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COMPLETE THEORIES IN 2-CATEGORIES

by Renato BETTI and Marco GRANDIS

RÉSUMÉ. On étudie les théories à valeurs dans une 2-catégorie concrète **A** sous forme de 2-foncteurs "des modèles" T: **A** → **CAT**, et leurs modèles biuniversels, sous forme de birépré-sentations de T. On donne des théorèmes d'existence pour ces derniers, à partir d'un théorème de l'objet biinitial étendant le théorème de Freyd à la dimension 2, ainsi que divers résultats sur les pseudolimites et les bilimites dans les 2-catégories.

En particulier, une théorie peut être définie par des conditions "syntaxiques" ayant sens dans les objets de **A**, par exemple des conditions de commutativité, de limite, de colimite, d'additivité, de majoration, etc. On retrouve alors, par une méthode générale procédant "d'en haut" au lieu de constructions syntaxiques "d'en bas", des résultats tels que l'existence du modèle générique d'une esqûisse de Bastiani-Ehresmann ou du topos libre engendré par un graphe.

0. INTRODUCTION.

0.1. Let **A** be a concrete 2-category, with structural 2-functor $|\ |$: **A** → **CAT**; typical examples (where $|\ |$ is the inclusion) will be **CAT** itself, the 2-category of finitely complete categories, of exact categories, of abelian categories, of toposes (and logical morphisms), and so on.

Notice that these categories are always pseudocomplete, hence bicomplete, but generally not complete: they lack equalizers (except for **CAT**). Thus we generally look for solutions of *bi*universal problems in **A** (e.g., the bifree abelian category generated by a graph, determined up to equivalence), and only exceptionally for solutions of 2-universal problems (e.g., the 2-free category generated by a graph, determined up to isomorphism).

0.2. According to our definition, a *theory* T on the (small) graph Δ, with values in $\mathbf{A}$, associates to every object A in $\mathbf{A}$ a (generally non small) set $T(A)$ of graph morphisms $t: \Delta \to |A|$ (the *models* of T in A), so that an obvious condition of stability under composition with the morphisms of $\mathbf{A}$ is satisfied (Def. 4.2). Thus T can be thought of as a 2-functor $T: \mathbf{A} \to \mathbf{CAT}$ assigning to each object A the category $T(A)$ of models $t: \Delta \to |A|$ (together with their natural transformations); it is a sub-2-functor of the *total* $\mathbf{A}$-theory T_Δ assigning to each A the whole set of graph morphisms $\Delta \to |A|$.

A *biuniversal model* (4.5) $t_0: \Delta \to |A_0|$ of T will be any model through which all the models (and their transformations) factor up to isocells: in other words, it is a birepresentation of T, or equivalently a biuniversal arrow from the category $\mathbf{1}$ to T. If existing, it is determined up to an equivalence of $\mathbf{A}$, and A_0 is called the *biclassifying object* of T. A 2-universal model is the corresponding strict notion.

After a general part on biuniversal problems in 2-categories, we give here solution set conditions for the existence of these models for *complete* theories, and apply them to the above recalled situations. Thus we get, by a general method "from above", such results as the existence of the classifying category of a Bastiani-Ehresmann sketch [BE], of the bifree topos generated by a graph or by a cartesian closed category [Bu; MS] and so on. Moreover the heavy syntactical constructions which are needed in proceeding "from below" are here replaced by lighter constructions proving the solution set condition; in the same way as, in the 1-dimensional case, it is simpler to prove, e.g., the existence of the free group generated by a set by means of the Freyd's Initial Object Theorem than actually construct it.

0.3. More precisely, the outline of this paper is the following.

Part I studies universal properties of 2-categories. In Chapter 1, birepresentations of 2-functors $T: \mathbf{A} \to \mathbf{CAT}$ and biuniversal arrows into a 2-functor $U: \mathbf{A} \to \mathbf{B}$ are considered: they are equivalent problems which, under suitable hypotheses on the bicotensor products with $\mathbf{2}$ in $\mathbf{A}$, reduce to the existence of a biinitial object (Thms. 1.8-9). Chapter 2 gives construction theorems (2.6, 2.8) for conical pseudolimits (from products, isoinserters and identifiers of endocells, which do exist in all our examples) and for conical bilimits. Chapter 3 supplies a "Biinitial Object Theorem" (3.1), extending Freyd's Theorem to the 2-dimensional case, and derives solution set conditions for the existence of birepresentations or biuniversal arrows. Strict universal properties are also considered.

