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# EXPONENTIABLE EMBEDDINGS IN TOTALLY REFLECTIVE SUBCATEGORIES OF *TOP*

*by F. CAGLIARI and S. MANTOVANI*

**RÉSUMÉ.** Dans une sous-catégorie ***P*** de ***Top***, ***P*** différente de ***Sing***, ***Sing***$_1$, les plongements exponentiables sont les compléments des monomorphismes réguliers. Si ***P*** ⊂ ***Haus***, ce sont aussi des plongements exponentiables dans ***Top***. L'inverse peut être faux (même pour ***P*** ⊂ ***Haus***), mais seulement si ***P*** n'est pas totalement réflective par rapport aux compléments de monomorphismes réguliers dans ***P***. Ceci donne une caractérisation des plongements exponentiables dans certaines sous-catégories usuelles de ***Top***, par exemple ***Haus, Tych, Reg, Bool.***

## INTRODUCTION.

In [6] Dyckhoff studied the reflective subcategories of ***Top*** which are totally reflective with respect to open embeddings, and he characterized them by means of Pasynkov's partial product [13]. In [7] Dyckhoff and Tholen showed that results in [6] can be obtained as an application of a general characterization of subcategories which are totally reflective with respect to a class of exponentiable embeddings in ***Top*** (which coincide with locally closed embeddings [12]).

We will show in two steps that the notion of open embedding is not always the proper one to describe total reflectivity.

First of all, if ***P*** is totally reflective with respect to a class S ⊂ Mor(***P***) of locally closed embeddings, then S consists of exponentiable embeddings of ***P*** (Proposition 1.3).

But every exponentiable embedding of ***P*** must be a ***P***-dominion open embedding of ***P***, that is a complement of a regular monomorphism in ***P*** (Proposition 1.2). So S must be contained in the class of ***P***-dominion open embeddings, which is contained in the class of open embeddings only if ***P*** ⊂ ***Haus***, and even when it is so, the two classes do not always coincide.

Our aim is to find conditions on ***P*** under which exponentiable embeddings in ***P*** are exactly the ***P***-dominion open embeddings in order to find a characterization of the exponentiable embeddings. When ***P*** ⊂ ***Haus***, ***P***-dominion open embeddings in ***P*** are open embeddings and then exponentiable in ***Top***. This fact allows us to use results of [7] to show that exponentiable embeddings in ***P*** are exactly the ***P***-dominion open embeddings, provided that ***P*** is totally reflective with respect to ***P***-dominion open embeddings. Furthermore, we give a characterization of such epireflective subcategories of ***Top***, which allows us to show that some epireflective subcategories of ***Top***, which are not totally reflective with respect to open embeddings, are nevertheless totally reflective with respect to ***P***-dominion open embeddings. This improves results given in [1]. As a final result we are able to give a complete description of exponentiable embeddings in ***Haus***, ***Reg***, ***Tych***, ***Bool***, Q(***Tych***), Q(***Bool***) and in every disconnectedness of ***Haus***, ***Reg***, ***Tych***.

Finally, we provide an example of a category ***P*** ⊂ ***Haus*** in which the class of exponentiable embeddings is strictly contained in the class of ***P***-dominion open embeddings.

We refer the reader to [8] for standard notations and definitions not explicitly given here.

We are very grateful to the referee for his useful comments and suggestions.

## 1. PRELIMINARIES.

Let ***C*** be a category with finite products. We recall some definitions and results from [3, 7, 12, 14]:

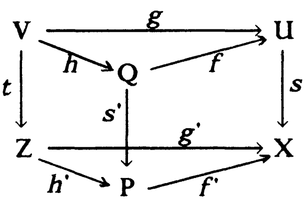
(a) Let $f\colon Q\to U$, $s\colon U\to X$ be morphisms of ***C***. A *pullback complement* of the composable pair $(f,s)$ is a pullback square

$$\begin{array}{ccc} Q & \xrightarrow{\quad f\quad} & U \\ {\scriptstyle s'}\downarrow & & \downarrow{\scriptstyle s} \\ P & \xrightarrow{\quad f'\quad} & X \end{array}$$

such that for every pullback square

$$\begin{array}{ccc} V & \xrightarrow{\quad g\quad} & U \\ {\scriptstyle t}\downarrow & & \downarrow{\scriptstyle s} \\ Z & \xrightarrow{\quad g'\quad} & X \end{array}$$

and any morphism $h: V \to Q$ with $fh = g$, there is a unique $h': Z \to P$ with $f'h' = g'$ and $s'h = h't$:




