

# CAHIERS DE TOPOLOGIE ET GÉOMÉTRIE DIFFÉRENTIELLE CATÉGORIQUES

GRAHAM J. ELLIS

## **Relative derived functors and the homology of groups**

*Cahiers de topologie et géométrie différentielle catégoriques*, tome 31, n° 2 (1990), p. 121-135

[http://www.numdam.org/item?id=CTGDC\\_1990\\_\\_31\\_2\\_121\\_0](http://www.numdam.org/item?id=CTGDC_1990__31_2_121_0)

© Andrée C. Ehresmann et les auteurs, 1990, tous droits réservés.

L'accès aux archives de la revue « Cahiers de topologie et géométrie différentielle catégoriques » implique l'accord avec les conditions générales d'utilisation (<http://www.numdam.org/conditions>). Toute utilisation commerciale ou impression systématique est constitutive d'une infraction pénale. Toute copie ou impression de ce fichier doit contenir la présente mention de copyright.

NUMDAM

Article numérisé dans le cadre du programme  
Numérisation de documents anciens mathématiques  
<http://www.numdam.org/>

**RELATIVE DERIVED FUNCTORS AND  
 THE HOMOLOGY OF GROUPS**

by *Graham J. ELLIS*

**RÉSUMÉ.** Dans cet article on étudie l'homologie d'un groupe relative à une famille de sous-groupes distingués. On obtient: une description du premier et du deuxième groupe d'homologie relative: deux généralisations de la formule de Hopf: une nouvelle suite exacte d'homologie.

**O. INTRODUCTION.**

Let  $N$  be a normal subgroup of a group  $G$ . It is not difficult to define relative homology groups  $H_n(G:N)$  for  $n \geq 1$  (see §2) which fit into a natural long exact sequence

$$\rightarrow H_{n+1}(G/N) \rightarrow H_n(G:N) \rightarrow H_n(G) \rightarrow H_n(G/N) \rightarrow \dots \rightarrow H_1(G/N) \rightarrow 0$$

where  $H_n(G)$  is the  $n^{\text{th}}$ -homology of  $G$  with integer coefficients.

More generally to  $m$  normal subgroups  $N_1, \dots, N_m$  of  $G$  one can associate hyper-relative homology groups  $H_n(G:N_1, \dots, N_m)$  for  $n \geq 1$  which fit into a natural long exact sequence

$$\begin{aligned} &\rightarrow H_{n+1}(G/N_m: N_1 N_m / N_m, \dots, N_{m-1} N_m / N_m) \rightarrow H_n(G: N_1, \dots, N_m) \\ &\rightarrow H_n(G: N_1, \dots, N_{m-1}) \rightarrow H_n(G/N_m: N_1 N_m / N_m, \dots, N_{m-1} N_m / N_m) \\ &\rightarrow \dots \rightarrow H_1(G/N_m: N_1 N_m / N_m, \dots, N_{m-1} N_m / N_m) \rightarrow 0. \end{aligned}$$

In [2] topological methods were used to give a computational description of the first hyper-relative homology group. In the present article we shall use algebraic methods to recover this description, and then to give a new computational description of the second hyper-relative homology group. Our methods are sufficiently general to apply in other algebraic settings such as the homology of Lie algebras and commutative algebras, and the details of this will be given in a subsequent article.

The description of the first hyper-relative homology group

which we reprove is:

**THEOREM 1** [2]. For  $m \geq 1$  there is a natural isomorphism

$$H_1(G; N_1, \dots, N_m) \cong \{ \prod_{i \in \langle m \rangle} N_i \} / \{ \prod_{\alpha \in \langle m \rangle} [ \prod_{i \in \alpha} N_i, \prod_{j \notin \alpha} N_j ] \}$$

where  $\langle m \rangle = \{1, \dots, m\}$ ,  $\prod_{i \in \emptyset} N_i = G$  and  $\prod$  denotes the group product in  $G$ .

Thus for  $m = 2$  this formula reads

$$H_1(G; N_1, N_2) \cong \{ N_1 \cap N_2 \} / \{ [N_1, N_2] [G, N_1 \cap N_2] \}.$$

The new description of the second hyper-relative homology is in terms of a "hyper-relative version" of the non-abelian exterior product of groups introduced in [3], which we denote by  $\wedge(G; N_1, \dots, N_m)$ .

The group  $\wedge(G; N_1, \dots, N_m)$  has a presentation with generators  $x \wedge y$  for  $x, y \in G$  such that  $x \in \prod_{i \in \alpha} N_i$  and  $y \in \prod_{j \notin \alpha} N_j$  for some  $\alpha \subset \langle m \rangle$ ; the relations are

$$\wedge x' \wedge y = (x x' x^{-1} \wedge x) x^{-1} (x \wedge y),$$

$$\wedge x \wedge y y' = (\wedge x \wedge y) (y \wedge y^{-1} \wedge y y' y^{-1}), \quad \wedge x \wedge y = (y \wedge x)^{-1},$$

$$(\wedge x \wedge y) (u \wedge v) (\wedge x \wedge y)^{-1} = [\wedge x, y] u [\wedge x, y]^{-1} [\wedge x, y] v [\wedge x, y]^{-1}, \quad z \wedge z = 1$$

for  $z \in \prod_{i \in \langle m \rangle} N_i$  and  $x, x' \in \prod_{i \in \alpha} N_i, y, y' \in \prod_{j \notin \alpha} N_j$ .

$$u \in \prod_{i \in \beta} N_i, v \in \prod_{j \notin \beta} N_j, \alpha, \beta \subset \langle m \rangle.$$

Here  $[\wedge x, y] = \wedge y x^{-1} y^{-1}$ . There is a homomorphism

$$\wedge(G; N_1, \dots, N_m) \rightarrow G$$

defined on generators by  $x \wedge y \mapsto [x, y]$ .

