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HOMOLOGY OF WEIGHTED SIMPLICIAL COMPLEXES

by Robert J. MacG. DAWSON

RÉSUMÉ. L'auteur généralise la notion de complexe simplicial en celle de complexe simplicial pondéré. Ces complexes sont particulièrement utiles pour construire des théories d'homologie non-standard pour des catégories de nature combinatoire.

1. DEFINITIONS.

Throughout this paper, we will maintain the convention that any natural number, including 0, divides 0 (0/0 = 0), while zero does not divide any other natural number. The natural numbers thus form a complete lattice, ordered by divisibility, with GCD as the meet operation, LCM as the join, 1 as bottom element and 0 as top element. A *weighted simplicial complex* (WSC) is a pair (K, w) consisting of a simplicial complex K and a function w from the simplexes of K to natural numbers, obeying

$$s_1 \subset s_2 \Rightarrow w(s_1) \mid w(s_2) .$$

A *morphism of* WSC's is a simplicial map

$$f\colon K \to L \text{ such that } w(f(s)) \mid w(s).$$

These form the morphisms of a category *WSC*.

EXAMPLES. For any simplicial complex K, and every natural number a, there is a WSC (K,a) in which every simplex has weight a. These constructions, which we call *constant weightings*, are functorial, and the constant weighting functor (-,1) and (-,0) are respectively right and left adjoint to the forgetful functor U from *WSC* to the category *SC* of simplicial complexes.

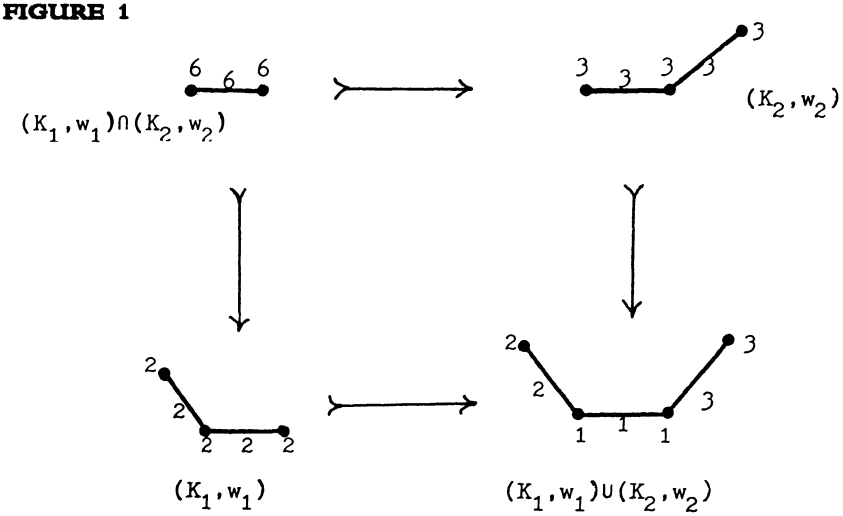
This implies that all limits and colimits that exist in *WSC* are preserved by U ([5], §V.5). In fact, more is true; any desired limit or colimit in *WSC* may be found by putting a suitable weighting on the corresponding limit or colimit in *SC* .

PROPOSITION 1.1. *WSC is complete and cocomplete.*

PROOF. A category is (co)complete if it has binary (co)equalizers and all small (co)products ([5], §5.2). The equalizer of a pair f, g: K$\rightrightarrows$ L is the subcomplex of K on which $f(x) = g(x)$, with weights inherited from K. The product $\prod(K_i, w_i)$ of a family of WSC's is the product complex, with $w(s) = \mathrm{LCM}\, w_i(s)$. Coproducts are disjoint unions with inherited weightings, while the coequalizer of f, g: K$\rightrightarrows$ L is the quotient complex $L/f(x)=g(x)$, with $w(s)$ equal to the GCD of the weights of its inverse images. ∎

In the context of homology, intersections and unions are particularly important instances of (co)limits. In topology, intersections and unions are the first and last corners, respectively, of bicartesian squares whose edges are subspace inclusions. We could define intersection and union in the same way here; but it will be useful to keep a little more generality, and define a *UI* square to be any bicartesian square whose edges are monomorphisms. (A monomorphism of WSC's is easily seen to be any WSC homomorphism that is 1-1 on the vertices, regardless of the weightings.) The first and last corners of a UI square will be called the *intersection* and *union*, respectively, of the other two corners. These operations are easily seen to be associative; so we can also define unions and intersections of more than two WSCs. Figure 1 shows an example of a UI square, which also illustrates the following corollary.