By a *partial product* of an object Y over a monomorphism $s: U \to X$ we will call the pullback complement of the pair $(\pi_U, s)$, where $\pi_U$ is the projection of $Y \times U$ over U. (For the general definition, see [7].)

(b) A morphism $s$: $U \to X$ of $\boldsymbol{C}$ is said to be *exponentiable* if it is exponentiable as an object of the comma category $\boldsymbol{C}/X$.

$(b)_1$ $\boldsymbol{C}$ has all pullback complements over $s$ (that is, for every $f$ in $\boldsymbol{C}$ composable with $s$, there exists the pullback complement of the pair $(f,s)$) if and only if $s$ is an exponentiable monomorphism.

$(b)_2$ Exponentiable embeddings in ***Top*** are the locally closed embeddings, whose class will be denoted by LC (see [12]).

(c) Let $\boldsymbol{P}$ be a full and replete reflective subcategory of ***Top*** with reflector $r$: $\boldsymbol{Top} \to \boldsymbol{P}$ and with reflection morphism $r^X$: $X \to rX$, for each X. Let S be a class of morphisms in ***Top***. The subcategory $\boldsymbol{P}$ is said to be *totally reflective with respect to* S if for all $s: U \to rX$ in S the pullback $s^*(r^X)$ or $r^X$ is uniquely $\boldsymbol{P}$-extendable (see [7]).

Notice that if $\boldsymbol{P}$ is epireflective, and S a class of embeddings, $\boldsymbol{P}$ is totally reflective with respect to S if and only if the restriction of every reflection morphism to S-subspaces is still a reflection morphism.

Let A, X be topological spaces and $\boldsymbol{P}$ a subcategory of ***Top***:

(d) Let A be a subspace of X. Denote by $K_X^{\boldsymbol{P}}(A)$ the *Salbany's closure operator* of A in X (see [14]), that is:

$$K_X^{\boldsymbol{P}}(A) = \left\{ x \in X \,\middle|\, \begin{array}{l} f(x) = g(x) \text{ for all pairs} \\ f, g: X \to P \text{ with } P \in \boldsymbol{P} \text{ and } f|_A = g|_A \end{array} \right\}$$

(The Salbany's definition is analogous to the Isbell's definition of dominion, given in [9] in cartegories of algebras.)

(e) The following properties of $K^P$ can be easily verified: Given $A \subset U \subset X$, then

(i) $A \subset K_X^P(A)$.

(ii) $K_X^P(A) \subset K_X^P(U)$.

(iii) $K_X^P(K_X^P(A)) = K_X^P(A)$.

(iv) $K_U^P(A) \subset K_X^P(A)$.

(v) If $f\colon X \to Y$ is a continuous map, $f(K_X^P(A)) \subset K_X^P(f(A))$.

(f) A subspace A of X is said to be ***P**-dominion closed* in X if $K_X^P(A) = A$.

A is **P**-dominion closed if and only if $A = \mathrm{Ker}(f,g)$, where $f, g\colon X \to P$, with $P \in \boldsymbol{P}$. It follows that $K_X^P(A)$ is a **P**-dominion closed subspace of X and it is the smallest one containing A.

(g) A subspace A of X is said to be ***P**-dominion open* in X if $X \setminus A$ is **P**-dominion closed.

Denote by $O_P$ the class of **P**-dominion open embeddings of ***Top***. Notice that, if $\boldsymbol{P} \subset \boldsymbol{Haus}$, $O_P$ is contained in the class of open embeddings (and then in LC), since regular monomorphisms in **P** are closed embeddings.

(h) **REMARK.** It follows from the definition of a **P**-dominion open embedding that $O_P$ is stable under pullbacks in ***Top***.

