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LANGUAGES FOR TRIPLES, BICATEGORIES AND BRAIDED MONOIDAL CATEGORIES

by C. Barry JAY

RÉSUMÉ. Les monades, les catégories monoïdales (ordinaires, symétriques ou tressées), les catégories ainsi que les bicatégories constituent toutes, chacune dans un sens approprié, des modèles de la théorie d'un (pseudo-)monoïde. Cette approche est utilisée pour créer des langages qui simplifient les calculs, par exemple, dans la théorie des monades ou des catégories relatives.

0. INTRODUCTION.

Many different kinds of categorical construction have been regarded, either formally or informally, as monoids (perhaps up to isomorphism), e.g. triples and monoidal categories (plain, symmetric or braided [4]), or as 'monoids-with-many-objects', e.g. categories, 2-categories and bicategories. This viewpoint will be developed rigorously by exhibiting in each case a structured category which is the theory of a monoid, so that the objects of interest appear as models, i.e., structure-preserving functors. For example, a triple on a category $\boldsymbol{A}$ is a tensor-preserving functor from Δ (the simplicial category) into $\mathbf{End}(\boldsymbol{A})$ while a monoidal category is a tensor-preserving functor from a 2-dimensional theory into **Cat**.

Following [3], this model-theoretic approach is used to create languages for triples (§1), bicategories (§2) and braided monoidal categories (§3) which simplify many calculations with triples or in enriched category theory. Consider the following example suggested by R. Paré and L. Grünenfelder.

PROPOSITION 0.1. *Let $(\boldsymbol{V}, \otimes, \mathrm{I}, a, l, r, c)$ be a symmetric monoidal category [7] with a commutative monoid $(\mathrm{R}, \cdot, e)$ in $\boldsymbol{V}$, and let $\lambda : \mathrm{R}\otimes\mathrm{M}\to\mathrm{M}$ be a left action of R on M. Then R has a right action ρ on M which commutes with λ.* ∎

The step of interest is the proof of commutativity, which is given by an equality of morphisms in $\boldsymbol{V}$, i.e.

$$\rho(\lambda\otimes 1) = \lambda(1\otimes\rho)a. \tag{0.1}$$

The diagrammatic proof requires at least five cells ([3], Ex. 2.4). If V = **Set**, however, then there is a familiar three-line proof (plus definitions) using elements. Let $\lambda(r,m) = r\cdot m$ when $r \in R$ and $m \in M$. Then $\rho(m,s) = s\cdot m$ and so

$$(r\cdot m)\cdot s = s\cdot(r\cdot m) = (s\cdot r)\cdot m = (r\cdot s)\cdot m = r\cdot(s\cdot m) = r\cdot(m\cdot s).$$

This proof with elements can be re-interpreted in the language of V, as a proof of the equivalence of the terms $(r\cdot m)\cdot s$ and $r\cdot(m\cdot s)$, which then implies (0.1).

Slightly modified, this proof yields the same result for a braided monoidal category. That their languages share many features with those for symmetric categories is the result of two factors. First, by choosing the monoidal structure of the *theory* of a braided monoidal category to be itself braided, rather then symmetric, the theory is *coherent* just as in the symmetric case. Second, among the braids are some which behave like permutations, including those which appear in the axioms. By tagging the variables with *memories*, it is possible to manipulate these 'permutations' just as in the symmetric case.

I would like to thank R. Paré and R. Wood for many useful discussions about these ideas.

1. LANGUAGES FOR MONOIDS.

Strict monoidal categories.

Lawvere showed that an algebraic theory is a category with finite products and a model of it in a category $\boldsymbol{A}$ is a product-preserving functor. Mac Lane generalised this idea to allow the theory to be a category with a tensor product, called a *prop* and a model to be a tensor-preserving functor, which we take here to mean a strong monoidal functor, i.e. one that preserves the tensor up to coherent isomorphism. For example, there is both an algebraic theory and a prop to describe monoids. The prop is the simplicial category $(\Delta,\oplus,0)$,which is a skeleton of the category of finite sets and order-preserving maps [7] with sum as the tensor. Equivalently, Δ has as objects the natural numbers, with tensor given by addition, and has morphisms, called *operations*, generated by $\otimes : 2\to 1$ and $I : 0\to 1$ subject to associativity and unit laws. The algebraic theory is generated by the same data, but now contains the projections and other paraphernalia of a finite product category, which are unnecessary to describe monoids.

If $M : \Delta\to$ **Cat** is a model (where **Cat** has the cartesian tensor) then $\boldsymbol{V} = M(1)$ is a strict monoidal category. Conversely, given a strict monoidal category $\boldsymbol{V}$ then it determines a *standard model* M, i.e. satisfying $M(n) = \boldsymbol{V}^n$.