**THEOREM 2.** There is a natural isomorphism

$$H_2(G; N_1, \dots, N_m) \cong \text{Ker}(\wedge(G; N_1, \dots, N_m) \rightarrow G).$$

On taking  $m = 2$  and  $N_1 = N_2 = G$  we find

$$H_2(G) \cong H_2(G; G, G) \cong \text{Ker}(\wedge(G; G, G) \rightarrow G)$$

where the first isomorphism follows immediately from the above long exact sequences, and the second follows from Theorem 2. This description of the second integral homology of  $G$  is also given in [3] where it is shown to be a reformulation of a result of Miller [11] (also cf. [6]).

**RELATIVE DERIVED FUNCTORS AND THE HOMOLOGY OF GROUPS**

With ingenuity one can use Theorems 1 and 2 to obtain other results on the ordinary integral homology of a group. We shall cite some examples, the proofs of which are given in §6.

**PROPOSITION 3.** *Let  $N_1, \dots, N_m$  be normal subgroups of  $G$  such that  $H_k(G/N_i N_j) = 0$  for all  $k = 1, 2, \dots, n+m-1$  and  $i \neq j$ . Then there is a natural exact sequence with  $(3n-1)$  terms:*

$$\begin{aligned}
 & H_n(G) - H_n(G/N_1) \oplus \dots \oplus H_n(G/N_m) - H_{n-1}(G:N_1, \dots, N_m) - \\
 & H_{n-1}(G) - H_{n-1}(G/N_1) \oplus \dots \oplus H_{n-1}(G/N_m) - H_{n-2}(G:N_1, \dots, N_m) - \\
 & \dots - H_1(G/N_1) \oplus \dots \oplus H_1(G/N_m) - 0. \quad \blacksquare
 \end{aligned}$$

This proposition together with Theorems 1 and 2 generalises the eight term exact sequence of [3] Theorem 4.5 in two respects: firstly our result is in terms of  $m$  normal subgroups of  $G$  and not just two; secondly for the case of two normal subgroups  $N_1$  and  $N_2$  the requirement in [3] that  $G = N_1 N_2$  can be weakened to a requirement that  $H_i(G/N_1 N_2) = 0$  for  $i = 1, 2, 3$ . Note that if  $G = N_1 N_2$  then our group  $\hat{\Delta}(G:N_1 N_2)$  coincides with the group  $N_1 \hat{\Delta} N_2$  in [3].

Proposition 3 together with Theorems 1 and 2 also generalises the five term exact sequence of [13] Theorem 2 in two respects: firstly it extends the five term sequence to eight terms; secondly the requirement in [13] that  $G = N_k (\bigcap_{j \neq k} N_j)$  for all  $k$  can be very much weakened to a requirement that

$$H_j(G/N_k N_j) = 0 \text{ for } i = 1, 2, 3, \text{ and all } k \neq j.$$

As further instances of how Theorems 1 and 2 can be used to give information about the ordinary integral homology of a group we cite two more theorems, the first of which has already appeared in [2]. Again we defer the proofs to §6.

**THEOREM 4** [2]. *Let  $R_1, \dots, R_n$  be normal subgroups of a group  $F$  such that*

$$H_2(F) = 0, \quad H_r(F/\prod_{i \in \alpha} R_i) = 0 \text{ for } r = |\alpha| + 1, \quad r = |\alpha| + 2$$

*with  $\alpha$  a non-empty proper subset of  $\langle n \rangle$  (for example these  $F/\prod_{i \in \alpha} R_i$  could be free) and  $F/\prod_{i \in \langle n \rangle} R_i \cong G$ . Then there is an isomorphism*

$$H_{n+1} \cong \{ \bigcap_{i \in \langle n \rangle} R_i \cap [F, F] \} / \{ \prod_{\alpha \subset \langle n \rangle} [ \bigcap_{i \in \alpha} R_i, \bigcap_{i \notin \alpha} R_i ] \}. \quad \blacksquare$$

**THEOREM 5.** *Let  $R_1, \dots, R_n$  be normal subgroups of a group  $F$*

such that

$$H_r(F/\prod_{i \in \alpha} R_i) = 0 \text{ for } r = |\alpha| + 2, r = |\alpha| + 3$$

with  $\alpha \neq \langle n \rangle$  a non-empty proper subset of  $\langle n \rangle$  and  $F/\prod_{i \in \langle n \rangle} R_i \approx G$ . Then there is an isomorphism

$$H_{n+2}(G) \approx \text{Ker}(\wedge(F: R_1, \dots, R_n) \rightarrow F). \quad \blacksquare$$

For  $n = 1$  Theorem 4 is the well-known Hopf formula for the second integral homology of a group, and Theorem 5 is the description of the third integral homology given in [3] Corollary 4.7. For  $n = 2$  Theorem 5 is new, and the isomorphisms of Theorems 4 and 5 read

$$H_3(G) \approx \{R_1 \cap R_2 \cap [F, F]\} / \{[R_1, R_2][F, R_1 \cap R_2]\},$$

$$H_4(G) \approx \text{Ker}(\wedge(F: R_1, R_2) - F).$$

As was pointed out in [2], for any group  $G$  an  $F$  and  $R_i$  can always be found to satisfy the hypothesis of Theorem 4 (or 5). One method is analogous to methods of [12, 14] and is best illustrated for  $n = 2$ . Choose any surjection  $F_i \rightarrow G$  with  $F_i$  free,  $i = 1, 2$ . Let  $P$  be the pullback of these surjections and choose a surjection  $F \rightarrow P$  with  $F$  free. Let  $R_i$  be the kernel of the composite map  $F \rightarrow P \rightarrow F_i$ . In general one constructs inductively an  $n$ -cube of groups  $F$  such that, for  $\alpha \subset \langle n \rangle$ :

- (i)  $F_{\langle n \rangle}$  is  $G$ , and
- (ii) the morphism  $F_\alpha \rightarrow \lim_{\beta \supset \alpha} F_\beta$  is surjective.