(k) Let **P** be a reflective subcategory of ***Top***, X a topological space and $r\colon X \to rX$ the reflection map, then

$$K_X^P(\{x\}) = r^{-1}(r(x))$$

for any $x$ in X (see [3]).

**1.1. PROPOSITION.** *Let **P** be a reflective subcategory of **Top** and let $r^X\colon X \to rX$ be a reflection map. Then a subspace U of X is **P**-dominion open in X if and only if there is a **P**-dominion open in rX such that $(r^X)^{-1}(V) = U$. In particular $V = r^X(U)$ when **P** is epireflective.*

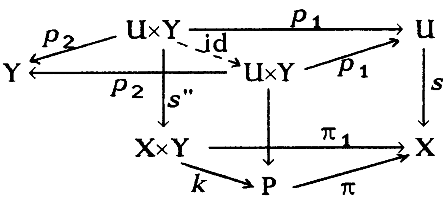
**PROOF.** Suppose U **P**-dominion open, that is $X \setminus U = \mathrm{Ker}(f,g)$, where $f, g\colon X \to P \in \boldsymbol{P}$. If $f', g'\colon rX \to P$ are the **P**-reflections of $f$ and $g$, respectively, and $V = X \setminus \mathrm{Ker}(f',g')$, using the definition of equalizer it is easy to see that $(r^X)^{-1}(V) = U$.

The converse follows from (v) and (i) of (e).

**1.2. PROPOSITION.** *Let **P** be a reflective subcategory of **Top** dif-*

*ferent from* ***Sing*** *and* $\boldsymbol{Sing}_1$ *(the subcategories of topological spaces which have, respectively, at most one point and exactly one point): if* $s: \mathrm{U} \to \mathrm{X}$ *is an exponentiable embedding in* ***P****, then* $s$ *is in* $\mathrm{O}_P$.

**PROOF.** Let Y be a space in $\boldsymbol{P}$ with at least two different points, say $y_1$, $y_2$. Since $s$ is exponentiable in $\boldsymbol{P}$, the partial product P($s$,Y) exists in $\boldsymbol{P}$ (by 2.1 of [7]) and it is the pullback complement of $p_1$: U×Y→U along $s$. Consider the pullback $p_1$ of the first projection $\pi_1$: X×Y→X along $s$. By the universal property of the pullback complement, there is a unique $k$: X×Y→P which makes the diagram




commute.

It can be easily proved, by the universal property of the pullback complement, that $k(\{x\}\times \mathrm{Y})$ is a singleton when $x \in \mathrm{X} \setminus \mathrm{U}$, while $k$ is injective on $s''(\mathrm{U}\times\mathrm{Y})$. If $i_1, i_2$ : X→X×Y are the embeddings of X into X×$\{y_1\}$ and X×$\{y_2\}$ respectively, then for $f = ki_1$, $g = ki_2$ we have $\mathrm{X}\setminus\mathrm{U} = \mathrm{Ker}(f,g)$, proving that $s$ is in $\mathrm{O}_P$.

**1.3. PROPOSITION.** *Let* ***P*** *be a reflective subcategory of* ***Top****, different from* ***Sing*** *and* $\boldsymbol{Sing}_1$*. Let* $\mathrm{S} \subset \mathrm{Mor}\,\boldsymbol{P}$ *be a class of locally closed embeddings stable under pullbacks in* ***P*** *and* ***P*** *be totally reflective with respect to* S*. Then every* $s$ *in* S *is exponentiable in* ***P*** *and is contained in* $\mathrm{O}_P$.

**PROOF.** From 3.1 of [7] (which works using the stability of S under pullbacks in $\boldsymbol{P}$) $\boldsymbol{P}$ is closed under partial products (over S). This implies that $\boldsymbol{P}$ admits partial products over S, because it is reflective, and then by 2.1 of [7], every $s$ in S is an exponentiable embedding in $\boldsymbol{P}$ ; the result then follows from 1.2.

## 2. TOTAL REFLECTIVITY WITH RESPECT TO DOMINION OPEN EMBEDDINGS.

In this section we look for necessary and sufficient conditions under which an epireflective subcategory $\boldsymbol{P}$ of ***Top***, contained in ***Haus***, is totally reflective with respect to the largest possible class of locally closed embeddings, in this case $O_P$.