Let $\boldsymbol{V}$ be a monoidal category, which is fixed for the rest of the section. Define a typed language $\boldsymbol{L(V)} = \boldsymbol{L}$ whose types are the objects of all $\boldsymbol{V}^n$. To each object V of $\boldsymbol{V}$ is associated a countable set of variables $v^k \in V$. Given variables $v_i \in V_i$ for $1 \le i \le n$, there is a *sequence of variables* $(v_i) \in (V_i)$. The empty sequence is denoted $! \in 1$, where 1 is the unique object of the terminal category **1**. The morphisms of all $\boldsymbol{V}^n$'s are the *function symbols* of the language. A *term* s consists of

(i) a sequence of variables $(v \in \boldsymbol{V}^m)$,

(ii) an operation $F: m \to n$ in Δ,

and (iii) a function symbol $f: FV \to V'$.

It is denoted $s = f(Fv)$. If $f = 1$ then s is a *basic term*: if $v = !$ then it is a *constant*. Given terms

$$s = f(Fv) \in V' \text{ and } t = g(Gw) \in W',$$

an operation H and a function symbol h then, whenever the right-hand side is defined, the following constructions are given:

$$(s,t) = (f,g)((F \oplus G)(v,w)) \in (V', W'),$$
$$Hs = Hf(HFv),$$
$$h(s) = hf(Fv)$$

where Hf denotes $M(H)f$. Further, let $I(!) = *$ and $\otimes(x,y) = x \otimes y$. From equations in Δ it follows that

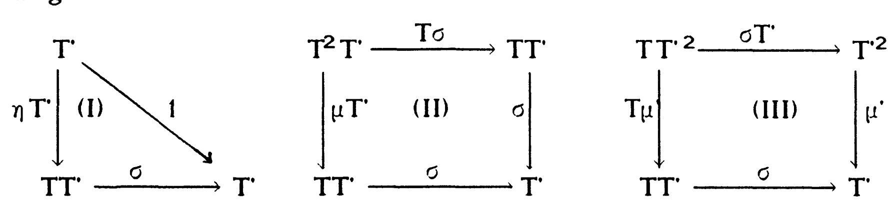
(1.1)
$$(x \otimes y) \otimes z = x \otimes (y \otimes z),$$
$$* \otimes x = x = x \otimes *$$

which can be used in calculations.

Triples.

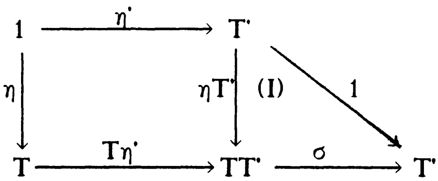
Consider the following proposition, found in [2].

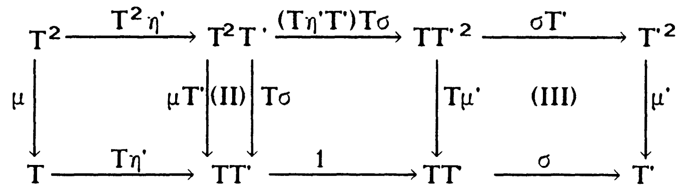
THEOREM 1.2. *Let* (T, η, μ) *and* (T', η', μ') *be triples on* $\boldsymbol{A}$ *and* σ: $TT' \to T'$ *be a natural transformation such that the following diagrams commute.*



Then $\alpha = \sigma \cdot T\eta' : T \to T'$ *is a morphism of triples.*

PROOF. The two diagrams are





■

This result and its proof become transparent on noticing that a triple on $\boldsymbol{A}$ is just a monoid in **End**($\boldsymbol{A}$), the strict monoidal category whose tensor is composition.

Let $(R,\cdot,1)$ and $(S,\cdot,1)$ be monoids in a strict monoidal category $\boldsymbol{V}$ and let $\sigma : R \otimes S \to S$. For variables $r, r' \in R$ and $s, s' \in S$ denote $\sigma(r \otimes s)$ by $r \cdot s$. Then σ is an R-action on S (compatible with its multiplication) if the following axioms hold:

$$1.s = s,$$
$$(r' \cdot r) \cdot s = r' \cdot (r \cdot s),$$
$$r \cdot (s \cdot s') = (r \cdot s) \cdot s'.$$

Now the theorem can be replaced by a more general result.

THEOREM 1.3. *Let* $\sigma : R \otimes S \to S$ *be an* R*-action. Then there is a monoid morphism* $\alpha : R \to S$ *given by* $\alpha(r) = r \cdot 1$.

PROOF. $\alpha(1) = 1 \cdot 1 = 1$ and

$$\alpha(r') \cdot \alpha(r) = (r' \cdot 1) \cdot (r \cdot 1) \underset{(III)}{=} r' \cdot (1 \cdot (r \cdot 1)) \underset{(IV)}{=} r' \cdot (r \cdot 1) \underset{(II)}{=} (r' \cdot r) \cdot 1 = \alpha(r' \cdot r)$$

■

The proof is, perhaps, familiar, since if $\boldsymbol{V} = \mathbf{Ab}$ (which is non-strict) then a monoid in $\boldsymbol{V}$ is just a ring and the corresponding result is that every R-algebra S induces a ring homomorphism $R \to S$.