The organization of this article is as follows. In §1 and §2 we recall basic definitions and facts on simplicial resolutions and derived functors. In §3 we prove some abstract technical results on relative derived functors. Theorems 1 and 2 are proved in §4 and §5 respectively. In §6 we prove Proposition 3 and Theorems 4 and 5.

## 1. DERIVED FUNCTORS.

In this section we recall from [1, 8] facts on cotriples and derived functors. A useful reference for the details of simplicial objects is [4].

Let  $\mathcal{C}$  be an arbitrary category and  $\mathbf{E} = (E, \varepsilon, \delta)$  a cotriple on  $\mathcal{C}$ . That is  $E: \mathcal{C} \rightarrow \mathcal{C}$  is an endofunctor and  $\varepsilon: E \rightarrow 1_{\mathcal{C}}$ ,  $\delta: E \rightarrow E^2$  are natural transformations such that

$$(E\varepsilon)\delta = \text{id. } (\varepsilon E)\delta = \text{id and } (E\delta)\delta = (\delta E)\delta.$$

**EXAMPLE 1.** Let  $U: Gp \rightarrow Set$  be the functor from groups to sets which takes a group to its underlying set, and let  $F: Set \rightarrow Gp$  be the functor which takes a set to the free group generated by the set. Then  $E = FU: Gp \rightarrow Gp$  is an endofunctor, and the obvious natural transformations provide us with a cotriple  $\mathbf{E} = (E, \varepsilon, \delta)$ .

To each object  $C$  of  $\mathbf{C}$  one can associate a simplicial object  $E(C)_\#$  in  $\mathbf{C}$  by setting

$$E(C)_n = E^{n+1}C. \quad \varepsilon_i = E^i \varepsilon E^{n-i}: E(C)_n \rightarrow E(C)_{n-1}.$$

$$\delta_i = E^i \delta E^{n-i}: E(C)_n \rightarrow E(C)_{n+1}, \text{ for } 0 \leq i \leq n.$$

Now let  $T: \mathbf{C} \rightarrow Gp$  be a functor from  $\mathbf{C}$  to groups. By applying  $T$  dimensionwise to the simplicial object  $E(C)_\#$  we obtain a simplicial group  $TE(C)_\#$ . The homotopy groups of this simplicial group are the *derived functors of  $T$  with respect to the cotriple  $\mathbf{E}$* , and we write

$$L_n^T(C) = \pi_n(TE(C)_\#).$$

The homotopy groups of  $TE(C)_\#$  are isomorphic to the homology groups of the associated *Moore complex* (cf. [4])

$$\dots \rightarrow M_n \xrightarrow{d_n} M_{n-1} \rightarrow \dots \xrightarrow{d_1} M_0 \xrightarrow{d_0} 0$$

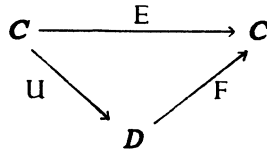
where  $M_0 = TE(C)_0$ ,  $M_n = \bigcap_{0 \leq i < n} \text{Ker}(\varepsilon_i: TE(C)_n \rightarrow TE(C)_{n-1})$  for  $n \geq 1$  and  $d_n$  is the restriction of  $\varepsilon_n$ . Thus

$$L_n^T(C) = \text{Ker } d_n / \text{Im } d_{n+1}, \quad n \geq 0.$$

**EXAMPLE 2.** Let  $\mathbf{E}$  be the cotriple described in Example 1, and let  $T = (-)^{ab}: Gp \rightarrow Gp$  be the functor which takes a group  $G$  to its abelianization  $G^{ab} = G/[G, G]$ . It is shown in [1] that  $L_n^T(G) = H_{n+1}(G)$  for  $n \geq 0$ .

## 2. HYPER-RELATIVE DERIVED FUNCTORS.

Throughout this and the next section we shall suppose that  $\mathbf{E} = (E, \varepsilon, \delta)$  is a cotriple on a category  $\mathbf{C}$  such that  $E: \mathbf{C} \rightarrow \mathbf{C}$  factors through a pair of functors



with  $U$  a right adjoint to  $F$ , with  $\varepsilon: FU \rightarrow 1_C$  the counit of the adjunction, with  $\nu: 1_D \rightarrow UF$  the unit of the adjunction and  $\delta = F\nu U$  (cf. [5]).

We shall say that a morphism  $f: C \rightarrow A$  in  $\mathcal{C}$  is a *fibration* if there exists a morphism  $\lambda: UA \rightarrow UC$  in  $\mathcal{D}$  such that  $(Uf)\lambda = 1_{UA}$ . It is readily seen that a fibration  $f: C \rightarrow A$  induces a dimensionwise surjective homomorphism of simplicial groups  $f_\# : TE(C)_\# \rightarrow TE(A)_\#$ . We denote the homotopy groups of the kernel of this simplicial map by

$$L_n^{RT}(f: C \rightarrow A) = \pi_n(\text{Ker } f_\#) \quad , \quad n \geq 0.$$

Since a surjection of simplicial groups yields a long exact sequence of homotopy groups (cf. [4]) we have immediately