From now on, let ***P*** be an epireflective subcategory of ***Top*** contained in ***Haus***. First of all, the hypothesis of total reflectivity with respect to $O_P$ (which is a class of open embeddings for ***P*** $\subset$ ***Haus***) allows us to use 1.3 to obtain the converse of 1.2.

**2.1. THEOREM.** *Let **P** be totally reflective with respect to* $O_P$. *Then* $s: U \to X$ *is an exponentiable embedding in **P** if and only if s is in* $O_P \cap \mathrm{Mor}\,\boldsymbol{P}$ .

**2.2. THEOREM.** *Let **P** be an epireflective subcategory of **Top**. The following are equivalent:*

(1) ***P** is totally reflective with respect to* $O_P$.

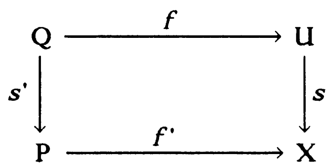
(2) (a) $O_P$ *is closed under composition*

*and* (b) ***P** is closed under pullback complements of monomorphisms over* $O_P$, *i.e., under amalgamations over* $O_P$. *In other words, if* $s: U \to X$, $f: Q \to U$, *f is a monomorphism,* $X, Q$ *are objects in **P** and s is in* $O_P$, *then the pullback complement of f over s is in **P**.*

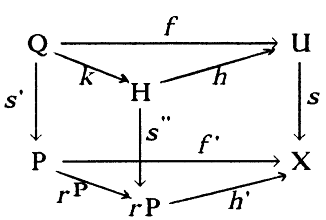
(3) ***P** is closed under pullback complements over* $O_P$.

**PROOF.** For ***P*** = ***Sing*** all the equivalences are trivial.

(1) $\Rightarrow$ (3): Let $f: Q \to U$, $s: U \to X$ be morphisms of ***P*** with $s$ in $O_P$, and




the pullback complement of the pair $(f, s)$. We have to show that P is in ***P***. Corresponding to the reflection $r^P: P \to rP$, there exists a map $h': rP \to X$ such that $h' r^P = f'$. Consider now the pullback $h: H \to U$ of $h'$ along $s$. We get a morphism $k: Q \to H$ making the following diagram commute:



Since $s''$ is in $O_{\mathbf{P}}$, by (h), and H is in $\mathbf{P}$, $k$ is the reflection of Q in $\mathbf{P}$, by total epireflectivity. But Q is in $\mathbf{P}$, so $k$ is a homeomorphism. By the universal property of the pullback complement, corresponding to $k^{-1}$, there is a map $z$: $r\mathrm{P}\to\mathrm{P}$ such that $z r^{\mathrm{P}} = \mathrm{id}_{\mathrm{P}}$. Since $r^{\mathrm{P}}$ is an epimorphism, so P is in $\mathbf{P}$.

(3)⇒(1): $\mathbf{P}$ is closed under partial products along $O_{\mathbf{P}}$, since we are dealing with monomorphisms (see (k)). The result then follows from 3.1 of [7].

(1)⇒(2)(a): Let $i_1$: Q'→X and $i_2$: Q→Q' be in $O_{\mathbf{P}}$. Then, if we denote by $r^j$ the restriction of $r^{\mathrm{X}}$ to $i_j$ for $j=1,2$, the following diagram

$$\begin{array}{ccc} \mathrm{Q} & \xrightarrow{\quad r^2 \quad} & r^{\mathrm{X}}(\mathrm{Q}) \\ {\scriptstyle i_2}\downarrow & & \downarrow{\scriptstyle s_2} \\ \mathrm{Q'} & \xrightarrow{\quad r^1 \quad} & r^{\mathrm{X}}(\mathrm{Q'}) \\ {\scriptstyle i_1}\downarrow & & \downarrow{\scriptstyle s_1} \\ \mathrm{X} & \xrightarrow{\quad r^{\mathrm{X}} \quad} & r\mathrm{X} \end{array}$$