The chief benefit of this proof is its familiarity and compactness. In other situations where there are many functoriality and naturality squares, instead of just two as here, the brevity of the variable proof is a significant benefit. Perhaps in this way some of the technical arguments of, say, projective resolutions may be simplified.

2-categories.

2-categories are usually described [5] as categories whose homs are categories, rather than just sets. We shall develop this viewpoint by looking at categories in a novel way.

A small category $\boldsymbol{C}$ consists of a graph $d_0, d_1: C_1 \to C_0$ in **Set**, together with identities $i: C_0 \to C_1$ and compositions $c: C_2 \to C_1$ satisfying some axioms, where C_2 is the object of composable pairs of morphisms, described by the pullback

$$\begin{array}{ccc} C_2 & \longrightarrow & C_1 \\ \big\downarrow & \text{p.b.} & \big\downarrow d_0 \\ C_1 & \xrightarrow{\;d_1\;} & C_0 \end{array}$$

Thus, a small category has been thought of as a 'monoid-with-many-objects' in **Set**. By passing from **Set** to another base, a category arises simply as a monoid.

Consider the bicategory **Span** of spans in **Set** with horizontal composition given by pullback. Let G be a span which is an endomorphism of some given object X, i.e. a graph on X. The full sub-bicategory of these is a monoidal category denoted $\mathbf{Graph}_X$. A monoid structure (with respect to span composition) on G is exactly the data required for a category. This yields a homomorphism of bicategories from Δ, as a one-object bicategory, to **Span**, or, more appropriate for our purposes, a strong monoidal functor $\Delta \to \mathbf{Graph}_X$.

Consider now a small 2-category $\boldsymbol{B}$. Let $B_0 = X$ be its set of objects and B_1 be its category of 1- and 2-cells. By regarding X as a discrete category, domain and codomain become functors $d_0, d_1: B_1 \to X$, respectively. Thus, underlying $\boldsymbol{B}$ is a graph in **Cat** whose category of objects is discrete. Let **SpanCat** denote the bicategory of spans in **Cat** and, given any category X, let $\mathbf{Graph}_X(\mathbf{Cat})$ denote the monoidal category of endomorphisms of X in **SpanCat**. Then a monoid in $\mathbf{Graph}_X(\mathbf{Cat})$ is a double category [5] in general, and a 2-category if X is a set. Thus, one can construct a language for $\boldsymbol{B}$ just as for strict monoidal categories, in which the types are the objects of B_1, i.e. the 1-cells.

2. LANGUAGES FOR PSEUDO-MONOIDS.

Monoidal categories.

In practice, monoidal categories such as **Ab** are much more prevalent than strict monoids in **Cat**, since often associativity and unit laws only hold up to isomorphism, e.g.

$$a_{A,B,C}: (A\otimes B)\otimes C \to A\otimes(B\otimes C),$$

rather than equality. Consequently the theory T must have all of the data of Δ and (in place of the equations of Δ) the 2-cell isomorphisms: associativity a: $\otimes(\otimes,1)\to\otimes(1,\otimes)$ and left and right units, l: $\otimes(I,1)\to 1$ and r: $\otimes(1,I)\to 1$, respectively, subject to the pentagon and triangle laws. This data is used to generate a (strict) monoidal 2-category. These are defined by substituting '2-functor' for 'functor' and '2-natural' for 'natural' in the definition of monoidal categories. For example, **Cat** equipped with its cartesian product is a monoidal 2-category. The 1-cells of T are still called operations: the 2-cells of T (and their images in models) are called *canonical transformations*. A standard model of T in **Cat** is, of course, a (non-symmetric) monoidal category.

The definition of the language L for a monoidal category is the same as in the strict case. However, (1.1) no longer holds. Nor is it true that

$$a_{X,Y,Z}: ((x\otimes y)\otimes z) = x\otimes(y\otimes z).$$

Instead, impose on the terms an equivalence relation $\equiv$ defined as follows. Let $s = f(Fv)$ with $v \in V$ and $t = g(Gw)$ be terms. Then $s \equiv t$ iff

(2.1) $v = w$,

(2.2) there is an α: $F\to G$ in T such that $f = g\circ\alpha_V$.

The equivalence is exemplified by its occurrence when α is a generating canonical transformation. Let $s \in X$, $t \in Y$ and $u \in Z$ be terms. Then

$$a_{X,Y,Z}((s\otimes t)\otimes u) \equiv s\otimes(t\otimes u),$$
$$l_X(*\otimes s) \equiv s,$$
$$r_X(s\otimes *) \equiv s.$$

Note that $\equiv$ is transitive since canonical transformations compose. It is also closed under tensoring and application of operations and function symbols.

Mac Lane's coherence theorem for monoidal categories [6], when interpreted as information about T, states that if F, G: $m\to 1$ in T then they are isomorphic in at most one (in fact, exactly one) way. The argument is easily extended to cover arbitrary codomains n (see [3]). Hence T is coherent, i.e. it is locally ordered.