**PROPOSITION 6.** *A fibration  $f: C \rightarrow A$  in  $\mathcal{C}$  yields a natural long exact sequence*

$$\cdots \rightarrow L_{n+1}^T(A) \rightarrow L_n^{RT}(C \rightarrow A) \rightarrow L_n^T(C) \rightarrow L_n^T(A) \rightarrow \cdots \rightarrow L_0^T(A) \rightarrow 0. \quad \blacksquare$$

Now the functors  $L_n^{RT}$  can be regarded as derived functors. To see this let  $\mathcal{RC}$  denote the category whose objects are the fibrations  $C \rightarrow A$  in  $\mathcal{C}$  and whose morphisms are commutative squares in  $\mathcal{C}$ :

$$\begin{array}{ccc} C & \xrightarrow{\quad} & A \\ \downarrow & & \downarrow \\ C' & \xrightarrow{\quad} & A' \end{array}$$

The cotriple  $\mathbf{E} = (E, \varepsilon, \delta)$  on  $\mathcal{C}$  extends in an obvious way to a cotriple  $\mathbf{RE} = (RE, \varepsilon, \delta)$  on  $\mathcal{RC}$ : on objects  $RE$  is defined by

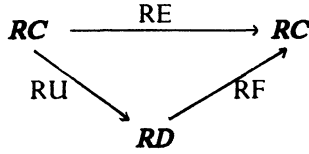
$$RE(C \rightarrow A) = EC \rightarrow EA.$$

Let  $RT: \mathcal{RC} \rightarrow \mathcal{Gp}$  be the functor which maps a fibration  $C \rightarrow A$  to the kernel of the induced group homomorphism  $TC \rightarrow TA$ , that is

$$RT(C \rightarrow A) = \text{Ker}(TC \rightarrow TA).$$

Clearly the functors  $L_n^{RT}$  defined above are the derived functors of the functor  $RT$  with respect to the cotriple  $\mathbf{RE}$  on  $\mathcal{RC}$ .

Note that the endofunctor  $RE$  factors through a pair of adjoint functors



where  $\mathbf{RD}$  is the category whose objects are the morphisms of  $\mathbf{D}$  which possess at least one splitting, and whose morphisms are commutative squares in  $\mathbf{D}$ : the adjoint pair is the obvious one. Thus we can define inductively, for  $m \geq 0$ ,

$$\mathbf{R}^{n+1}\mathbf{C} = \mathbf{R}(\mathbf{R}^n\mathbf{C}), \mathbf{R}^0\mathbf{C} = \mathbf{C}, \mathbf{R}^{m+1}\mathbf{T} = \mathbf{R}(\mathbf{R}^m\mathbf{T}), \mathbf{R}^0\mathbf{T} = \mathbf{T}.$$

Therefore we have derived functors  $L_n^{\mathbf{R}^m\mathbf{T}}: \mathbf{R}^m\mathbf{C} \rightarrow \mathbf{Gp}$ ,  $m, n \geq 0$ . We call these functors the *hyper-relative derived functors* of  $\mathbf{T}$ .

**EXAMPLE 3.** Let  $G$  be a group with normal subgroups  $N_1, \dots, N_m$ . This data gives rise to an  $m$ -cubical diagram consisting of groups  $G_\alpha = G/\prod_{i \in \alpha} N_i$  for each  $\alpha \subseteq \langle m \rangle$ , and of quotient homomorphisms  $G_\alpha \rightarrow G_{\alpha \cup \langle i \rangle}$  for each  $i \in \alpha$ . We denote this  $m$ -cube by  $\{G_\alpha\}$ . Thus for example if  $m=2$  we have

$$\{G_\alpha\} = \begin{array}{ccc}
 G & \xrightarrow{\quad} & G/N_1 \\
 \downarrow & & \downarrow \\
 G/N_2 & \xrightarrow{\quad} & G/N_1N_2
 \end{array}$$

It is clear that in general  $\{G_\alpha\}$  is an object in the category  $\mathbf{R}^m\mathbf{Gp}$ .

Letting  $\mathbf{T} = (-)^{\text{ab}}$  as in Example 2, we define

$$H_{n+1}(G; N_1, \dots, N_m) = L_n^{\mathbf{R}^m(-)^{\text{ab}}} \{G_\alpha\}.$$

It follows immediately from Proposition 6 that these hyper-relative homology groups fit together in exact sequences as described in the Introduction.

**3. SOME TECHNICAL RESULTS;**

Extending a definition of [9] we shall say that a diagram

$$(*) \quad \mathbf{C}'' \begin{array}{c} \xrightarrow{q_1} \\ \xrightarrow{q_2} \end{array} \mathbf{C}' \xrightarrow{p} \mathbf{C}$$

in  $\mathbf{C}$  satisfying  $q_1 p = q_2 p$  is *split over  $\mathbf{D}$*  if there exist mor-



phisms  $\lambda: UC \rightarrow UC'$  and  $\mu: UC' \rightarrow UC''$  in  $\mathcal{D}$  such that

$$(U\rho)\lambda = 1_{UC}, (Uq_1)\mu = 1_{UC'} \text{ and } (Uq_2)\mu = \lambda(U\rho).$$

We shall say that a functor  $W: \mathcal{C} \rightarrow Gp$  is *right exact* if it maps any split diagram (\*) to a diagram in  $Gp$  which is split over *Set* with set maps  $\lambda: WC \rightarrow WC'$ ,  $\mu: WC' \rightarrow WC''$  which preserve the group identity elements.

**LEMMA 7.** For any functor  $T: \mathcal{C} \rightarrow Gp$  the zeroth derived functor  $L_0^T: \mathcal{C} \rightarrow Gp$  is *right exact*.