Part II introduces theories and their 2-universal, or biuniversal, model (Ch. 4). 2-complete, pseudocomplete and bicomplete theories are considered in Chapter 5, and solution set conditions for the existence of the 2-universal or biuniversal model are derived from the results of Part I. Chapter 6 concerns *reflective* theories (satisfying a property of reflections of models) in *well adapted* 2-categories, proving that such theories are always bicomplete and provided with a biuniversal model.

Applications are given in Part III:

Chapter 7 proves that the 2-category **FLM** of finitely complete categories, with finitely continuous functors and natural transformations is well adapted. Thus each reflective theory in it has a biuniversal model; in particular this holds for the theory $T(\Delta,K,\Gamma)$ defined by a projective sketch (Δ,K,Γ) where Δ is a small graph, K is a set of commutativity conditions on Δ and Γ is a set of finite limit conditions on Δ. Analogously for the 2-category **FP** of categories with finite products. More generally, analogous conclusions hold for the 2-category ***FF'*LM** of F-complete, F'-cocomplete categories, where F and F' are small sets of small graphs.

Chapters 8 and 9 prove analogous results on the 2-categories: **Λ-Cat** of Λ-linear categories (Λ a small ring), **EX** of exact categories (in the sense of Puppe-Mitchell [Pu; Mi]), **AB** of abelian categories, **RG** of regular categories (in the sense of Grillet [Gr]) and **TPL** of toposes and logical morphisms. In all these cases reflective theories can be defined by suitable syntactic conditions on a small graph Δ; moreover, by means of the "change of base" for theories (4.9-10), "intermediate steps can be chained": e.g., the bifree topos on the small graph Δ is the bifree topos on the bifree closed category on Δ, and so on.

Last, Chapter 10 concerns theories with values in involutive ordered categories, already considered in [G2]; since the "good" transformations in this case are just lax-natural, these theories live in a 2-category A which is only "1-concrete", and some adaptations in terminology are required.

0.4. We would stress the following point: in defining theories we adopt here a *semantical* approach: a theory is given by assigning its models. This approach differs from definitions based on "partial syntaxis", e.g., a Bastiani-Ehresmann sketch, or on a "global syntaxis" as in Lawvere's functorial semantics [La].

Actually the definition of an **A**-theory by syntactic conditions on the graph Δ (e.g., by a mixed sketch for **A** = ***FF'* LX**) is a very useful tool when working in some *specified* **A** ; however the type of syntactic conditions which may be imposed (commutativity, limit and colimit conditions in the above case; linearity conditions for **A** = **A-CAT**, exponentiation conditions for cartesian closed categories, inequality conditions for ordered categories, etc.) depends on **A**, and it seems hard to give a general treatment from this point of view.

On the other hand the global syntactic definition (an **A**-theory is an object A_0 of **A**, corresponding to its biclassifying object in our formulation) is simple and general from a theoretical point of view, but needs theorems of existence of the biuniversal model in order to be applied in particular cases. Such results as we aim to give here.

The approach we follow here has already been used by one of the authors, in a work concerning reflective (homological) theories in **EX** [G1-3]; the existence of the biuniversal model for a reflective **EX**-theory was proved through an associated theory in a 2-*complete* 2-category, **RE**, and the (strict) initial object Theorem. The present results would allow to reach the goal directly in **EX**.

0.5. General conventions. We generally use Mac Lane's terminology [Ma] for categories and Kelly-Street's [K1, KS, S1, S2] for 2-categories.

A universe $\mathcal{U}$ is chosen once for all, whose elements are called small sets. A $\mathcal{U}$-category is assumed to have objects and morphisms belonging to $\mathcal{U}$. **CAT** will always denote the 2-category of (large or small) $\mathcal{U}$-categories. The cardinal of a small set is assumed to be small.