THEOREM 2.1. *Let* V *be a monoidal category and* $x = Fv$ *be a basic term with* $v \in V$ *and* $f, g: FV\to W$ *in* V*. Then*

$$f(x) \equiv g(x) \text{ iff } f = g.$$

PROOF. By definition, there is a canonical transformation $\alpha: F \to F$ such that $f = g \circ \alpha_V$. Now the coherence of $\boldsymbol{T}$ forces $\alpha = 1$. ∎

EXAMPLE 2.2. Let R be an object of $\boldsymbol{V}$ with an associative, binary operation $m: R \to R$, i.e.

$$m(m \otimes 1) = m(1 \otimes m) a_{R,R,R}.$$

This is equivalent, on writing $x \cdot y$ for $m\otimes(x,y)$ to

$$(x \cdot y) \cdot z \equiv x \cdot (y \cdot z)$$

since, as normal forms, these terms are

$$m(m.1) \otimes (\otimes.1)(x.y.z) \equiv m(1.m) \otimes (1.\otimes)(x.y.z)$$

and a is the unique canonical transformation $\otimes(\otimes.1) \Rightarrow \otimes(1.\otimes)$. Similarly, $e: I \to R$ is a right unit for m if (writing e for $e(+)$) we have $x \cdot e \equiv x$. Further, the term $e \cdot x$, when written formally, yields a function symbol $f: R - R$ which is idempotent since

$$e \cdot (e \cdot x) \underset{\text{assoc}}{\equiv} (e \cdot e) \cdot x \underset{\text{r. unit}}{\equiv} e \cdot x.$$

The diagrammatic proof of this result requires a six cell diagram [3].

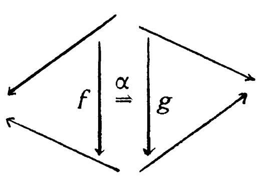
EXAMPLE 2.3. Let $\boldsymbol{V}$ be a monoidal category and $\boldsymbol{A}$ a $\boldsymbol{V}$-category. For f a variable of type $\boldsymbol{A}(A,B)$ and $g \in \boldsymbol{A}(B,C)$ write $g \circ f$ in $\boldsymbol{A}(A,C)$ for the effect of composition on $g \otimes f$. Similarly, let $1_A \in \boldsymbol{A}(A,A)$ denote the identity $j_A: I - \boldsymbol{A}(A,A)$ applied to $+$. Then the category axioms are

$$h \circ (g \circ f) = (h \circ g) \circ f, \; 1_B \circ f = f = f \circ 1_A.$$

Now one can calculate as with ordinary categories.

Bicategories.

Note that **SpanCat**, as defined above, made no use of the 2-cells (natural transformations) of **Cat**. Let f and g be a pair of span morphisms in **SpanCat** with $\alpha: f \to g$ a natural transformation satisfying $d_0\alpha = 1$ and $d_1\alpha = 1$.



Then α is a 3-*cell* of **SpanCat**. Compositions are defined in the obvious way. Presumably, **SpanCat** is now a tricategory.

though, since they have not been defined in general, we shall confine ourselves to $\mathbf{Graph}_X(\mathbf{Cat})$, now equipped with 2-cells, so that it is a monoidal 2-category. Observe that here a standard model of $\boldsymbol{T}$ is just a bicategory. It is straightforward now to define a language for a bicategory

$$B_1 \rightrightarrows B_0$$

whose types and function symbols are generated by the objects and morphisms of B_1, i.e. the 1-cells and 2-cells of $\boldsymbol{B}$. Further, the coherence theorem for bicategories proved by Mac Lane and Paré in [8] is now seen to be just the (known) coherence of $\boldsymbol{T}$. Hence, the appropriate modification of Theorem 2.1 follows for this language. The theory of categories enriched over a bicategory can now be developed just as for ordinary categories.

Symmetric monoidal categories.

To define, say, a commutative monoid in $\boldsymbol{V}$ requires a symmetry. The usual ones are considered here: braid symmetries will be dealt with in §3. A symmetry on a monoidal 2-category is a 2-natural transformation C satisfying $C^2 = 1$. The iterates of C (i.e. the operations generated by C under tensoring and composition) are called *permutations*, and may be identified with the usual permutations of the natural numbers.

Let $\boldsymbol{S}$ be the symmetric, strict monoidal 2-category (called a *2-prop*) generated by the data and equations of $\boldsymbol{T}$ and a canonical transformation $c:\otimes\to\otimes C$ satisfying $c_C c = 1$ and the hexagon law

$$\begin{array}{ccccc}
\otimes(\otimes,1) & \xrightarrow{\;a\;} \otimes(1,\otimes) & \xrightarrow{c(1,\otimes)} & \otimes C(1,\otimes) = \otimes(\otimes,1)C_{1,2} & \\
{\scriptstyle \otimes(c,1)}\downarrow & & & \downarrow{\scriptstyle aC_{1,2}} & (2.1) \\
\otimes(\otimes C,1) & \xrightarrow{a(C,1)} \otimes(1,\otimes)(C,1) & \xrightarrow{\otimes(1,c)(C,1)} & \otimes(1,\otimes)(1,C)(C,1) &
\end{array}$$

A model of $\boldsymbol{S}$ is a strong, symmetric, monoidal 2-functor: a standard model in **Cat** is a symmetric monoidal category $\boldsymbol{V}$. It is important to distinguish c, whose image is the symmetry of $\boldsymbol{V}$, from C whose image is the switch functor $\boldsymbol{V}^2 - \boldsymbol{V}^2$.