is such that the bottom square is a pullback square and $s_1$ is in $O_{\mathbf{P}}$ by 1.1. By total reflectivity, $r^1$ is uniquely $\mathbf{P}$-extendable and its codomain is in $\mathbf{P}$, so $r^1 = r^{\mathrm{Q'}}$. So also the top square is a pullback square and $s_2$ is in $O_{\mathbf{P}}$, again by 1.1. From 2.1, it follows that $s_1$ and $s_2$ are in the class of exponentiable embeddings in $\mathbf{P}$, which is closed under composition (see [12]). By 1.2, $s_1 s_2$ is then in $O_{\mathbf{P}}$. Since the outward diagram is a pullback and $O_{\mathbf{P}}$ is stable under pullbacks, we have that $i_1 i_2$ is in $O_{\mathbf{P}}$.

(3)⇒(2)(b): is trivial.

(2)⇒(1): Let X be a topological space. We need to show that if $s$: U→ $r$X is in $O_{\mathbf{P}}$, the pullback $s^*(r^{\mathrm{X}}) = r^{\mathrm{X}}_1$ of $r^{\mathrm{X}}$ along $s$

$$\begin{array}{ccc} \mathrm{V} & \xrightarrow{\quad r^{\mathrm{X}}_1 \quad} & \mathrm{U} \\ {\scriptstyle s'}\downarrow & & \downarrow{\scriptstyle s} \\ \mathrm{X} & \xrightarrow{\quad r^{\mathrm{X}} \quad} & r\mathrm{X} \end{array}$$

coincides with $r^{\mathrm{V}}$. Since U is in $\mathbf{P}$, there is a morphism $f$: $r$V→U such that $r^{\mathrm{V}} f = r^{\mathrm{X}}_1$. We want to show that $f$ is a monomorphism (and then a bimorphism). Let $x$, $y$ be two points of V such that $r^{\mathrm{X}}_1(x) = r^{\mathrm{X}}_1(y)$. The points must be in the same $\mathbf{P}$-quasicomponent of X, that is $\mathrm{K}^{\mathbf{P}}_{\mathrm{X}}(\{x\}) = \mathrm{K}^{\mathbf{P}}_{\mathrm{X}}(\{y\})$ (see (k)). By (e),

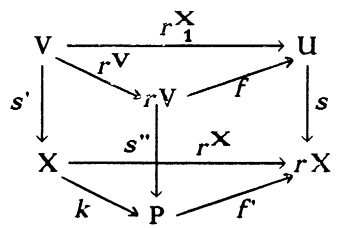
$$K_V^P(\{x\}) \subset K_X^P(\{y\}) \subset V.$$

As $V = (r^X)^{-1}(U)$, $V \backslash K_V^P(\{x\})$ is ***P***-dominion open in V, which is ***P***-dominion open in X by (h). Then, by (2)(a), $V \backslash K_V^P(\{x\})$ is ***P***-dominion open in X. So $X \backslash (V \backslash K_V^P(\{x\}))$ is ***P***-dominion closed in X and contains $x$. Hence it must contain $K_X^P(\{x\})$ by (f). Consequently $K_X^P(\{x\}) = K_V^P(\{x\})$ and, by the same argument,

$$K_X^P(\{y\}) = K_V^P(\{y\}), \text{ so } r^V(x) = r^V(y),$$

thus $f$ is a monomorphism.

Consider the pullback complement P of $f$ over $s$. By the universal property, there is a morphism $k$: X→P such that the following diagram commutes:




since the pullback complement of a monomorphism is a monomorphism, $f'$ is mono. Moreover, since P ∊ ***P*** by (2)(b), there is a morphism $k'$: $rX \to P$ such that $k' r^X = k$ and so $f' k' = id_{rX}$. This implies that, since $f'$ is a monomorphism, it is an isomorphism, and then $f$ is an isomorphism too; thus $r_1^X = r^V$.

**2.3. REMARK.** If ***P*** is not contained in ***Haus***, the proof of the implication (2)⇒(1) still works when we substitute $O_P$ by $O_P \cap LC$.