A cell of a 2-category **A** will be typically written

$$\alpha\colon a_1 \to a_2 : A \to A' \quad \text{or also} \quad \alpha\colon A \Rightarrow A' \ ;$$

notice that the double arrow always concerns the *horizontal* domain and codomain of the cell; $\alpha\colon a_1 \simeq a_2$ denotes an isocell from a_1 to a_2.

A 2-functor U: **A** $\to$ **B** is called 2-*full* (resp. *bifull*) if all the functors **A**(A,A') $\to$ **B**(UA,UA') are surjective (resp. representative and full); a sub-2-category is 2-full or bifull whenever its embedding is so.

In a 2-category $\mathbf{A}$ the object A_0 is said to be *2-initial* if:
(1) for each object A there is a unique morphism $a: A_0 \to A$ and a unique cell $a \to a$ (the identity of a).

A_0 is said to be *biinitial* if:
(2) for each object A there is some morphism $a: A_0 \to A$; for every pair of morphisms $a_1, a_2: A_0 \to A$ there is a unique cell $\alpha: a_1 \to a_2$ (an isocell).

The 2-initial (resp. biinitial) object is the 2-colimit (resp. the bi-colimit) of the empty diagram (see Ch. 2) and it is determined up to isomorphism (resp. up to equivalence). Notice that **AB**, the 2-category of abelian $\mathcal{U}$-categories, has a biinitial object (the category $\underline{1}$) but no 2-initial (or initial) one.

If $\mathbf{A}$ and $\mathbf{B}$ are 2-categories, we shall write $(\mathbf{A},\mathbf{B})$ the 2-category of 2-functors $\mathbf{A} \to \mathbf{B}$, their *natural* transformations and modifications, while $[\mathbf{A},\mathbf{B}]$ will denote the 2-category of 2-functors, *pseudo*transformations and modifications.

0.6. Last we fix notations for pseudotransformations and their modifications.

A *pseudotransformation* (of 2-functors) $\phi: F \to G: \mathbf{A} \to \mathbf{B}$ is a collection:

(1) $\phi = ((\phi A),(\phi a)), \quad \phi A: FA \to GA, \quad \phi a: \phi A'.Fa \simeq Ga.\phi A: FA \to GA',$

for A and $a: A \to A'$ varying in $\mathbf{A}$, with coherence conditions:

(PT.1) For all A, $\phi 1_A = 1_{\phi A}$,

(PT.2) For all composable a, a' in $\mathbf{A}$:

(2)
$$\begin{tikzcd} FA \arrow[r, "\phi A"] \arrow[d, "Fa"'] \arrow[dr, phantom, "\overset{\phi a}{\to}"] & GA \arrow[d, "Ga"] \\ FA' \arrow[r, "\phi A'"] \arrow[d, "Fa'"'] \arrow[dr, phantom, "\overset{\phi a'}{\to}"] & GA' \arrow[d, "Ga'"] \\ GA'' \arrow[r, "\phi A''"'] & GA'' \end{tikzcd} \quad = \quad \phi(a'a)$$

(PT.3) For all $\alpha: a \to a': A \to A'$ in $\mathbf{A}$:

$$
\begin{tikzcd}
FA \arrow[r, "\phi A"] \arrow[d, "Fa"'] \arrow[dr, phantom, "\overset{\phi a}{\to}"] & GA \arrow[d, "Ga"'] \arrow[d, shift left=4, "Ga'"] \\
FA' \arrow[r, "\phi A'"'] & GA'
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
FA \arrow[r, "\phi A"] \arrow[d, "Fa"'] \arrow[d, shift left=4, "Fa'"] \arrow[dr, phantom, "\overset{\phi A'}{\to}"] & GA \arrow[d, "Ga'"] \\
FA' \arrow[r, "\phi A'"'] & GA'
\end{tikzcd}
$$

A *modification* of pseudotransformations $\Phi\colon \phi \to \psi\colon F \to G\colon \mathbf{A} \to \mathbf{B}$ is a family:

(4) $$\Phi = (\Phi A), \quad \Phi A\colon \phi A \to \psi A\colon FA \to GA,$$

for A in $\mathbf{A}$, with coherence condition:

(MD) For all $a\colon A \to A'$ in $\mathbf{A}$, the following square of $\mathbf{B}$-cells commutes:

