_0}) \#_q A^{X_0})$.

Proof. (i) As $X_0 = A^{X_0}\cdot R$, we have that

$$t \in T \Rightarrow t = y\cdot r , y \in A^{X_0} , r \in R , d(y) = d(r)^{-1} .$$

But

$d(y) \in N , d(r) \in M \Rightarrow \exists\, k \in (R \cap S)' , d(k) = d(y) = d(r)^{-1}$.

Hence, if $t \in T , x \in A^{X_0} \cap B^{X_0}$, then

$$[t, x] = [yk^{-1}kr, x] = {}^{yk^{-1}}[kr, x]\cdot[yk^{-1}, x]$$

where $kr \in T \cap B^{X_0}$, $yk \in T \cap A^{X_0}$, $x \in A^{X_0} \cap B^{X_0}$.
(ii) follows from (i).

Proposition 5. *If* V *is the variety of abelian groups of exponent* q , *and* V^q *is the verbal subgroup functor, then, with the above notation*

$$\text{i) } L_0V_2(\alpha,\gamma) = \frac{A \#_q B}{((T \cap A^{X_0}) \#_q B^{X_0})\cdot((T \cap B^{X_0}) \#_q A^{X_0})} = \frac{B \#_q A}{D}$$

ii) $$L_0V_2(\alpha,\gamma) = \frac{T \cap (A \#_q B)}{((T \cap A^{X_0}) \#_q B^{X_0})\cdot((T \cap B^{X_0}) \#_q A^{X_0})} = \frac{T \cap (B \#_q A)}{D}.$$

Proof. This follows from the previous lemma and proposition 5.6 and 5.8 of [B-R].

Lemma 6. *If* $m,m' \in M$, $n,n' \in N$, $k,k' \in M \cap N$, $a \in A^{X_0}$, $b \in B^{X_0}$, *then*

i) a) $[b,\mu_2(nn')]\cdot D = [b,\mu_2(n)\mu_2(n')]\cdot D.$
b) $[\mu_1(mm'),a]\cdot D = [\mu_1(m)\mu_1(m'),a]\cdot D.$
c) $[b,\mu_3(kk')]\cdot D = [b,\mu_3(k)\mu_3(k')]\cdot D.$
d) $[\mu_3(kk'),a]\cdot D = [\mu_3(k)\mu_3(k'),a]\cdot D.$

ii) a) $[b,\mu_2({}^{m}n)]\cdot D = [b,{}^{\mu_1(m)}\mu_2(n)]\cdot D.$
b) $[\mu_1({}^{n}m),a]\cdot D = [{}^{\mu_2(n)}\mu_1(m),a]\cdot D.$
c) $[b,\mu_2({}^{k}n)]\cdot D = [b,{}^{\mu_3(k)}\mu_2(n)]\cdot D.$
d) $[\mu_1({}^{k}m),a]\cdot D = [{}^{\mu_3(k)}\mu_1(m),a]\cdot D.$

iii) a) $[b,\mu_2(k)]\cdot D = [b,\mu_3(k)]\cdot D.$
b) $[\mu_1(k),a]\cdot D = [\mu_3(k),a]\cdot D.$

iv) ${}^{\mu_3(k^q)}[b,a]\cdot D = {}^{(\mu_3(k))^q}[b,a]\cdot D.$

v) $[\mu_3(kk')^{-1},\mu_3(k)\mu_3(k')]\cdot D = D.$

vi) $\mu_3([m,n])^q\cdot D = [\mu_1(m),\mu_2(n)]^q\cdot D.$

Proof. i), ii) and iii) are proved in a similar way. We do i) a):

$[b,\mu_2(nn')]\cdot D = [b,\mu_2(n)\mu_2(n')(\mu_2(n)\mu_2(n'))^{-1}\mu_2(nn')]\cdot D =$

$= [b,\mu_2(n)\mu_2(n')]\cdot {}^{\mu_2(n)\mu_2(n')}[b,(\mu_2(n)\mu_2(n'))^{-1}\mu_2(nn')]\cdot D =$
$= [b,\mu_2(n)\mu_2(n')]\cdot D$ as

$(\mu_2(n)\mu_2(n'))^{-1}\mu_2(nn') \in T \cap A^{X_0}.$

iv) ${}^{\mu_3(k^q)}[b,a]\cdot D = {}^{(\mu_3(k^q)\mu_3(k)^{-q}\mu_3(k)^q}[b,a]\cdot D =$