**2.4. COROLLARY.** *If* ***P*** *is totally reflective with respect to* $O_P$ *and* $(X,\tau)$ *is a topological space, then* ***P****-dominion open subspaces of* X *form a topology (coarser than* $\tau$ *).*

**PROOF.** $O_P$ is stable under pullbacks by (h) and closed under composition by (2)(a) of Theorem 2.2, so $O_P$ is closed under finite intersections. The other axioms for a topology on X are trivially verified.

**2.5. COROLLARY.** *Let* ***P*** *be quotient reflective in* ***Top***. *Then* ***P*** *is totally reflective with respect to* $O_P$ *if and only if* $O_P$ *is closed under composition.*

**PROOF.** (2)(b) of Theorem 2.2 is trivially verified, since the pullback complement $f': P \to X$ of a monomorphism $f$ is a monomorphism. Thus the corollary follows 2.2.

**2.6. REMARK.** It can be easily proved that in ***Top*** the ***P***-reflection is hereditary with respect to $O_P$ if and only if ***P*** is totally epireflective with respect to $O_P$.

As a consequence, Theorem 2.2 characterizes the reflective subcategories of ***Top*** whose ***P***-reflections are hereditary with respect to $O_P$.

**2.7. COROLLARY.** *Let* ***P*** *be as in* 2.2 *and let* Q(***P***) *denote the quotient reflective hull of* ***P*** *in* ***Top***. *If* ***P*** *is totally reflective with respect to* $O_P$, *then* Q(***P***) *is totally reflective with respect to* $O_P$.

**PROOF.** This follows from $K^P = K^{Q(P)}$ (see [3]).

**2.8. REMARK.** We recall (see [3]) that the Salbany's closure operators $K^P$ are in a bijective correspondence with the quotient reflective subcategories of ***Top***:

$$K^P = K^{P'} \text{ if and only if } Q(P) = Q(P').$$

Salbany showed that these operators are algebraic closure operators and in [2] there is an example of a category ***P*** whose operator $K^P$ is not a Kuratowski operator. Nevertheless, the question of when these operators are topological is still open. Corollary 2.4 implies the following:

**2.9. COROLLARY.** *If* Q(***P***) *is totally reflective with respect to* $O_P$, *then* $K^P$ *is a Kuratowski closure operator.*

**EXAMPLES.** For ***P*** = ***Haus, Reg, Tych, Bool*** it is proved in [6] that ***P*** is totally reflective with respect to open embeddings. Consequently, by 1.2 open embeddings in ***P*** are in $O_P$ (and vice versa, since ***P*** ⊂ ***Haus***). Then ***P*** is totally reflective with respect to $O_P$.

**2.10. COROLLARY.** *For* ***P*** = Q(***Reg***), Q(***Tych***), Q(***Bool***) = Q($D_2$), ***P*** *is totally reflective with respect to* $O_P$, *but not with respect to open embeddings.*

**PROOF.** The first assertion follows from 2.7 and the previous observation. The latter one follows from the fact that the class of all regular monomorphisms in these categories is strictly contained in the class of closed embeddings (see [1]).

**2.11. COROLLARY.** *The category* ***Ury*** *of Urysohn spaces (a space is Urysohn if any two distinct points have disjoint closed neighborhoods) is totally reflective with respect to* $O_{Ury}$.

**PROOF.** One can easily check that Corollary 2.3 applies to ***Ury***-dominion open embeddings, which are characterized by Schroeder in [15].

**2.12. REMARK.** If we consider a strongly rigid Hausdorff space R, the reflective hull of R in ***Top*** is not totally reflective with respect to open embeddings (this is essentially the example given by Kennison in [10] to give a negative answer to Dyckoff's conjecture in [6]). But even the epireflective hull and the quotient reflective hull of R (whose reflection maps are surjective and therefore dense) are not totally reflective with respect to ***P***-dominion open embeddings, as these are not closed under composition (see [2]).

## 3. EXPONENTIABLE EMBEDDINGS IN TOTALLY REFLECTIVE SUBCATEGORIES OF *Top*.

In the last part of Section 2 we have given examples of subcategories of ***Top*** totally reflective with respect to $O_P$. So we are able to apply Theorem 2.1 in order to obtain a characterization of exponentiable embeddings in these categories, which are nothing more than ***P***-dominion open embeddings inside ***P***.