The equivalence relation defined by (2.1) and (2.2) yields $c(x\otimes y) \equiv \otimes C(x,y)$ which is not quite satisfactory. It must be extended to include $C(x,y) \equiv (y,x)$ so that $c(x\otimes y) \equiv (y\otimes x)$. Formally, define $f(Fv) \equiv g(Gw)$ in this language if

(2.3) v and w have the same number of variables,

(2.4) there is a permutation P such that $v_{P_i} = w_i$.

and (2.5) there is an $\alpha: F \rightarrow GP$ in $\boldsymbol{S}$ such that $f = g \circ \alpha_V$.
As before, the relation is an equivalence, closed under tensoring and the application of operations and function symbols.

THEOREM 2.4. *Let $\boldsymbol{V}$ be a symmetric monoidal category and let $x = Fv$ be a basic term with $v \in V$ and $f,g: FV \rightarrow W$ in $\boldsymbol{V}$. Then $f(x) \equiv g(x)$ iff $f = g$.*

PROOF. $P = 1$ is the unique permutation such that $Pv \equiv v$ and, by coherence, $\alpha = 1$ is the unique canonical transformation $F \Rightarrow F = FP$. Hence $f = g \circ \alpha_V = g$. ∎

Now the general proof of Proposition 0.1 is given by applying the theorem to the result for $\boldsymbol{V} = \mathbf{Set}$, now reinterpreted as a proof that

$$\rho(\lambda \otimes 1)((r \otimes m) \otimes s) \equiv \lambda(1 \otimes \rho) a((r \otimes m) \otimes s).$$

3. LANGUAGES FOR BRAIDED PSEUDO-MONOIDS.

The braid category.

Recall that the *braid group* $\mathbf{B}_n$ on n strings is generated algebraically by τ_i for $i = 1, \ldots, n-1$ subject to the relations

$$\tau_i \tau_{i+1} \tau_i = \tau_{i+1} \tau_i \tau_{i+1} \text{ for } i = 1, \ldots, n-2.$$
$$\tau_i \tau_j = \tau_j \tau_i \text{ for } |i-j| \geq 2.$$

For a geometric interpretation see [1] or [4]. By imposing the extra relation $\tau_i^2 = 1$ one obtains the permutation group $\mathbf{P}_n$. Thus there is a canonical surjective homomorphism $q_n: \mathbf{B}_n \rightarrow \mathbf{P}_n$. For $S \in \mathbf{B}_n$ and $1 \leq i \leq n$ write Si for $(q_n S)i$. The *j-pure elements* of $\mathbf{B}_n$ form the smallest subset closed under

(i) τ_i^ε is j-pure if $i = j$ or $i = j-1$ (for $\varepsilon = 1, -1$).

(ii) if S is j-pure and T is Sj-pure then TS is j-pure.

Consequently, $1 = \tau_j^{-1} \tau_j$ is j-pure.

THEOREM 3.1 (Artin). *Every element S of $\mathbf{B}_n$ may be expressed. in a unique way, as a normal form $S_n S_{n-1} \ldots S_2$ where S_j is uniquely expressible as a reduced j-pure power product of τ_i's satisfying $i < j$ (where reduced means successive terms in the power product are powers of distinct τ_i's).* ∎

Consider the subset (not a submonoid) K_n of $\mathbf{B}_n$ consisting of those braids B whose normal form only employs first powers, i.e. $Sj = \tau_{j-1} \tau_{j-2} \ldots \tau_i$ for some i. Now every permutation is the image of such a form. Identify the elements of Kn with their images and call them *permutations*.

Now consider $\mathbf{B}_n$ as a category with one object n. Define the *braid category* to be $\mathbf{B} = \Sigma \mathbf{B}_n$. It is a strict monoidal category with tensor ($\oplus$) given on objects by addition and determined for morphisms by

$$\tau_i \oplus \tau_j = \tau_{m+j} \cdot \tau_i \text{ for } \tau_i \in \mathbf{B}_m \text{ and } \tau_j \in \mathbf{B}_n.$$

Braiding is defined as follows. Given m and n let

$$\gamma = \tau_n \tau_{n-1} \ldots \tau_2 \tau_1 \in \mathbf{B}_{n+1} \text{ and } \gamma_p = (m-p.\gamma.p-1) \in \mathbf{B}_{m+n}.$$

Then $C_{m,n} = \gamma_m \ldots \gamma_2 \gamma_1$. $\mathbf{B}$ plays the same role for braid theories that $\mathbf{P}$ (the permutation category) plays for props. Adding identity 2-cells to $\mathbf{B}$ yields a 2-category also denoted by $\mathbf{B}$, which is a strict, braided monoidal 2-category, i.e. $\oplus$ is a 2-functor and C is a 2-natural isomorphism, satisfying the usual braid axioms.