$= [\mu_3(k^q)\mu_3(k)^{-q},{}^{\mu_3(k^q)}[b,a]]\cdot {}^{\mu_3(k)^q}[b,a]\cdot D = {}^{\mu_3(k)^q}[b,a]\cdot D$

since
$(\mu_3(k^q)\mu_3(k)^{-q} \in T \cap A^{X_0} \cap B^{X_0}\ ,{}^{\mu_3(k)^q}[b,a] \in A^{X_0} \cap B^{X_0}$.

v) In a similar way to the previous i), ii) and iii), the following more general result can be proved

$$[b,\mu_3(k)\mu_3(k')\cdot D = [b,\mu_3(kk')]\cdot D\ .$$

vi) $\mu_3([m,n])\ \cdot D =$
$= ([\mu_1(m),\mu_2(n)][\mu_1(m),\mu_2(n)]^{-1}\mu_3([m,n]))^q\cdot D =$
$= [\mu_1(m),\mu_2(n)]^q\cdot D$ since

$$[\mu_1(m),\mu_2(n)]^{-1}\mu_3([m,n]) \in T \cap A^{X_0} \cap B^{X_0}\ .$$

3. THE "EXTERIOR PRODUCT MODULO q" .

Definition 7. *Let* M *and* N *be two normal subgroups of a group* G *(not necessarily* MN = G*) . The exterior product modulo* q , $M\Delta^qN$, *is the group generated by symbols* $m\wedge n$, $\{k\}$, $m \in M$, $n \in N$, $k \in M \cap N$, *with relations*

(1) $m\wedge nn' = (m\wedge n)({}^m m\wedge {}^m n')$.
(2) $mm'\wedge n = ({}^m m'\wedge {}^m n)(m\wedge n)$.
(3) $k\wedge k = 1$.
(4) $\{k\}(m\wedge n)\{k\}^{-1} = {}^{k^q}m\wedge {}^{k^q}n$.
(5) $\{kk'\} = \left[\prod_{i=1}^{q-1} ({}^{k^i}(k'^{i})\wedge k)\right]\{k\}\{k'\}$.
(6) $[\{k\},\{k'\}] = k^q\wedge k'^q$.
(7) $(m\wedge n)^q = \{[m,n]\}$
for $m \in M$, $n \in N$, $k \in M \cap N$.

Proposition 8. *If* $\lambda : M\Delta^qN \longrightarrow G$ *is the homomorphism defined by* $\lambda(m\wedge n) = [m,n]$ *and* $\lambda(\{k\}) = k^q$ *(it is a routine to check that* λ *is well defined) and if* $\ell,\ell' \in M\Delta^qN$, *with* $\ell' = \prod_{i=1}^{s} m_i\wedge n_i$ *we have the following*

i) $\ell\ell'\ell^{-1} = {}^{\lambda(\ell)}\ell'$ *and* $[\ell,\ell'] = \lambda(\ell)\wedge\lambda(\ell')$,
ii) $(\ell')^q = \{\lambda(\ell')\}$

where ${}^{g}\ell' = \prod_{i=1}^{s} ({}^{g}m_i \wedge {}^{g}n_i)$.

Proof. i). We have to prove that

$$\{k\}(m \wedge n)\{k\}^{-1} = {}^{k^q}m \wedge {}^{k^q}n ,$$
$$(m \wedge n)(m' \wedge n')(m \wedge n)^{-1} = {}^{\lambda(m \wedge n)}(m' \wedge n') ,$$
$$m \wedge \lambda(\ell') = {}^{m}\ell' \, \ell'^{-1} .$$

The first equality is the relation (4) and the others follow from proposition 2.3 of [B-L].

ii) is relation (7) in the case $\ell' = m \wedge n$.

By induction, if we denote $\ell = \prod_{i=1}^{s-1} m_i \wedge n_i$, then we have

$$(\ell')^q = \left[\prod_{i=1}^{q-1} [{}^{\ell^i}((m_s \wedge n_s)^i) \ , \ \ell] \right] \ell^q (m_s \wedge n_s)^q$$

$$= \left[\prod_{i=1}^{q-1} ({}^{\lambda(\ell)^i}[m_s,n_s]^i) \wedge \lambda(\ell) \right] \{\lambda(\ell)\}\{[m_s,n_s]\}$$

$$= \{\lambda(\ell)[m_s,n_s]\} = \{\lambda(\ell')\} .$$

(Brown in [B] shows that the group G acts on the tensor product $M \otimes^q N$ with $N = G$. This result can be generalized to the case $M \otimes^q N$ with $MN = G$. The present equalities could be proved using these results but we decided to show them directly.)