The following characterizations are then obtained describing ***P***-dominion open embeddings inside ***P*** by means of Theorem 1.1 and the nature of regular monomorphisms of ***P*** given in [2].

**3.1.** In ***Haus, Reg, Tych, Bool***, exponentiable embeddings are exactly the open embeddings.

**3.2.** In any disconnectedness of ***Haus, Reg, Tych***, e.g. in the category of totally disconnected Hausdorff spaces, exponentiable embeddings are exactly the open embeddings.

**PROOF.** This follows from 2.1, 3.1 and Remark 2.5 of [4].

**3.3.** In Q(***Tych***) exponentiable embeddings are the embeddings whose image is a union of cozero subsets of the codomain.

**3.4.** In Q(***Bool***) exponentiable embeddings are the embeddings whose image is a union of clopen subsets of the codomain.

**3.5.** In ***Ury*** exponentiable embeddings are the embeddings whose image contains a closed neighborhood of each of its points (see [15]).

**3.6. REMARK.** We are going to show the existence of a category ***P*** contained in ***Haus*** in which the class of exponentiable embeddings is strictly contained in $O_P$. This implies that ***P*** is not totally reflective with respect to $O_P$. This fact will show that the condition (2)(a) of Theorem 2.2, which is satisfied by ***P***, is not sufficient for the total reflectivity with respect to $O_P$ (if ***P*** is not quotient reflective).

The spaces involved in this example are:

$\mathbb{R}_1$: the real line with the standard topology;

$\mathbb{R}_2$: the real line with the topology whose base is formed by $]a,b[$ and $]a,b[\cap\mathbb{Q}$, for all $a,b\in\mathbb{R}$.

**3.7. LEMMA.** *Every continuous map from* $\mathbb{R}_1$ *to* $\mathbb{R}_2$ *is constant.*

**PROOF.** Let $f:\mathbb{R}_1\to\mathbb{R}_2$ be continuous and consider $A=\mathbb{Q}\cap f(\mathbb{R}_1)$. If A has more than one point, then $f^{-1}(A)$ is an open subset of $\mathbb{R}_1$ which is the disjoint countable union of closed sets and this is impossible. So A is empty or $A=\{a\}$. In any case $f(\mathbb{R}_1)$ is contained in a totally disconnected space and $f$ must be constant.

**3.8. LEMMA.** $C(\mathbb{R}_2,\mathbb{R}_1)$ *coincides with* $C(\mathbb{R}_1,\mathbb{R}_1)$, *the* ***Tych****-reflection of* $\mathbb{R}_2$ *is* $\mathbb{R}_1$.

**PROOF.** Suppose $f:\mathbb{R}_2\to\mathbb{R}_1$ continuous. We will prove that $f:\mathbb{R}_1\to\mathbb{R}_1$ is continuous and then, since $\mathbb{R}_1$ is a cogenerator of ***Tych***, this will prove the lemma. If $x$ is irrational, $f$ is continuous in $x$, since the set of symmetric neighborhoods of $x$ is a base at $x$ for $\mathbb{R}_1$ as well as for $\mathbb{R}_2$. If $x$ is rational, then for every $\varepsilon>0$ there is a $\delta>0$ such that

$$f(]x-\delta,x+\delta[\cap\mathbb{Q})\subset\,]f(x)-\varepsilon,f(x)+\varepsilon[.$$

Thus, since in $\mathbb{R}_2$ the closure of $]x-\delta,x+\delta[\cap\mathbb{Q}$ is $[x-\delta,x+\delta]$ and $f$ preserves the adherence,

$$f([x-\delta, x+\delta]) \subset [f(x)-\varepsilon, f(x)+\varepsilon];$$

this means $f\colon \mathbb{R}_1 \to \mathbb{R}_1$ continuous.

**3.9. LEMMA.** *If* $f\colon \mathbb{R}_1\times\mathbb{R}_2 \to \mathbb{R}_1$ *is continuous, then* $f\colon \mathbb{R}_1\times\mathbb{R}_1\to\mathbb{R}_1$ *is continuous.*

**PROOF.** Similar to the proof of Lemma 3.8 considering the two cases in which the second coordinate is rational and irrational.