FIGURE 1



COROLLARY 1.1.1. $(K_1, w_1) \cup (K_2, w_2)$ *is the union of* K_1 *and* K_2 *with weighting* w_1 *in* $K_1 \backslash K_2$, w_2 *in* $K_2 \backslash K_1$ *and* $\mathrm{GDC}(w_1, w_2)$ *in* $K_1 \cap K_2$. $(K_1, w_1) \cap (K_2, w_2)$ *is the intersection of* K_1 *and* K_2, *with weighting* $\mathrm{LCM}(w_1, w_2)$. ∎

COROLLARY 1.1.2. *Intersections and unions of WSC's obey the distributive laws:*

$$(K_1, w_1) \cup [(K_2, w_2) \cap (K_3, w_3)] = [(K_1, w_1) \cap (K_2, w_2)] \cup [(K_1, w_1) \cap (K_3, w_3)],$$

$$(K_1, w_1) \cap [(K_2, w_2) \cup (K_3, w_3)] = [(K_1, w_1) \cup (K_2, w_2)] \cup [(K_1, w_1) \cap (K_3, w_3)].$$ ∎

A WSC with no zero weights will be called *positive* or *positively weighted.* Let *PWSC* be the full subcategory of positively weighted simplicial complexes. The operation $(-)^+$, which takes (K, w) to its subcomplex of positively weighted simplexes, is easily seen to be functorial and right adjoint to the inclusion *PWSC* ⊂ *WSC*. Hence, *PWSC* is complete and cocomplete, with colimits reflected by the inclusion *PWSC* ⊂ *WSC* and limits preserved by $(-)^+$.

PROPOSITION 1.2. *The following conditions on a WSC are equivalent:*

i) The weight of any simplex is the LCM of the weights of its faces;

ii) The weight of any simplex is the LCM of the weights of its subsimplexes:

iii) The weight of any simplex is the LCM of the weights of its vertices;

iv) The weight of any simplex is the LCM of the weights of some two of its faces.

PROOF. (i)⇒(ii) by induction, and (ii) implies (iii) trivially. (iii)⇒(i), as the LCM is an associative and absorptive function. Any two different faces of a simplex contain, between them, all its vertices; so (iii)⇔(iv). ∎

We will call any weighted simplicial complex that satisfies the conditions of Proposition 1.2 *economical.* An economical weighted simplicial complex may equivalently be represented as a simplicial complex with a weighting function v defined on its vertices. The full subcategory of (positive) economical weighted simplicial complexes will be designated $(P)EWSC$; and the morphisms of $(P)EWSC$ may be equivalently defined to be simplicial maps such that $v(f(x)) \mid v(x)$ for any vertex x.

There is a functor E: $WSC \to EWSC$, which preserves the underlying simplicial complex K and the weightings on its vertices, but assigns to each higher-dimensional simplex the LCM of the weights of its vertices. As $\mathrm{LCM}\{w(x):\ x \in s\} \mid w(s)$, E is right adjoint to the inclusion $EWSC \subset WSC$; hence $EWSC$ and $PEWSC$ are complete and cocomplete, with limits agreeing with those in WSC. Colimits in $EWSC$ are the images under E of the corresponding colimits in WSC.

(Categorical-structuralist note: Any simplicial complex K is partially ordered by inclusion, and the natural numbers N are partially ordered by divisibility. As any partially ordered set may be considered as a category, a weighting may be considered as a functor $K \to N$. Furthermore, we may consider the simplexes of K as analogous to the open sets of a topological space, and a weighting as a "presheaf" of integers over K (strictly, over K^{op}). This is not a particularly illuminating analogy, but becomes more interesting when we recognize that economical weighted simplicial complexes correspond to sheaves!