Proposition 9. *There exists a morphism* $h : M \Delta^q N \longrightarrow L_0 V_2^q(\alpha,\gamma)$ *defined by* $h(m \wedge n) = [\mu_1(m),\mu_2(n)] \cdot D$, $h(\{k\}) = \mu_3(k)^q \cdot D$.

Proof. We must show that h preserves the relations (1)-(7).

h clearly preserves (1)-(3) by [B-R].

Preservation of (4) follows from ii) c) and d) and iv) of Lemma 6.

Relation (5) is preserved, as we have

$$\left[\prod_{i=1}^{q-1} [{}^{k^i}(\mu_1(k'^i)) \;,\; \mu_2(k)]\right]\mu_3(k)^q\mu_3(k')^q\cdot D$$

$$= \left[\prod_{i=1}^{q-1} [{}^{k^i}(\mu_3(k'^i)) \;,\; \mu_3(k)]\right]\mu_3(k)^q\mu_3(k')^q\cdot D$$

$$= \mu_3(kk')^q\mu_3(kk')^{-q}(\mu_3(k)\mu_3(k'))^q\cdot D$$

$$= \mu_3(kk')^q(\mu_3(kk')^{-1}\mu_3(k)\mu_3(k'))^q\cdot D = \mu_3(kk')^q\cdot D \;,$$

since $\mu_3(kk')^{-1}\mu_3(k)\mu_3(k') \in T \cap A^{X_0} \cap B^{X_0}$.

Relation (6) follows from iii) a) and b) of Lemma 6 and from the fact that

$$[\mu_3(k^q) \;,\; \mu_3(k'^q)]\cdot D = [\mu_3(k)^q \;,\; \mu_3(k')^q]\cdot D \;.$$

As for relation (7), preservation follows from Lemma 6 vi).

Proposition 10. *With the notation as above*

$$\tau : (R \cap S)' \#_q (R \cap S)' \longrightarrow M\Delta^q N \;,$$

given by $\tau([k,k']k''^q) = (d(k)\wedge d(k'))\{d(k'')\}$, *is a group homomorphism.* (*Recall that* $d = \begin{pmatrix}\varepsilon k\\ \varepsilon i\\ \varepsilon j\end{pmatrix} : X_0 \to G$) .

Proof. $\partial : (R \cap S)' \#_q (R \cap S)' \to (R \cap S)' \Delta^q (R \cap S)'$, defined by $\partial([k,k']k''^q) = (k\wedge k')\{k''\}$ is an isomorphism [E-R] and τ is the composite of ∂ with the morphism

$$(R \cap S)' \Delta^q (R \cap S)' \longrightarrow M\Delta^q N$$

induced by d .

Remark 11. If $x \in B^{X_0} \#_q A^{X_0}$, then

$$x = \prod_{i=1}^{n} k_ir_is_i \;,\; k_i \in (R \cap S)' \;,\; r_i \in R \;,\; s_i \in S \;,$$

$$\prod_{i=1}^{n} s_i = 1 \ , \ \prod_{i=1}^{n} r_i = 1$$

and as a consequence

$$x = \left[\prod_{i=1}^{n-1} {}^{\left[\prod_{j=1}^{i} k_j r_j\right]} \left[\prod_{j=1}^{i} s_j \ , k_{i+1} r_{i+1} \right] \right] \cdot$$

$$\cdot \left[\prod_{i=1}^{n-1} {}^{\left[\prod_{j=1}^{i} k_j\right]} \left[\prod_{j=1}^{i} r_j \ , \ k_{i+1} \right] \right] \left[\prod_{i=1}^{n} k_i \right]$$

with $\prod_{i=1}^{n} k_i \in (R \cap S)' \#_q (R \cap S)'$.