Braided monoidal categories.

A *braided monoidal category* is a monoidal category $\boldsymbol{V}$ with a natural isomorphism c: $\otimes \Rightarrow \otimes C$ satisfying the hexagon law (Diagram 2.1) and its dual. In short, the hexagon law is $aca = (1 \otimes c) a (c \otimes 1)$ while its dual is $a^{-1} c a^{-1} = (c \otimes 1) a^{-1} (1 \otimes c)$. Note that c is not, in general, an involution. For examples and applications see [4]. Thus, there is a 2-prop whose models in **Cat** are the braided monoidal categories. However, to precisely formulate the coherence theorem in this context is rather difficult: in [4] it is stated as

> *to test whether a diagram built up from a, r, l and c commutes in all braided monoidal categories it suffices to see that each leg of the diagram has the same underlying braid.*

Note, however, that in the 2-prop, no use is made of the equations $C^2 = 1$. By removing it, c^2 and 1 no longer have the same codomain, which will make the coherence theorem easy to state.

A *braided theory* is a braided, monoidal 2-category $(\boldsymbol{T}, \oplus, 0, C)$. The iterates of C are called *braids*. A *model* of $\boldsymbol{T}$ in **Cat** is a strong, (braid) symmetric, monoidal 2-functor M from $\boldsymbol{T}$ to **Cat**. A model is *standard* if $M(n) = \boldsymbol{V}^n$ for some category $\boldsymbol{V}$. As with 2-props, every model is isomorphic to a standard one. For example, the (braided) theory $\boldsymbol{T}_1$ of a braided pseudo-monoid is generated by the data for a (non-symmetric) pseudo-monoid, and a canonical isomorphism c: $\otimes \Rightarrow \otimes C$ which satisfies the hexagon law and its dual. Standard models in **Cat** are just braided monoidal categories and so their coherence can be expressed as a property of $\boldsymbol{T}_1$. Translating the result of [4], or proving directly yields

THEOREM 3.2. *T_1 is coherent.*

Languages for braid theories.

Let M: $T_1 \to$ **Cat** be a standard model, with M(1) = ***V***. Construct a language ***L*** for M just as before. By imposing the same relation as on ***T***, one can repeat Theorem 2.1 in this context. This does not yet make calculations any easier, however, because there isn't, in general, a good notation for Bv where B is a braid and v is a sequence of variables (call such terms *braids of variables*). If one simply write $C(x,y) \equiv (y,x)$ then $C^2(x,y) \equiv (x,y)$ with the consequent loss of information. Note, however, that many calculations employing braids, including all those adapted from the study of the usual symmetry, only involve the permutations of **B**. For these a better notation is possible.

Modify the language by assigning to each variable v_i in a term s, as part of its data, a *set of variables $m(v_i)$* called its *memory*. Define the memory of s to be $m(s) = \cup m(v_i)$. Also

$$|s| = |v| = \{v_i \mid 1 \le i \le n\}$$

is the set of variables in s. Given a variable x of s and a set of variables w satisfying $|w| \cap (|s| \cup m(s)) = \varnothing$ then x^w is the variable x, but now with w added to its memory. Further, $s^w = f(F(v_i{}^w))$. Finally, given a term t with $|t| = w$ then $s^t = s^w$. Consequently, we have

$$(s,t)^u = (s^u, t^u), \tag{3.1}$$

$$(s^t)^u = s^{tu}. \tag{3.2}$$

$$(u, t^u, s^{tu}) = (u, t^u, s^{ut}) \tag{3.3}$$

where s^{tu} denotes $s^{(t,u)}$. Note that if e is a constant then

$$m(e) = |e| = \varnothing.$$

Hence $s^e = s$ and $e^s = e$.

Now define an order $\gg$, called *reduction*, here for the permuted variables, and later for all terms. For sequences of variables v and w we have

$$C(v,w) \gg (w, v^w) \text{ if } |w| \cup m(v) = \varnothing. \tag{3.4}$$

Close this relation under ordering, tensoring and application of C. This order will later be extended to an equivalence. Note that (3.1) and (3.2) represent the hexagon law and its dual while (3.3) expresses the naturality of C with respect to itself.

LEMMA 3.3. *Let v and w denote sequences of variables of* *length n and* P, Q $\in \mathbf{B}_n$ *denote braids.*

(i) If $m(v) = \emptyset$ *then* P *is a permutation iff there is a* w *such that* $Pv \gg w$.

(ii) If $Pv \gg w$ *and* $Qv \gg w$ *then* $P = Q$.