In orthodox sheaf theory, a sheaf is defined to be a presheaf F with the additional property that the set of sections over any open set U is the intersection of the sets of sections over the open subsets of U. In other words, after adjustment for contravariance, a sheaf is a presheaf which preserves pushouts; but, by Proposition 1.2, this is precisely what an economical weighting does.)

Note that every constant weighted simplicial complex is economical. This has a converse for functorial weightings on SC:

PROPOSITION 1.3. *Every functorial economical weighting on SC is constant over all of SC.*

PROOF. Given two vertices $a \in A$, $b \in B$, there is a SC-morphism mapping a to b. Thus $w(b) \mid w(a)$; but similarly $w(a) \mid w(b)$, so the weights are equal, from which the proposition follows. ∎

PROPOSITION 1.4. *For any WSC* (K, w), *the following are equivalent:*

i) Every simplex $s \in K$ *of dimension greater than* 1 *has a proper subsimplex* t *such that* $w(s) = w(t)$;

ii) Every simplex $s \in K$ *has a vertex* x *such that*
$w(s) = w(x)$:

iii) (K, w) *is economically weighted, and the vertices* $\{x_i\}$ *of every simplex can be ordered so that* $w(x_i) \mid w(x_{i+1})$.

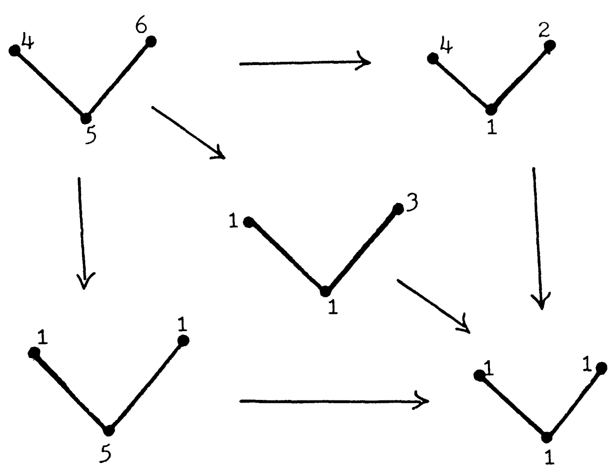
PROOF. (i)$\Rightarrow$(ii) by a simple induction. (ii) implies that (K, w) is economically weighted, and if x is a vertex of s such that $w(s) = w(x)$, the weight of every other vertex of s must divide $w(x)$; so (ii)$\Rightarrow$(iii). Finally, given (iii), every simplex has a subsimplex containing its vertex of maximal weight, and so (i) is satisfied. ■

We will call any WSC satisfying the conditions of Proposition 1.4 *divisible*. The full subcategory *DWSC* of divisible WSC's is not complete, and there is no canonical way to make a WSC divisible. If the weights of a WSC (K, w) are all powers of some integer n, we will call (K, w) *n-ary*. The full subcategory of n-ary WSC's for a given n will be denoted n*WSC* (so, for instance, 2*WSC* , 5*WSC*): and the full subcategory of all n-ary WSC's will be denoted N*WSC*. These categories are subcategories of *DWSC*, and for any n, n*WSC* is a coreflective subcategory of *EWSC* (the left adjoint to the inclusion n*WSC* $\subset$ *EWSC* being the functor $(-,-_n)$ which does not affect K, but takes each weight to the largest power of n dividing it.)

PROPOSITION 1.5. *Every EWSC is an intersection of* N*WSC's.*

PROOF. (K, w) is the intersection of WSC's (K, w_p), where p runs over all primes, in a diagram whose edges are induced by the identity map K$\to$K; their union is (K,1) (Figure 2). ■

FIGURE 2



COROLLARY 1.5.1. N*WSC and DWSC are dense in EWSC.* ■

2. HOMOLOGY OF WSC's.

The *chain group,* C(K, w) (or, more succinctly, where unambiguous, C(K)) is the graded free abelian group generated by the positively-weighted simplexes of K. Given a function f: K→L, the induced homomorphism C(f): C(K)→C(L) is defined on the generators of C(K) as:

$$C(f):\ s \mapsto (w(s)/w(f(s)))f(s).$$

Note that this is well-defined, as, if $w(s) > 0$, $w(f(s)) > 0$.