Proposition 12. *The map* $g : B^{X_0} \#_q A^{X_0} \longrightarrow M\Delta^q N$, *defined by*

$$g(x) = \prod_{i=1}^{n-1} \left[{}^{d\left[\prod_{j=1}^{i} k_j r_j\right]} \left[d(k_{i+1} r_{i+1}) \wedge d\left[\prod_{j=1}^{i} s_j \right] \right] \right]^{-1} \cdot$$

$$\prod_{i=1}^{n-1} \left[{}^{d\left[\prod_{j=1}^{i} k_j\right]} \left[d\left[\prod_{j=1}^{i} r_j \right] \wedge d(k_{i+1}) \right] \cdot \tau \left[\prod_{i=1}^{n} k_i \right] \right]$$

for $x = \prod_{i=1}^{n} k_i r_i s_i \in B^{X_0} \#_q A^{X_0}$, *is a group homomorphism.*

Proof. g is an application due to uniqueness of image after Remark 11. Moreover for

$$x = \prod_{i=1}^{m} k_i r_i s_i \ , \ x' = \prod_{i=m+1}^{n} k_i r_i s_i \in B^{X_0} \#_q A^{X_0} \ ,$$

we denote

$$x_1 = \prod_{i=1}^{m-1} \left[{}^{d\left(\prod_{j=1}^{i} k_j r_j\right)} \left[d(k_{i+1} r_{i+1}) \wedge d\left(\prod_{j=1}^{i} s_j\right)\right]\right]^{-1} ,$$

$$y_1 = \prod_{i=1}^{m-1} \left[{}^{d\left(\prod_{j=1}^{i} k_j\right)} \left[d\left(\prod_{j=1}^{i} r_j\right) \wedge d(k_{i+1})\right]\right] ,$$

$$z_1 = \tau\left(\prod_{i=1}^{m} k_i\right)$$

and we have

$$g(x) = x_1 y_1 z_1 \ .$$

Similarly

$$g(x') = x_1' y_1' z_1'$$

and then

$$g(x)g(x') = x_1 y_1 z_1 x_1' y_1' z_1' = x_1 (y_1 z_1) x_1' (y_1 z_1)^{-1} y_1 z_1 y_1' z_1^{-1} z_1 z_1'$$

which after Proposition 8 gives

$$g(x)g(x') = x_1 \cdot {}^{\lambda(y_1 z_1)} x_1' \cdot y_1 \cdot {}^{\lambda(z_1)} y_1' \cdot z_1 z_1' =$$

$$= x_1 \cdot {}^{d\left(\prod_{i=1}^{m} k_i r_i\right)} x_1' \cdot y_1 \cdot {}^{d\left(\prod_{i=1}^{m} k_i r_i\right)} y_1' \cdot z_1 z$$

and finally, since $\prod_{i=1}^{m} s_i = 1 = \prod_{i=1}^{m} r_i$ we have

$$g(x)g(x') = g(xx') .$$

Lemma 13. *With the above notations,* $[B,A]$ *is generated by the elements of the form*
$[b,s]$, $[r,k]$, $[k,k']$, $b \in B$, $s \in S$, $r \in R$, $k,k' \in (R \cap S)'$.

Lemma 14. If $b \in B$, $s \in S$, $r \in R$, $k,k' \in (R \cap S)'$, $x \in X_0$, then

i) $g([b,s]) = d(b) \wedge d(s)$.
ii) $g([r,k]) = d(r) \wedge d(k)$.
iii) $g([k,k']) = d(k) \wedge d(k')$.
iv) $g({}^x[b,s]) = {}^{d(x)}d(b) \wedge {}^{d(x)}d(s)$.
v) $g({}^x[r,k]) = {}^{d(x)}d(r) \wedge {}^{d(x)}d(k)$.
vi) $g({}^x[k,k'] = {}^{d(x)}d(k) \wedge {}^{d(x)}d(k')$.

Lemma 15. *If* $y \in [B,A]$, $x \in X_0$, *then* $g({}^x y) = {}^{d(x)}g(y)$.

Proposition 16. *If* $b \in B^{X_0}$, $a \in A^{X_0}$, *then*

$$g([b,a]) = d(b) \wedge d(a) .$$

Proof. $a \in A^{X_0} \Rightarrow a = ysk$, $y \in [B,A]$, $s \in S$, $k \in (R \cap S)'$
$b \in B^{X_0} \Rightarrow b = y'rk'$, $y \in [B,A]$, $r \in R$, $k' \in (R \cap S)'$
taking into account Lemmas 13 and 14 and Proposition 2.3 of [B-L] and the equality $\lambda g = d$ $(\lambda : M\Delta^q N \longrightarrow G)$ we obtain