**PROOF.** A split diagram (\*) gives rise to a diagram of group homomorphisms

$$TFUC'' \begin{array}{c} \longleftarrow \\ \xrightarrow{\quad} \\ \xrightarrow{\quad} \end{array} TFUC' \begin{array}{c} \longleftarrow \\ \xrightarrow{\quad} \end{array} TFUC.$$

Since  $L_0^T(C''')$  is a quotient of  $TFUC'''$  for  $C''' = C.C'.C''$  it follows that the diagram induced by the arrows going from left to right

$$L_0^T(C'') \begin{array}{c} \xrightarrow{L_0^T(q_1)} \\ \xrightarrow{L_0^T(q_2)} \end{array} L_0^T(C') \xrightarrow{L_0^T(\rho)} L_0^T(C)$$

is split over *Set*, and that the splittings can be chosen so that they preserve the group identity elements.

**LEMMA 8.** If

$$G'' \begin{array}{c} \xrightarrow{q_1} \\ \xrightarrow{q_2} \end{array} G' \xrightarrow{p} G$$

is a diagram of groups such that  $q_1 p = q_2 p$  and which is split over *Set* with splittings which preserve identity elements, then  $q_1(\text{Ker } q_2)$  is a normal subgroup of  $G'$  and  $G$  is isomorphic to  $G'/q_1(\text{Ker } q_2)$ .

**PROOF.** The homomorphism  $q_1$  is surjective as it has a set theoretic splitting  $\mu$ . Thus the image of any normal subgroup of  $G''$  is normal in  $G'$ . Now if  $g$  is in  $\text{Ker } p$  then

$$q_1(\mu g) = g \text{ and } q_2(\mu g) = \lambda p(g) = 1.$$

Hence  $\text{Ker } p = q_1(\text{Ker } q_2)$ . Finally  $p$  is surjective as it has a set theoretic splitting  $\lambda$ . ■

Let us now suppose that the category  $\mathcal{C}$  contains pull-backs. Thus given a fibration  $C \rightarrow A$  in  $\mathcal{C}$  we can consider the

diagram

$$C \times_A C \times_A C \begin{array}{c} \xrightarrow{q_1} \\ \xrightarrow{q_2} \\ \xrightarrow{q_3} \end{array} C \times_A C \xrightarrow[p_2]{p_1} C$$

where  $p_i$  and  $q_i$  "forget the  $i^{\text{th}}$  copy of  $C$ ". (Thus for example  $p_1$  denotes the projection onto the second copy of  $C$ .) On applying the functor  $L_0^T$  we obtain a diagram

$$L_0^T(C \times_A C \times_A C) \begin{array}{c} \xrightarrow{\bar{q}_1} \\ \xrightarrow{\bar{q}_2} \\ \xrightarrow{\bar{q}_3} \end{array} L_0^T(C \times_A C) \xrightarrow[\bar{p}_2]{\bar{p}_1} L_0^T(C)$$

where we write  $\bar{p}_i$  and  $\bar{q}_i$  instead of  $L_0^T(p_i)$  and  $L_0^T(q_i)$ . With this notation we have

**LEMMA 9.** For any fibration  $f: C \rightarrow A$  there is an isomorphism

$$L_0^{\text{RT}}(f: C \rightarrow A) \approx \{ \text{Ker } \bar{p}_2 \} / \{ \bar{q}_1(\text{Ker } \bar{q}_2 \cap \text{Ker } \bar{q}_3) \}.$$

**PROOF.** We can consider

$$\begin{array}{ccccc} C \times_A C \times_A C & \begin{array}{c} \xrightarrow{q_1} \\ \xrightarrow{q_2} \end{array} & C \times_A C & \xrightarrow{p_1} & C \\ \downarrow & & \downarrow & & \downarrow \\ C \times_A C & \begin{array}{c} \xrightarrow{p_1} \\ \xrightarrow{p_2} \end{array} & C & \xrightarrow{f} & A \end{array}$$

as a diagram in  $\mathbf{RC}$  with the vertical maps the objects. As such the diagram is split over  $\mathbf{RD}$ . To see this note that a map  $\lambda: \mathbf{UA} \rightarrow \mathbf{UC}$  exists by virtue of the fact that  $f$  is a fibration: the other maps of the splitting are

$$(\lambda \cup f, 1): \mathbf{UC} \rightarrow \mathbf{U}(C \times_A C) \text{ and } (\lambda \cup f, 1) \times 1: \mathbf{U}(C \times_A C) \rightarrow \mathbf{U}(C \times_A C \times_A C).$$

Since  $\bar{q}_3$  and  $\bar{p}_2$  are split homomorphisms we have

$$L_0^{\text{RT}}(\bar{q}_3) \approx \text{Ker } \bar{q}_3 \text{ and } L_0^{\text{RT}}(\bar{p}_2) \approx \text{Ker } \bar{p}_2.$$

Hence the proof is completed by applying Lemmas 7 and 8 to the following diagram of groups:

$$\text{Ker} \begin{bmatrix} L_0^T(C \times_A C \times_A C) \\ \bar{q}_3 \downarrow \\ L_0^T(C \times_A C) \end{bmatrix} \begin{array}{c} \xrightarrow{\bar{q}_1} \\ \xrightarrow{\bar{q}_2} \end{array} \text{Ker} \begin{bmatrix} L_0^T(C \times_A C) \\ \bar{p}_2 \downarrow \\ L_0^T(C) \end{bmatrix} \xrightarrow{\bar{p}_1} L_0^{\text{RT}}(C \rightarrow A).$$