The boundary operator $\partial: C_n(K) \to C_{n-1}(K)$ is the map:

$$\partial:\ s \mapsto \sum_{j=0}^{n}(w(s)/w(\partial_j(s)))(-1)^j\partial_j(s)$$

where the face operators ∂_j are defined as in standard simplicial homology. Again, if $w(s) > 0$, $w(\partial_j(s)) > 0$.

PROPOSITION 2.1. $\partial\partial = 0$.

PROOF. This is proved as in standard simplicial homology (see, e.g., [6], p. 159) except that instead of pairing and cancelling terms of the form $\pm\partial_i\partial_j$, we pair and cancel terms of the form

$$\pm(w(s)/w(\partial_i\partial_j(s)))\partial_i\partial_j(s).\ ■$$

PROPOSITION 2.2. *For any WSC homomorphism f,* C(f)∂ = ∂C(f).

PROOF.

$$\begin{aligned} C(f)\partial(s) &= C(f)(\sum_{j=0}^{n}(w(s)/w(\partial_j(s)))(-1)^j\partial_j(s) \\ &= \sum_{j=0}^{n}(w(\partial_j(s))/w(f(\partial_j(s))))(w(s)/w(\partial_j(s)))(-1)^j C(f)\partial_j(s) \\ &= \sum_{j=0}^{n}(w(f(s))/w(f(\partial_j(s))))(w(s)/w(f(s)))(-1)^j\,\partial_j f(s) \\ &= \partial C(f)(s).\ ■ \end{aligned}$$

Given these two properties, it follows immediately that we can define a "homology functor" H from *WSC* to *GrAb*, the category of graded abelian groups. Furthermore, if we let *WSCPair* be the category of monomorphic pairs in *WSC* (where the monomorphism does not necessarily preserve weights), we can extend H to a relative homology functor, with the exactness and excision properties. It is evident that, for any constant positive weighting, $H(K,a)$ is the usual homology of the complex K.

THEOREM 2.3. *Let ∂ be the boundary operator described above, and let* H *be the associated relative homology operation. Then*

(i) H *is functorial.*

(ii) (H,∂) *has the exactness property: for all monomorphic pairs* $A \rightarrowtail K$, *there is a long exact sequence*

$$\cdots \to H_n(A) \to H_n(K) \to H_n(K,A) \to H_{n-1}(A) \to \cdots$$

in homology;

(iii) has the excision property: for all UI squares

$$\begin{array}{ccc} (K_0, w_0) & \rightarrowtail & (K_1, w_1) \\ \downarrow & & \downarrow \\ (K_2, w_2) & \rightarrowtail & (K_3, w_3) \end{array}$$

$$H_n((K_1, w_1), (K_0, w_0)) = H_n((K_3, w_3), (K_2, w_2)).$$

PROOF. (i) follows from the naturality of ∂; (ii) follows from Proposition 2.1 and the usual diagram chase. To prove (iii), observe that by Corollary 1.1.1, $w_0(s)/w_1(s) = w_2(s)/w_3(s)$, and so, in any dimension,

$$\begin{aligned} C_n(K_1, w_1)/C_n(K_0, w_0) &= \sum_{K_0}(Z/(w_0(s)/w_1(s))Z) \oplus \sum_{K_1/K_0} Z \\ &= \sum_{K_0}(Z/(w_2(s)/w_3(s))Z) \oplus \sum_{K_3/K_2} Z \\ &= C_n(K_3, w_3)/C_n(K_2, w_2) \ . \quad \blacksquare \end{aligned}$$

COROLLARY 2.3.1. *The Mayer-Vietoris sequence is exact for any UI square.* ■

PROPOSITION 2.4. *(Dimension axiom) For any 1-point positive WSC* $(\{x\}, w)$, $H_0 = Z$ *and all other homology groups are trivial.*

PROOF. $\{x\}$ and its multiples are the only 0-cycles, and there are no nontrivial n-simplexes for $n > 0$, and hence no nontrivial n-cycles or $(n-1)$-boundaries. ■

We have so far verified analogues of all but one of the Eilenberg-Steenrod axioms [4]. It is natural to ask whether (H,∂) has the homotopy property, or rather its simplicial equivalent, the contiguity property. If we define "contiguity" correctly, the answer is yes; but "contiguities" do not exist in *WSC* as often as we might expect from our experience in other categories.