$$\begin{aligned} d(b) \wedge d(a) &= d(y'rk') \wedge d(ysk) \\ &= {}^{d(y')}(d(rk') \wedge d(y)) \cdot {}^{d(y'y)}(d(rk') \wedge d(sk)) \cdot \\ &\quad \cdot (d(y') \wedge d(y)) \cdot {}^{d(y)}(d(y') \wedge d(sk)) \\ &= {}^{d(y')}\left[{}^{d(rk')}g(y) \cdot g(y)^{-1}\right] \cdot {}^{d(y'y)}(d(rk') \wedge d(s)) \cdot \\ &\quad \cdot {}^{d(y'ys)}(d(rk') \wedge d(k)) \cdot \\ &\quad \cdot [g(y'), g(y)] \cdot {}^{d(y)}\left[g(y') \cdot ({}^{d(sk)}g(y'))^{-1}\right] \end{aligned}$$

$$= g({}^{y'}[rk', y])\cdot g({}^{y'y}[rk', s])\cdot$$

$$\cdot {}^{d(y'ysr)}(d(k')\wedge d(k))\cdot {}^{d(y'ys)}(d(r)\wedge d(k))\cdot$$

$$\cdot g\left[[y',y]\cdot {}^{y}y'\cdot ({}^{ysk}y')^{-1}\right]$$

$$= g({}^{y'}[rk', ys])\cdot g({}^{y'ys}[k', k])$$

$$\cdot g({}^{y'ys}[r , k])\cdot g([y', ysk])$$

$$= g([b,a]) .$$

Proposition 17. *If* $c \in B^{X_0} \cap A^{X_0}$, *then*

$$g(c^q) = \{d(c)\} .$$

Proof.

$$c = \prod_{i=1}^{n} k_i r_i s_i \Rightarrow c = x\cdot z , \ x \in [B^{X_0},A^{X_0}] , \ z = \prod_{i=1}^{n} k_i .$$

After Proposition 8 we have $g(x)^q = \{\lambda g(x)\} = \{d(x)\}$, hence

$$g(c^q) = g((xz)^q) = g\left[\left[\prod_{i=1}^{q-1} [(z^i)^{x^i} , x]\right]\cdot x^q\cdot z^q\right]$$

$$= g\left[\prod_{i=1}^{q-1} [(z^i)^{x^i} , x]\right]\cdot g(x^q)\cdot g(z^q)$$

$$= g\left[\prod_{i=1}^{q-1} ((d(z)^i)^{d(x^i)}\wedge d(x))\right]\cdot \{d(x)\}\cdot \{d(z)\}$$

$$= \{d(x)\cdot d(z)\} = \{d(xz)\} = \{d(c)\} .$$

Corollary 18. *If* $a \in A^{X_0}$, $b \in B^{X_0}$, $c \in A^{X_0} \cap B^{X_0}$, *then*

$$d([b,a]c^q) = (d(b)\wedge d(a))\{d(c)\} .$$

Theorem 19. *The morphism*

$$h : M\Delta^q N \longrightarrow L_0V_2^q(\alpha,\gamma)$$

is an isomorphism.

Proof. From the corollary above it is clear that $g(D) = 1$, and therefore g induces $\varphi : L_0V_2^q(\alpha,\gamma) \longrightarrow M\Delta^q N$.

It is now trivial to check that φ and h are inverse to each other.

4. A FREE PRESENTATION OF THE NON-ABELIAN TENSOR PRODUCT.

Proposition 20. *With the above notation, if* $X = R*S$ *and* T' *denotes the kernel of the morphism* $d' = \begin{pmatrix}\varepsilon i\\ \varepsilon j\end{pmatrix} : R*S \longrightarrow G$ *we have*

$$M \otimes N \cong \frac{[R^X , S^X]}{[T' \cap R^X, S][R , T' \cap S^X]} .$$

Proof. $R^X \cap S^X \cap [X,X] = [R^X,S^X] = [R,S]$ (Prop. 3).

Given that $[R,S]$ is the free group over the set

$$\{[r,s] \mid r \in R , s \in S\}$$

we have that

$$\varphi([r,s]) = d'(r) \otimes d'(s)$$

defines a group homomorphism

$$\varphi : [R,S] \longrightarrow M \otimes N$$

with

$$\varphi({}^{r'}[r,s]) = {}^{d'(r')}\varphi([r,s])$$

because

$$\varphi({}^{r'}[r,s]) = \varphi([r'r,s]\cdot[r',s]^{-1}) =$$
$$= (d'(r'r) \otimes d'(s))\cdot(d'(r') \otimes d'(ss))^{-1} =$$
$$= {}^{d'(r')}(d(r) \otimes d(s)) = {}^{d'(r')}\varphi([r,s]) .$$