PROOF. (i) Let $P = S_n S_{n-1} \ldots S_2$ be a permutation in normal form and let $P_j = S_j S_{j-1} \ldots S_2$. By induction it follows that $P_j v$ reduces to a sequence of variables with

$$m(P_j v) \subset \{v_2, v_3, \ldots, v_j\}$$

since each S_i, being i-pure, only adds v_i to memories. The converse is trivial.

(ii) From (i), P and Q are permutations, and hence equal, since their images in $\mathbf{P}_n$ are. ∎

Reduction is defined in general by $f(Fv) \gg g(Gw)$ if:

(3.5) there is a permutation P such that $Pv \gg w$,

and (3.6) there is an $\alpha: F \to GP$ in $\boldsymbol{T}_1$ such that $f = g \circ \alpha_V$.

As before, it is closed under tensoring and the application of operations and function symbols.

Now the main results for §2 can be duplicated for braid theories, with the permutations playing the same role as before.

THEOREM 3.4. *Let f and g be function symbols, x be a basic term and r be a term. Then* $f(x) \gg r$ *and* $g(x) \gg r$ *imply* $f = g$.

PROOF. Let $x = Fv$ and $r = h(Hu)$. By the lemma there is a unique permutation P such that $Pv \gg u$ and also $f = h \circ \alpha_V$ and $g = h \circ \beta_V$ for some $\alpha, \beta: F \Rightarrow HP$. Since $\boldsymbol{T}_1$ is coherent, $\alpha = \beta$ and so $f = g$. ∎

EXAMPLE 3.5. That Proposition 0.1 holds for $\boldsymbol{V}$ a braided monoidal category can be shown as follows. Let $r, s \in R$ and $m \in M$ all be variables with empty memories. Then apply the corollary to

$$(r \cdot m) \cdot s \gg s \cdot (r \cdot m)^s$$

and

$$r \cdot (m \cdot s) \gg r \cdot (s \cdot m^s) \equiv (r \cdot s) \cdot m^s \gg (s \cdot r^s) \cdot m^s$$
$$\equiv s \cdot (r^s \cdot m^s) = s \cdot (r \cdot m)^s.$$

One cannot extend the reduction process from permutations to arbitrary braids if memories are to be assigned to variables, rather than sequences of variables, since (3.3) forces $(x^y)^z = (x^z)^y$ while removing the restriction on memories to obtain $C^2(x, y) = (x^y, y^x)$ yields

$$(C^2.1)(1.C^2)(x, y, z) \gg (C^2.1)(x, y^z, z^y) \gg (x^{y^z}, y^{zx}, z^y)$$

and

$$(1,C^2)(C^2,1)(x,y,z) \gg (x^y, y^{xz}, z^y)$$

although

$$(C^2,1)(1,C^2) \neq (1,C^2)(C^2,1).$$

Similar problems occur in attempting to arbitrarily reduce C^{-1}. One might try to put signs on the elements of the memory and allow

$$C^{-1}(x,y) \geq (y^{-x}, x)$$

but $(C,1)(1,C^{-1})(C,1)(x,y,z)$ and $(1,C)(C^{-1},1)(1,C)(x,y,z)$ both reduce to (z^{-x}, y^{z}, x^y) although

$$(C,1)(1,C^{-1})(1,C) \neq (1,C)(C^{-1},1)(1,C).$$

One possible extension of $\gg$ remains: without allowing negative entries in the memory, make (3.4) an equivalence, i.e. extend $\gg$ to the smallest equivalence relation $\equiv$ containing it. We must, however, restrict our reductions to terms without memory. (Otherwise, on replacing z above by z^x the old problems recur.)

PROPOSITION 3.6. *If $s \equiv t$ and $m(s) = \emptyset$ then $s \gg t$.*

PROOF. Without loss of generality, there is a term r such that $s, t \gg r$ with $s = Pv$ and $t = Qw$ with v, w and r sequences of variables. Thus the problem reduces to showing that $P = QP'$ for some permutation P', and this for $Q = (p, C, q)$. Hence

$$r = (w_1, w_2, \ldots, w_{i-1}, w_{i+1}, w_i^{i+1}, \ldots, w_n).$$

Now, $m(v) = \emptyset$ implies that P acts on the variables w_i and w_{i+1} by some expansion of C, denoted P_2, i.e. $P = P_3P_2P_1$ as a product of normal forms. Since P is a permutation and w_i and w_{i+1} are adjacent in r, it follows that $P_3P_2 = QP_3$ and so $P = Q(P_3P_1)$ as required. ∎

The results above for $\gg$ can now be extended to $\equiv$. Consequently, the two-part proof in Example 3.6, for example, can now be compressed to a single string of equivalences.

Examples.

EXAMPLE 3.7. *Braid monoidal identitites.* A representative sample of the calculations required for enriched category theory using braidings is given by B3-B7 of [4]. They are proved here using the language, since they all rely on braids which are permutations. Note, however, that by using the language, explicit reference to these results is not required in applications. Equivalence of terms will now be written as equality, since the need for the distinction has passed.