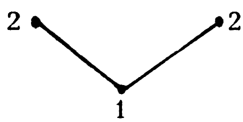
DEFINITION. Two *WSC* morphisms $f,g\colon (K,w)\rightrightarrows(L,v)$ are *contiguous* if there exists a *WSC* morphism $h\colon (K,w)\times I\rightrightarrows(L,v)$, where I is the 1-simplex with vertices $\{0,1\}$ and all we hts 1, such that

$$h(x,0) = f(x) \text{ and } h(x,1) = g(x).$$

DEFINITION. A WSC (K,w) is *contractible* if there exists a sequence $f_0, f_1,\ldots, f_n\colon (K,w)\rightrightarrows(K,w)$ of *WSC* homomorphisms such that f_0 is the identity on (K,w), f_i is contiguous to f_{i-1}, and f_n is a constant map.

EXAMPLE. In Figure 3, the complex shown is contractible to the center vertex; but the complex shown in Figure 4 is not contractible, as any endomorphism contiguous to the identity must fix both end vertices.

FIGURE 3



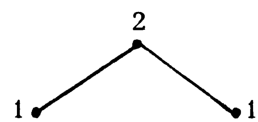


FIGURE 4

Notice that if we had used the 1-simplex with all weights 0 as our "unit interval" in the definition of contiguity, two maps would have been contiguous in *WSC* iff they were contiguous in *SC*. However, that definition would not give us the following desirable result:

THEOREM 2.5. (*Homotopy axiom). Contiguous WSC homomorphisms induce equal maps in homology.*

PROOF. Let the vertices (k_i) of K be numbered, and in any simplex s of K let the vertices (s_i) appear in a restriction of that order. Then let $H(K,w)$ be the weighted simplicial complex whose $(n+1)$-simplexes are those of the form

$$s(i) = ((s_0,0),(s_1,0),\ldots,(s_i,0),(s_i,1),\ldots,(s_n,1)),\qquad 0\le i\le n$$

where $s=(s_0,s_1,\ldots,s_n)$ is a n-simplex of K with $w(s(i)) = w(s)$. This is a subcomplex of $(K,w)\times I$; thus the contiguity map h restricts to a *WSC* morphism $h\colon H(K,w)\to(L,v)$ with $h(x,0) = f(x)$, and $h(x,1) = g(x)$. Furthermore, the simplexes that appear in ∂hH are precisely those that appear in the unweighted boundary. As $w(s(i)) = w(s)$, those that appear in f(K) or g(K) appear with the same multiplicity as in those images; while those that appear in hH∂(K) appear with the same multiplicity that they do there. Thus, hH is a chain homotopy in the usual sense, and the result follows. ■

COROLLARY 2.5.1. *If* (K, w) *is positive and contractible, then* H(K, w) = Z *in dimension* 0 *and* 0 *in all other dimensions.* ■

3. CALCULATION OF HOMOLOGY GROUPS IN *WSC*.

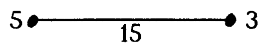
The homology functor that we have just constructed differs in several ways from the standard simplicial homology functor. For instance, it is possible for H_0 of a weighted simplicial complex to have torsion, as the following example shows.

EXAMPLE. Let (K, w) be as in Figure 4. There are no nontrivial cycles in dimension greater than zero. H_0 is generated by the three 0-cycles corresponding to the points x, y and z, and has boundaries of the two edges as relators. Thus

$$H_0 = \langle x,y,z \mid 2x = y,\ y = 2z \rangle \approx \langle x,z \mid 2x = 2z \rangle \approx Z \oplus Z_2.$$

It is just as easy to weight any n-simplex so that it will have any homology group up to the (n-1)-st nonzero; if an n-simplex has weight p, and all of its proper subsimplexes have weight 1, it will have $H_{n-1} \approx Z_p$.