Similarly

$$\varphi({}^{s'}[r,s]) = {}^{d'(s')}\varphi([r,s]) .$$

Moreover for

$$x \in R^X , x' \in S^X$$

we have

$$x = \prod_{i=1}^{n} r_i s_i , \quad \prod_{i=1}^{n} s_i = 1 , \quad x' = \prod_{i=1}^{m} r'_i s'_i , \quad \prod_{i=1}^{m} r'_i = 1$$

and if we denote

$$z = \prod_{i=1}^{n-1} \left[\prod_{j=1}^{i} r_j , \prod_{j=1}^{i} s_j\right] \cdot \left[\prod_{j=1}^{i} s_j , \prod_{j=1}^{i+1} r_j\right] , \quad r = \prod_{i=1}^{n} r_i$$

$$z' = \prod_{i=1}^{m-1} \left[\prod_{j=1}^{i} r'_j , \prod_{j=1}^{i} s'_j\right] \cdot \left[\prod_{j=1}^{i} s'_j , \prod_{j=1}^{i+1} r'_j\right] , \quad s' = \prod_{i=1}^{n} s'_i$$

then

$$z, z' \in [R,S], \ x = zr , \ x' = z's'$$

and from Prop. 2.3 of [B-L], since

$$\lambda \cdot \varphi = d' \quad (\lambda : M \otimes N \longrightarrow [M,N] \subset M \cap N) ,$$

we obtain

$$\begin{aligned}
\varphi([x,x']) &= \varphi(z \cdot {}^{r}z' \cdot [r,s'] \cdot {}^{s'}z^{-1} \cdot z'^{-1}) \\
&= \varphi(z) \cdot \varphi({}^{r}z') \cdot \varphi([r,s']) \cdot \varphi({}^{s'}z^{-1}) \cdot \varphi(z'^{-1}) \\
&= \varphi(z) \cdot {}^{d'(r)}\varphi(z') \cdot (d'(r) \otimes d'(s')) \cdot {}^{d'(s')}\varphi(z^{-1}) \cdot \varphi(z'^{-1}) \\
&= \varphi(z) \cdot (d'(r) \otimes d'(z')) \cdot \varphi(z') \cdot (d'(r) \otimes d'(s')) \cdot \\
&\qquad \cdot \varphi(z^{-1}) \cdot (d(z) \otimes d(s')) \cdot \varphi(z'^{-1}) \\
&= \varphi(z) \cdot (d'(r) \otimes d'(z')) \cdot \varphi(z^{-1}) \cdot \\
&\qquad \cdot \varphi(zz') \cdot (d'(r) \otimes d'(s')) \cdot \varphi(zz')^{-1} \cdot \\
&\qquad \cdot \varphi(z) \cdot \varphi(z') \cdot \varphi(z)^{-1} \cdot \varphi(z')^{-1} \cdot \\
&\cdot \varphi(z') \cdot (d(z) \otimes d(s')) \cdot \varphi(z'^{-1}) =
\end{aligned}$$

$$= {}^{\lambda\varphi(z)}(d'(r) \otimes d'(z')) \cdot {}^{\lambda\varphi(zz')}(d'(r) \otimes d'(s')) \cdot$$

$$\cdot (\lambda\varphi(z) \otimes \lambda\varphi(z')) \cdot {}^{\lambda\varphi(z')}(d(z) \otimes d(s')) =$$

$$= {}^{d'(z)}\left[(d'(r) \otimes d'(z')) \cdot {}^{d'(z')}(d'(r) \otimes d'(s'))\right] \cdot$$

$$\cdot (d'(z) \otimes d'(z')) \cdot {}^{d'(z')}(d(z) \otimes d(s')) =$$

$$= {}^{d'(z)}(d'(r) \otimes d'(z's')) \cdot (d'(z) \otimes d'(z's')) =$$

$$= (d'(zr) \otimes d'(z's')) = d'(x) \otimes d'(x')$$

and therefore

$$d'([T' \cap R^X, S] \cdot [R, T' \cap S^X]) = 1$$

and there exists

$$\varphi : \frac{[R^X \ , \ S^X]}{[T' \cap R^X, S][R \ , \ T' \cap S^X]} \longrightarrow M \otimes N$$

Its inverse is given by

$$\Psi(m \otimes n) = [\mu_1(m), \mu_2(n)] \cdot D'$$

where

$$D' = [T' \cap R^X, S][R, T' \cap S^X]$$

and μ_1 and μ_2 are the above sections (see Proposition 9).

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