4. PROOF OF THEOREM 4.

If  $N$  is a normal subgroup of a group  $G$  and if  $Q = G/N$ , then the diagram

$$G \times_Q G \times_Q G \begin{array}{c} \xrightarrow{\quad} \\ \xrightarrow{\quad} \\ \xrightarrow{\quad} \end{array} G \times_Q G \begin{array}{c} \xrightarrow{\quad} \\ \xrightarrow{\quad} \\ \xrightarrow{\quad} \end{array} G$$

is isomorphic to the diagram

$$N \hat{\curvearrowright} (N \hat{\curvearrowright} G) \begin{array}{c} \xrightarrow{\bar{q}_1} \\ \xrightarrow{\bar{q}_2} \\ \xrightarrow{\bar{q}_3} \end{array} N \hat{\curvearrowright} G \begin{array}{c} \xrightarrow{\bar{p}_1} \\ \xrightarrow{\bar{p}_2} \end{array} G$$

where  $\hat{\curvearrowright}$  denotes a semi-direct product: the action of  $G$  on  $N$  is  $gn = gng^{-1}$ ; the action of  $N \hat{\curvearrowright} G$  on  $N$  is  $(n.g)n' = gn'n^{-1}g^{-1}$ ; the homomorphisms are

$$p_1(n.g) = ng, \quad p_2(n.g) = g, \quad q_1(n'.n.g) = (n'n^{-1}.ng), \\ q_2(n'.n.g) = (n'.g) \text{ and } q_3(n'.n.g) = (n.g).$$

Now the induced maps

$$\bar{q}_i: \{N \hat{\curvearrowright} N \hat{\curvearrowright} G\}^{ab} \rightarrow \{N \hat{\curvearrowright} G\}^{ab}$$

clearly satisfy  $\text{Ker } \bar{q}_2 \cap \text{Ker } \bar{q}_3 = 1$ . Thus from Examples 2 and 3 and Lemma 9 we get an isomorphism

$$H_1(G;N) \approx \text{Ker } \{\bar{p}_2: (N \hat{\curvearrowright} G)/[N \hat{\curvearrowright} G, N \hat{\curvearrowright} G] \rightarrow G/[G,G]\}.$$

It is thus readily seen that  $H_1(G;N) \approx N/[G,N]$  and this proves Theorem 1 for  $m=1$ .

More generally suppose that  $N_1, \dots, N_m$  are normal subgroups of a group  $G$  ( $m \geq 2$ ). For  $\alpha \in \langle m-1 \rangle$  let

$$G_\alpha = G/(\prod_{i \in \alpha} N_i), \quad \bar{G}_\alpha = G/(\prod_{i \in \alpha} N_m \prod_{j \in \alpha} N_j).$$

Then the following pullback diagram in  $\mathbf{R}^{m-1}Gp$

$$\{G_\alpha\} \times \langle \bar{G}_\alpha \rangle \{G_\alpha\} \times \langle \bar{G}_\alpha \rangle \{G_\alpha\} \begin{array}{c} \xrightarrow{\quad} \\ \xrightarrow{\quad} \\ \xrightarrow{\quad} \end{array} \{G_\alpha\} \times \langle \bar{G}_\alpha \rangle \{G_\alpha\} \begin{array}{c} \xrightarrow{\quad} \\ \xrightarrow{\quad} \\ \xrightarrow{\quad} \end{array} \{G_\alpha\}$$

is isomorphic to the diagram

$$\{(N_\alpha \cap G_\alpha) \hat{\curvearrowright} (N_\alpha \hat{\curvearrowright} G) \hat{\curvearrowright} G_\alpha\} \begin{array}{c} \xrightarrow{\bar{q}_1} \\ \xrightarrow{\bar{q}_2} \\ \xrightarrow{\bar{q}_3} \end{array} \{(N_\alpha \cap G_\alpha) \hat{\curvearrowright} G_\alpha\} \begin{array}{c} \xrightarrow{\bar{p}_1} \\ \xrightarrow{\bar{p}_2} \end{array} \{G_\alpha\}.$$

Let us suppose that Theorem 1 has been proved for  $m-1$  normal subgroups of  $G$ . Clearly  $\text{Ker } \bar{q}_2 \cap \text{Ker } \bar{q}_3 = 1$ . So Lemma 9 gives us an isomorphism

$$\begin{aligned}
 H_1(G; N_1, \dots, N_m) &\approx \text{Ker}(\bar{p}_2: \\
 H_1(N_m \hat{\times} G; (N_m \cap N_1) \hat{\times} N_1, \dots, (N_m \cap N_{m-1}) \times N_m) &\rightarrow H_1(G; N_1, \dots, N_m)) \\
 \text{and this kernel is isomorphic to the kernel of} \\
 (\bigcap_{i \in \langle m-1 \rangle} (N_m \cap N_i) \hat{\times} N_i) / \prod_{\alpha \in \langle m-1 \rangle} [\bigcap_{i \in \alpha} (N_m \cap N_i) \hat{\times} N_i, \bigcap_{i \notin \alpha} (N_m \cap N_i) \hat{\times} N_i] \\
 \rightarrow \bigcap_{i \in \langle m-1 \rangle} N_i / \prod_{\alpha \in \langle m-1 \rangle} [\bigcap_{i \in \alpha} N_i, \bigcap_{i \notin \alpha} N_i].
 \end{aligned}$$

Theorem 1 follows by induction. ■

**5. PROOF OF THEOREM 2.**

Let  $G$  be a group and let  $E(G)_\#$  be the simplicial group obtained from the cotriple of Example 1. Applying the functors  $(-)^{ab}$  and  $[.,.]$ :  $Gp \rightarrow Gp$  dimensionwise to  $E(G)_\#$  yields a short exact sequence of simplicial groups

$$[E(G)_\#, E(G)_\#] \rightarrow E(G)_\# \rightarrow E(G)^{ab}.$$

Since  $\pi_0(E(G)_\#) \approx G$ ,  $\pi_n(E(G)_\#) = 0$  for  $n \geq 1$ , and

$$\pi_n(E(G)_\#^{ab}) \approx H_{n+1}(G) \text{ for } n \geq 0.$$

the resulting long exact homotopy sequence provides us with isomorphisms

$$H_2(G) \approx \text{Ker}(L_0^{[.,.]}(G) \rightarrow G), H_{n+2}(G) \approx L_n^{[.,.]}(G), n \geq 1.$$