On the other hand, it is possible to find conditions under which a simplex will have the "expected" homology groups, Z in dimension 0 and 0 in all other dimensions. One such condition is given in Corollary 2.5.1; all contractible simplexes have these homology groups. This is not a necessary condition however, in light of the next theorem and the fact that not every positive economically weighted simplex is contractible (Figure 5).



THEOREM 3.1. *A positive economically weighted simplex* S *has* $H_0 \approx Z$ *and all other homology groups trivial; and if the weighting on a complex is not economical and positive. some simplex in the complex will fail to have these homology groups.*

PROOF. Any divisibly-weighted simplex is contractible, and by Proposition 1.5, any economically weighted N-simplex is the intersection of a family of p-ary weighted N-simplexes, where p ranges over the prime factors of the weight of the simplex. For notational convenience, we will prove the case in detail for which there are two distinct values of p; repeating this proves the general case. We can calculate the homology of the intersection using the Mayer-Vietoris long exact sequence (Corollary 2.3.1):

$$\cdots \to H_n(S,w) \to H_n(S,w_p) \oplus H_n(S,w_q) \xrightarrow{f+g} H_n(S,1) \to \cdots$$

For $n > 0$, $H_n(S,w_p)$, $H_n(S,w_q)$ and $H_n(S,1)$ are all zero, by Corollary 2.5.1, so $H_n(S,w)$ is zero too. $H_0(S,w_p)$, $H_0(S,w_q)$, and $H_0(S,1)$ are all isomorphic to Z. f: $H_0(S,w_p) \to H_0(S,1)$ must take the generator of its domain to p^i times the generator of its codomain; and $g: H_0(S,w_q) \to H_0(S,1)$ must take the generator of its domain to q^j times the generator of its codomain. If (S,w) is positive, p and q are nonzero and p^i, q^j are coprime. Thus, $(p^i, -q^j)$, which generates $\ker(f+g)$, is a prime element of $H_0(S,w_p) \oplus H_0(S,w_q) \approx Z \oplus Z$. Hence, the range of the previous map in the sequence is isomorphic to Z; and as $H_i(S,1) = 0$, we conclude that $H_0(S,w) = Z$.

It now remains only to show that if w is not economical, or not positive, some subsimplex of S has homology groups different from those given. Let s be a non-economically weighted subsimplex of S of minimal dimension; then its subsimplexes are economically weighted, and

$$\sum_{i=0}^{n} (\mathrm{LCM}(w(s_0), w(s_1), \ldots, w(s_n))/w(s_i))\, s_i$$

is a cycle, but not a boundary. Finally, if w is not positive, but is economical, some vertex has weight 0, and thus has $H_0 = 0$. ∎

This theorem corresponds to the *strong dimension axiom* of [1], asserting (in this case for the category *PEWSC*) that all simplexes have $H_0 = Z$ and all other homology groups trivial. The corresponding axiom was shown, along with the excision and exactness axioms, to characterize the homology of finite simplicial complexes. However, these axioms are insufficient to characterize the homology of PEWSC's, as they are also satisfied by the usual homology functor H on the underlying complexes, and these homology functors are not the same. Consider, for instance, Figure 4, in which $\mathrm{H}(K) \approx Z \oplus Z_2$, while $H(K) \approx Z$.

This apparent paradox is easily explained, when it is remembered that the forgetful functor $U: PEWSC \to SC$ is not full. In particular, various complexes, such as that in Figure 4, are contractible in *SC* but not contractible in *WSC*. As the homology of contractible complexes is an essential element of the proof in [1], it is unsurprising that the result does not generalize.

4. CONSTRUCTION OF WEIGHTED SIMPLICIAL COMPLEXES IN OTHER CATEGORIES.

If we can construct a functor from a category C to the category of simplicial complexes, we have constructed a homology for C . Furthermore, if we can construct a functor from C to *WSC* or (better) to *PEWSC*, we have constructed a "nonstandard" homology theory which obeys similar axioms but may not agree with the standard one. Under what circumstances is this a genuine generalization?