Consider now the epireflective hull $\boldsymbol{P}$ of $\mathbb{R}_1\times\mathbb{R}_2$ in ***Top*** (which is contained in Q(***Tych***)) and the following spaces: $\mathbb{R}_1\times\mathbb{R}_1$, $\mathbb{R}_1\times\mathbb{R}_2$, the subspace $A = ]0,1[\times]0,1[$ of $\mathbb{R}_1\times\mathbb{R}_1$, and the subspace $B = ]0,1[\times]0,1[$ of $\mathbb{R}_1\times\mathbb{R}_2$. The inclusion open map $s\colon A\to\mathbb{R}_1\times\mathbb{R}_1$ is a $\boldsymbol{P}$-dominion open embedding, since A is a cozero set in $\mathbb{R}_1\times\mathbb{R}_1$ and we have (see [3]):

$$\textit{\textbf{Tych}} \subset \boldsymbol{P} \subset \mathrm{Q}(\textit{\textbf{Tych}}) \; .$$

We will now show $s$ is not exponentiable in $\boldsymbol{P}$. Suppose, to the contrary, that $s$ is exponentiable in $\boldsymbol{P}$. There then exists the pullback complement of the pair $(i,s)$, where $i\colon B\to A$ denotes the identity map:

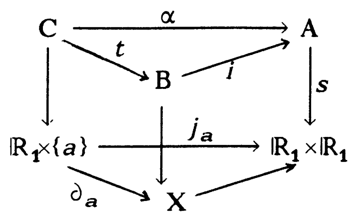
$$\begin{array}{ccc} B & \xrightarrow{\;\;i\;\;} & A \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle s} \\ X & \xrightarrow[\;\;k\;\;]{} & \mathbb{R}_1\times\mathbb{R}_1 \end{array}$$

We will prove that X is not in $\boldsymbol{P}$.

**PROOF.** Since $i$ is a bijection, $k$ is a bijection too (by properties of pullback complement). Let $a$ be a real number and $j_a$ the inclusion: $\mathbb{R}_1\times\{a\}\to\mathbb{R}_1\times\mathbb{R}_1$; consider the following pullback

$$\begin{array}{ccc} C & \xrightarrow{\;\;\alpha\;\;} & A \\ \downarrow & & \downarrow{\scriptstyle s} \\ \mathbb{R}_1\times\{a\} & \xrightarrow[\;\;j_a\;\;]{} & \mathbb{R}_1\times\mathbb{R}_1 \end{array}$$

where C may be the empty set or a copy of $]0,1[\times\{a\}$, when $a$ is in $]0,1[$. In any case there is a unique $t\colon C\to B$ such that $it = \alpha$ and so there is an embedding $\partial_a\colon \mathbb{R}_1\times\{a\}\to X$ by the universal property of the pullback complement




By Lemma 3.7, for any $f: X\to\mathbb{R}_2$ the composition map $f\partial_a$ is constant, as $a$ varies in the real numbers.

In a similar way we can prove that $f\partial'_b$ is constant, for the embedding $\partial'_b$ of $\{b\}\times\mathbb{R}_1$ in X, for any $b$ not belonging to $]0,1[$. Since $k$ is a bijection, this means that $f$ must be constant on X.

So the initial topology induced on X by $C(X,\mathbb{R}_1)$ and $C(X,\mathbb{R}_2)$ coincides with the topology induced by the $C(X,\mathbb{R}_1)$ alone; thus the topology on X should be completely regular, in case $X\in\boldsymbol{P}$. But this is not, since X contains as a subspace a copy of $\mathbb{R}_2$; in fact $h: B\to X$ is an embedding and B is homeomorphic to $\mathbb{R}_1\times\mathbb{R}_2$.

$\boldsymbol{P}$ satisfies (2)(a) of Theorem 2.2, since ***Tych*** $\subset \boldsymbol{P} \subset$ Q(***Tych***) and then

$$O_{\boldsymbol{P}} = O_{\mathit{Tych}} = O_{Q(\mathit{Tych})},$$

and ***Tych*** and Q(***Tych***) satisfy (2)(a).

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