Recall from the Introduction the definition of the group  $\wedge(G; N_1, \dots, N_m)$  where  $N_i$  are normal subgroups of  $G$ . The special case  $\wedge(G; G)$  coincides with the group denoted in [3] by  $G \wedge G$ . It is known from [3] (see [6] for an algebraic proof) that for a free group  $F$  there is an isomorphism

$$\wedge(F; F) \approx [F, F], x \wedge y \mapsto [x, y].$$

Thus if we let  $\wedge: Gp \rightarrow Gp$  be the functor which takes a group  $G$  to the group  $\wedge(G; G)$ , then we have an isomorphism  $L_n^\wedge(G) \approx L_n^{[.,.]}(G)$  for  $n \geq 0$ . So in particular

$$H_2(G) \approx \text{Ker}(L_0^\wedge(G) \rightarrow G).$$

If  $\{G_\alpha\}$  is the  $m$ -cubical diagram of Example 3 arising from the subgroups  $N_1, \dots, N_m$  then we find that

$$H_2(G; N_1, \dots, N_m) \approx \text{Ker}(L_0^{R^m \wedge \{G_\alpha\}} \rightarrow G).$$

Thus to prove Theorem 2 we need to show that

$$L_0^{R^m \wedge \{G_\alpha\}} \approx \wedge(G; N_1, \dots, N_m).$$

This isomorphism follows from an inductive use of Lemma 9 and

Proposition 11 below, together with the fact (proved in Proposition 3 of [7]) that  $L_0^\wedge(G) \approx \wedge(G;G)$ . The proof of Proposition 11 uses the following lemma.

**LEMMA 10.** For any normal subgroups  $N_1, \dots, N_m$  of  $G$  ( $m \geq 2$ ) the canonical sequence of groups

$$\begin{aligned} \wedge(G; N_1, \dots, N_m) &\xrightarrow{\iota} \wedge(G; N_1, \dots, N_{m-1}) \xrightarrow{\pi} \\ \wedge(G/N_m; N_1 N_m/N_m, \dots, N_{m-1} N_m/N_m) &\rightarrow 0. \end{aligned}$$

is exact.

**PROOF.** Clearly  $\pi: x \wedge y \mapsto x N_m \wedge y N_m$  is surjective. The image of the homomorphism  $\iota: x \wedge y \mapsto y$  clearly lies in the kernel of  $\pi$ , and moreover  $\text{Im } \iota$  is normal in  $\wedge(G; N_1, \dots, N_{m-1})$ . Finally it is readily verified (cf. Proposition 1 in [6]) that there is a homomorphism

$$\begin{aligned} \psi: \wedge(G/N_m; N_1 N_m/N_m, \dots, N_{m-1} N_m/N_m) &\rightarrow \wedge(G; N_1, \dots, N_{m-1}) / \text{Im } \iota \\ x N_m \wedge y N_m \mapsto \{x \wedge y\} \text{Im } \iota \end{aligned}$$

and that  $\pi$  induces an inverse to  $\psi$ . ■

Now the homomorphisms  $q_i$  and  $p_i$  of §4 induce homomorphisms

$$\begin{aligned} \bar{q}_i: \wedge(N_m \hat{\times} N_m \hat{\times} G; \\ \{N_1 \cap N_m\} \hat{\times} \{N_1 \cap N_m\} \hat{\times} N_1, \dots, \{N_{m-1} \cap N_m\} \hat{\times} \{N_{m-1} \cap N_m\} \hat{\times} N_{m-1}) \\ \rightarrow \wedge(N_m \hat{\times} G; \{N_1 \cap N_m\} \hat{\times} N_1, \dots, \{N_{m-1} \cap N_m\} \hat{\times} N_{m-1}), \\ \bar{p}_i: \wedge(N_m \hat{\times} G; \{N_1 \cap N_m\} \hat{\times} N_1, \dots, N_{m-1} \cap N_m \hat{\times} N_{m-1}) \\ \rightarrow \wedge(G; N_1, \dots, N_{m-1}). \end{aligned}$$

**PROPOSITION 11.** With the above notation and  $m \geq 2$ , there is an isomorphism

$$\wedge(G; N_1, \dots, N_m) \xrightarrow{\cong} \text{Ker } \bar{p}_2 / \bar{q}_1 (\text{Ker } \bar{q}_2 \cap \text{Ker } \bar{q}_3).$$

**PROOF.** From Lemma 10 we see that  $\text{Ker } \bar{q}_2$  is generated by the elements of the form  $(n, 1, 1) \wedge (n', n'', x)$ . Also  $\text{Ker } \bar{q}_3$  is generated by the elements of the form  $(1, n, 1) \wedge (n', n'', x)$ . Hence  $\text{Ker } \bar{q}_2 \cap \text{Ker } \bar{q}_3$  is generated by the elements of the form  $(n, 1, 1) \wedge (1, m, 1)$ . It follows that  $\bar{q}_1(\text{Ker } \bar{q}_2 \cap \text{Ker } \bar{q}_3)$  is generated by the elements of the form  $(n, 1) \wedge (m^{-1}, m)$ . Now  $\text{Ker } \bar{p}_2$  is generated by elements of the form  $(n, 1) \wedge (m, x)$ , and there is a surjective homomorphism

$$\varphi: \text{Ker } \bar{\rho}_2 \longrightarrow \wedge(G; N_1, \dots, N_m), (n, 1) \wedge (m, g) \mapsto n \wedge m g.$$