We have seen, in Proposition 1.3, that we cannot obtain a non-constant economical functorial weighting on *SC*; and it has been observed that H(K,a) is the usual homology on K for any constant weighting a. Thus, weighting the entire category of simplicial complexes will not yield any new generalized homology theories. The category *Top* of topological spaces also has the property that any point of any space can be mapped to any other point of any other space; so we cannot obtain any new results by weighting, say, the singular simplicial complexes of *Top*.

(It is possible to find functorial noneconomical and/or nonpositive weightings on the singular simplicial complexes of *Top*; for instance, if we assign weight 1 to every 0-simplex and weight 0 to every higher-dimensional simplex, we obtain a functor that takes every space to the free abelian group generated by its points. This is not really even "generalized" homology, though, as it fails to have anything resembling a homotopy property.)

However, the category Top_* of pointed topological spaces and base-point preserving continuous functions allows somewhat more interesting weightings. (We must, of course, define a singular simplex of a pointed topological space X to be any continuous $f: \Delta_n \to X$, not necessarily base-point preserving.) It is not possible, in Top_*, to map the base-point to any other point, or to map a point in the principal path-component to a point in any other path-component. Therefore, we are at liberty to assign (for instance) weight 1 to any simplex which maps Δ_n into the principal path-component, and weight 2 to all other simplexes. Note that this weighting is positive economical.

Such a weighting generates a homology functor that is generally well-behaved, although it does not agree with the standard one. For instance, any space which can be contracted to its base-point has $H_0 \approx Z$ and all other homology groups trivial; the proof is essentially the same as in the unweighted case. However, if we let X be the discrete 2-point space, Y the 1-point space, and $f: X \to Y$ the unique function, then

$$H(X) \approx Z^2, \quad H(Y) \approx Z, \quad \text{but} \quad H(f)(a,b) = a+2b,$$

rather than $a+b$ as we would usually expect.

For a less trivial example, consider the category of *preconvexity spaces* [2,3]. A preconvexity space (X,K) is a set X, together with an intersectionally closed family K of subsets ("convex sets") which need not include X itself; if X is an element of the preconvexity, (X,K) is called a *convex preconvexity space*. The morphisms of the category *Precxy* of preconvexity spaces are the convex-set-preserving ("Darboux") functions, under which the direct image of a convex set is convex.

FIGURE 6

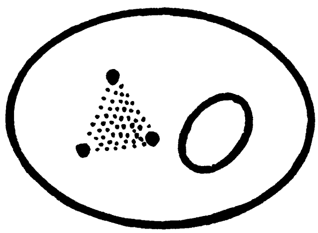
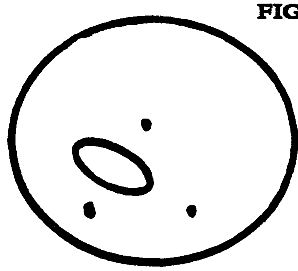


FIGURE 7

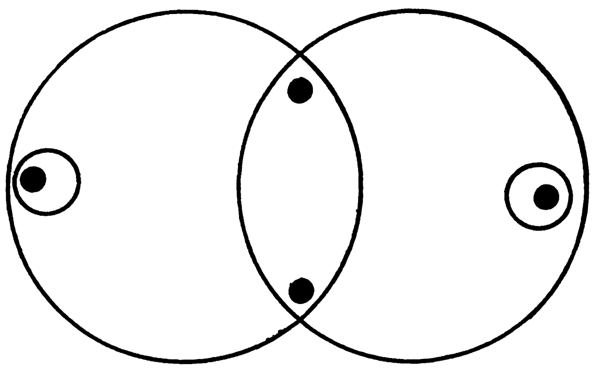


We can define a homology functor on *Precxy*, derived from a simplicial complex whose simplexes are finite ordered sets of points which have a convex hull in the space. Thus, the points shown in Figure 6 form a simplex, while those shown in Figure 7 do not. It is shown in [1] that the exactness, excision, and strong dimension axioms (the latter here stating that the homology group of every convex preconvexity space is Z in dimension 0 and trivial otherwise) characterize this homology functor on the full subcategory of finite downclosed preconvexity spaces. Replacing the strong dimension axiom by an "ultra-strong dimension axiom" which not only gives the homology groups of every convex preconvexity space, but states that every Darboux function between convex preconvexity spaces induces an isomorphism in homology, and adding an "inductivity axiom" that states that the homology functor preserves filtered colimits, we can characterize the homology functor on all of *Precxy*. The concept of a weighted simplicial complex was originally devised to show that this ultrastrong dimension axiom is necessary, and that the strong dimension axiom does not suffice.