B3. $$lc(x\otimes *) = l(*\otimes x) = x = r(x\otimes *).$$

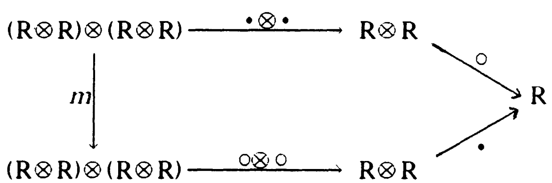
B4. $rc = l$. Dual to B3.

B5. $ac(1\otimes c)a((x\otimes y)\otimes z) = ac(x\otimes(z\otimes y^{z})) = z\otimes(y^{z}\otimes x^{zy}) =$
$= z\otimes(y^{z}\otimes x^{yz}) = c((y\otimes x^{y})\otimes z) = c(c\otimes 1)((x\otimes y)\otimes z)$.

B6. $$a((aca)\otimes 1)a^{-1}(1\otimes c)((x\otimes y)\otimes(z\otimes t))$$
$$= a((aca)\otimes 1)(((x\otimes y)\otimes t)\otimes z^{t} = y\otimes((t\otimes x^{yt})\otimes z^{t})$$
$$= (1\otimes(a^{-1}ca^{-1}))(y\otimes(x^{y}\otimes(z\otimes t)))$$
$$= (1\otimes(a^{-1}ca^{-1}))a(c\otimes 1)((x\otimes y)(z\otimes t)).$$

B7. $$(1\otimes c)a(c\otimes 1)a^{-1}(1\otimes c)a = a^{-1}(c\otimes 1)a^{-1}(1\otimes c)a(c\otimes 1).$$
Left to the reader.

EXAMPLE 3.8 (*Eckmann-Hilton*). Let R be an object in ***V*** with two associative binary operations $\bullet$ and $\circ$ having the same unit $e\colon I\to R$ and satisfying the interchange law



where
$$m((x\otimes y)\otimes(z\otimes t)) \equiv (x\otimes z)\otimes(y^{z}\otimes t)$$
so that the interchange law says
$$(x\cdot y)\circ(z\cdot t) = (x\circ z)\cdot(y^{z}\circ t).$$
Writing e for the constant $e(*)$, we have
$$x\cdot y = (x\circ e)\cdot(e\circ y) = (x\cdot e)\circ(e^{e}\cdot y) = x\circ y.$$
Note that $e^{e} = e$ since e is a constant. Hence the two operations are the same. The proof of commutativity is similar.

EXAMPLE 3.9. ***V*-categories.** Given a braiding on ***V*** and a ***V***-category ***A***, define $\boldsymbol{A}^{op}$ to have the same objects as ***A*** but with $\boldsymbol{A}^{op}(A,B) = \boldsymbol{A}(B,A)$. For variables $f\in\boldsymbol{A}^{op}(A,B)$ and $g\in\boldsymbol{A}^{op}(B,C)$ composition is defined by
$$g\circ f = mc(g\otimes f) = f\cdot g^{f}$$
where m is the composition of ***A***. Then for $h\in\boldsymbol{A}^{op}(C,D)$,
$$h\circ(g\circ f) = h\circ(f\cdot g^{f}) = (f\cdot g^{f})\cdot h^{fg}$$
$$= f\cdot(g\cdot h^{g})^{f} = f\cdot(h\circ g)^{f} = (h\circ g)\circ f.$$
The identity laws are handled similarly.

Given $\boldsymbol{V}$-categories $\boldsymbol{A}$ and $\boldsymbol{B}$, define $\boldsymbol{A}\otimes\boldsymbol{B}$ to be the $\boldsymbol{V}$-category which has as objects pairs (A,B) of objects from $\boldsymbol{A}$ to $\boldsymbol{B}$, respectively, and as hom-objects

$$\boldsymbol{A}\otimes\boldsymbol{B}((A,B),(A',B')) = \boldsymbol{A}(A,A')\otimes\boldsymbol{B}(B,B').$$

The identity of (A,B) is $(1_A,1_B)$ and composition is determined by (the unique morphism such that)

$$(h,k)\cdot(f,g) = (h\cdot f, k^{f}\cdot g).$$

BIBLIOGRAPHY.

1. E. ARTIN, Theory of braids, *Ann. of Math.* 48 (1947), 101-136.
2. M. BARR & C. WELLS, *Toposes, triples and theories.* Springer 1985.
3. C. B. JAY, Languages for monoidal categories, *J. Pure & Appl. Alg.* 59 (1989), 61-85.
4. A. JOYAL & R. STREET, Braided monoidal categories, *Macquarie Math. Reports,* 1986.
5. G.M. KELLY & R. STREET, Review of the elements of 2-categories. *Lecture Notes in Math.* 420, Springer (1974).
6. S. MAC LANE, Natural associativity and commutativity, *Rice Univ. Studies* 49 (1963), 28-46.
7. S. MAC LANE, *Categories for the working mathematician.* Springer 1971.
8. S. MAC LANE & R. PARE, Coherence for bicategories and indexed categories, *J. Pure & Appl. Alg.* 37 (1985), 59-80.

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