By Lemma 10 we have  $\text{Ker } \varphi = \bar{q}_1(\text{Ker } \bar{q}_2 \cap \text{Ker } \bar{q}_3)$ .     ■

**6. REMAINING PROOFS.**

Proposition 3 is certainly true for  $m = 1$ . Suppose then that for some  $m \geq 2$  the proposition is true for any suitable collection of  $m-1$  subgroups. Let  $N_1, \dots, N_m$  be subgroups of  $G$  satisfying the hypothesis of the proposition, and consider the following commutative diagram in which  $k \leq n$  and the rows and columns are exact:

$$\begin{array}{ccc} 0 & = & \bigoplus_{1 \leq i < m} H_{k+1}(G/N_m N_i) \\ & & \downarrow \\ H_k(G; N_1, \dots, N_m) & \longrightarrow & H_k(G/N_m; N_1 N_m/N_m, \dots, N_{m-1} N_m/N_m) \\ & & \downarrow \approx \\ & & H_k(G/N_m) \\ & & \downarrow \\ 0 & = & \bigoplus_{1 \leq i < m} H_k(G/N_m N_i) \end{array} \quad \begin{array}{l} \\ \\ \longrightarrow H_{k-1}(G; N_1, \dots, N_m) \\ \\ \\ \end{array}$$

We thus obtain an isomorphism

$$\varphi_m: H_k(G/N_m) \longrightarrow H_{k-1}(G; N_1, \dots, N_m).$$

Similarly we obtain homomorphisms

$$\varphi_i: H_k(G/N_i) \longrightarrow H_{k-1}(G; N_1, \dots, N_m)$$

which we can combine to get a homomorphism

$$\begin{aligned} \varphi: H_k(G/N_1) \oplus \dots \oplus H_k(G/N_m) &\longrightarrow H_{k-1}(G; N_1, \dots, N_m), \\ (x_1, \dots, x_m) &\mapsto \varphi_1(x_1) \dots \varphi_m(x_m). \end{aligned}$$

The canonical homomorphisms  $\psi_j: H_k(G) \rightarrow H_k(G/N_j)$  combine to give a homomorphism

$$(\psi_1, \dots, \psi_m): H_k(G) \longrightarrow H_k(G/N_1) \oplus \dots \oplus H_k(G/N_m).$$

Finally, using our inductive hypothesis, there is a composite map

$$H_k(G:N_1, \dots, N_m) \rightarrow H_k(G:N_1, \dots, N_{m-1}) \rightarrow H_k(G).$$

Thus the sequence of Proposition 3 exists for  $m$  subgroups. The verification of its exactness is a routine exercise.

Theorems 4 and 5 follow from Theorems 1 and 2 together with the following lemmas.

**LEMMA 12.** *Let  $R_1, \dots, R_n$  be normal subgroups of a group  $F$  such that the hypothesis of Theorem 4 is satisfied. Then there is an exact sequence*

$$0 \rightarrow H_{n+1}(G) \rightarrow H_1(F; R_1, \dots, R_n) \rightarrow H_1(F). \quad \blacksquare$$

**LEMMA 13.** *Let  $R_1, \dots, R_n$  be normal subgroups of a group  $F$  such that the hypothesis of Theorem 5 is satisfied. Then there is an exact sequence*

$$0 \rightarrow H_{n+2}(G) \rightarrow H_1(F; R_1, \dots, R_n) \rightarrow H_2(F) = 0. \quad \blacksquare$$

Lemma 12 is precisely Proposition 5 of [2], and Lemma 13 is proved in a similar fashion.

REFERENCES.

1. M. BARR & J. BECK, Homology and standard constructions, *Lecture Notes in Math.* 80, Springer (1969), 245-335.
2. R. BROWN & G.J. ELLIS, Hopf formulæ for the higher homology of a group, *Bull. London Math. Soc.* 20 (1988), 124-128.
3. R. BROWN & J.-L. LODAY, Van Kampen Theorems for diagrams of spaces, *Topology* 26 (3) (1987), 311-335.
4. E.B. CURTIS, Simplicial homotopy theory, *Advances in Math.* 6 (1971), 107-204.
5. Introduction to *Seminar on triples and categorical homology theory*, Lecture Notes in Math. 80. Springer (1969).
6. G.J. ELLIS. Non-abelian exterior products of groups and exact sequences in the homology of groups, *Glasgow Math. J.* 29 (1987), 13-19.
7. G.J. ELLIS. An algebraic derivation of a certain exact sequence, *Galway Math. Preprint* 88/1 (1988).
8. F. KEUNE, Homotopical algebra and algebraic K-theory, Thesis. Amsterdam 1972.
9. F. KEUNE. The relativisation of  $K_2$ , *J. Algebra* 54 (1978), 159-177.
10. J.-L. LODAY, Spaces with finitely many homotopy groups, *J. Pure & Appl. Algebra* 24 (1982), 179-202.
11. C. MILLER. The second homology of a group. *Proc. A.M.S.* 3 (1952), 588-595.
12. G.S. RINEHART, Satellites and cohomology, *J. Algebra* 12 (1969), 295-332; Errata. *ibid.* 14 (1970), 125-127.
13. A.G. RODICIO. A homological exact sequence associated with a family of normal subgroups, *Cahiers Top. & Géom. Diff. Cat.* to appear.
14. R.J. STEINER. Resolutions of spaces by cubes of fibrations, *J. London Math. Soc.* (2) 34 (1986), 169-176.

DEPARTMENT OF MATHEMATICS  
 UNIVERSITY COLLEGE GALWAY.  
 IRELAND