In contrast to *Top*, *Precxy* does not have the property that given spaces A, B and points $a \in A$, $b \in B$, there exists a morphism mapping a to b. In particular, it is not necessary that a point should be convex, and a convex point clearly cannot map to a nonconvex one. Thus, for instance, by assigning a weight 1 to every simplex consisting wholly of convex points, and a weight of 2 to every other simplex, a nontrivial weighted simplicial complex is obtained which induces a homology functor. Being a weighted simplicial homology functor, it obeys the exactness, excision, and strong dimension axioms: but it fails to agree with the usual *Precxy* homology functor.

An example is shown in Figure 8; with the usual homology functor, this space has $H_0 = Z$, but with the weighting given above, and the *WSC* homology, $H_0 \approx Z \oplus Z_2$. This may be compared to the example in Figure 4.

FIGURE 8



It is not entirely true that whenever we have points that do not map to each other there exist nonstandard homology theories obeying the exactness, excision, and strong dimension axioms: for example, consider the category consisting of two disjoint copies of *SC*. There are no morphisms from "blue complexes" to "green complexes" or vice versa; but the axioms can be used independently in each half of the category to establish a unique homology functor. However, if there is a pair of objects A, B in a category C of simplicial complexes, with a morphism $f: A \to B$ but no morphism $g: B \to A$, there cannot exist a unique simplicial homology theory on C defined by the axioms. For let $w(p) = 1$ if p is in the image of B under some C-morphism, and otherwise 2; this gives rise to a homology functor that differs from the standard one (for instance, f_* takes a generator of $H_0(A)$ to twice a generator of $H_0(B)$, which is impossible in standard homology.) Both this and the usual homology theory obey the excision, exactness, and strong dimension axioms.

Finding a converse to this result is more difficult. We would like to show that if a category C of finite simplicial complexes "has enough maps", the excision, exactness, and strong dimension axioms suffice to characterize the unique homology functor on C. However, it seems likely that this will require much more than just a map from every vertex in the category to every other vertex. Following [1], it can be shown that if C contains, for every object K, all of K's subobjects and their inclusion maps, and all barycentric divisions of K and the canonical epimorphisms to K, as well as constant maps from K to every vertex of every other object, then C has a unique homology theory obeying the excision, exactness, and strong dimension axioms; however, it remains to be seen whether these conditions are necessary.

Finally, it is easy to show that the excision, exactness, contiguity and strong dimension properties of *PEWSC* do not suffice to characterize its own homology. One example of a homology functor on *PEWSC* that obeys these axioms but differs from the standard one is, of course, the *SC* homology of the underlying complexes. Another example is provided by first applying the functor from *PEWSC* to itself that preserves the underlying simplicial complexes and squares every weight. $H(X, w^2)$ obeys all four axioms; but if X is the complex of Figure 4, $H_0(X, w^2) \approx Z \oplus Z_4$.

REFERENCES.

1. R. J. MacG. DAWSON, A simplification of the Eilenberg-Steenrod axioms for the category of finite simplicial complexes, *J. Pure & Appl. Algebra* 53 (1988), 257-265.
2. R. J. MacG. DAWSON, Limits and colimits of preconvexity spaces, *Cahiers de Topologie et Géométrie Différentielle* XXVIII (1987), 307-328.
3. R. J. MacG. DAWSON, Homology of preconvexity spaces, *Cahiers de Topologie et Géométrie Différentielle*, to appear.
4. EILENBERG & STEENROD, *Foundations of algebraic Topology*, Princeton 1952.
5. S. MAC LANE, *Categories for the working mathematician*, Springer 1971.
6. E. SPANIER, *Algebraic Topology*. Springer 1966.

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