(5) $$
\begin{tikzcd}
\phi A'.Fa \arrow[r, "\phi a"] \arrow[d, "\Phi A'.Fa"'] & Ga.\phi A \arrow[d, "Ga.\Phi A"] \\
\psi A'.Fa \arrow[r, "\psi a"] & Ga.\psi A.
\end{tikzcd}
$$

Part I. BIUNIVERSAL PROPERTIES FOR 2-CATEGORIES

1. Birepresentations, biuniversal arrows and biinitial objects.

Weak universal properties in 2-categories can be introduced as birepresentations of a 2-functor $T\colon \mathbf{A} \to \mathbf{CAT}$ or equivalently as biuniversal arrows with respect to a 2-functor $U\colon \mathbf{A} \to \mathbf{B}$. In contrast with the 1-dimensional case, these problems seem not to have a simple and general formulation in terms of biinitial objects. This fact, however, becomes possible under suitable assumptions on the existence and preservation of bicotensors [2¢A] in $\mathbf{A}$.

U and T are always as above and B_0 is an object of $\mathbf{B}$.

1.1. Recall that the comma 2-category $(B_0 \downarrow U)$ has objects of the form $(A, b\colon B_0 \to UA)$ and cells $\alpha\colon (A,b) \Rightarrow (A',b')$ given by $\mathbf{A}$-cells

α: A $\Rightarrow$ A' such that Uα.b = b'.

We also consider the 2-category $[B_0{\downarrow}U]$ (¹); the *objects* are the same as in $(B_0{\downarrow}U)$, a *morphism* (a,σ): $(A,b) \to (A',b')$ is given by an $\mathbf{A}$-morphism a: A $\to$ A' with a $\mathbf{B}$-isocell σ: $b' \to Ua.b$: $B_0 \to UA'$; a *cell* α: $(a_1,\sigma_1) \to (a_2,\sigma_2)$: $(A,b) \to (A',b')$ is given by an $\mathbf{A}$-cell α: $a_1 \to a_2$: A $\to$ A' such that $\sigma_2 = (U\alpha.b)\sigma_1$:

(1)
$$\begin{tikzcd} B_0 \arrow[r, "b"] \arrow[d, equal] \arrow[dr, phantom, "\overset{\sigma_1}{\to}"] & UA \arrow[d, shift right=2] \arrow[d, shift left=2] \arrow[d, phantom, "\overset{U\alpha}{\to}"] \\ B_0 \arrow[r, "b'"'] & UA' \end{tikzcd} \quad = \sigma_2$$

In particular, for the 2-functor T: $\mathbf{A} \to$ **CAT**, we shall use the 2-categories $(\mathbb{1}{\downarrow}T)$ and $[\mathbb{1}{\downarrow}T]$ determined by the trivial one-object category $\mathbb{1}$ and contained in the Grothendieck 2-category $\mathbf{El}_0(T)$ of *elements* of T (in the notation of Street [S2]).

1.2. A *birepresentation* [S2] of the 2-functor T: $\mathbf{A} \to$ **CAT** is an object A_0 of $\mathbf{A}$ provided with a family of equivalences of categories (²), natural for A in $\mathbf{A}$:

(1)
$$\lambda A\colon \mathbf{A}(A_0,A) \simeq TA.$$

A *strict* solution of this problem, with λ an *isomorphism* of 2-functors (i.e., all the components λA are isomorphisms of categories), will be called a *2-representation* of T; these will be shortly considered at the end of this chapter (1.10).

1.3. More explicitly, by Yoneda, a birepresentation of T is given by an object (A_0,t_0) of the 2-category $[\mathbb{1}{\downarrow}T]$ $(t_0 = \lambda A_0(1_{A_0}) \in \mathrm{Ob}\, TA_0)$ verifying:

(BR.1) For every $\mathbf{A}$-object A and every $t \in \mathrm{Ob}\, TA$ there is some $\mathbf{A}$-morphism a: $A_0 \to A$ such that $t \simeq Ta(t_0)$;

(¹) It is a comma in the 2-category of 2-categories, 2-functors and pseudotransformations.

(²) This family produces an equivalence λ: $\mathbf{A}(A_0,-) \simeq T$ in the 2-category $[\mathbf{A},\mathbf{CAT}]$ of 2-functors from $\mathbf{A}$ to CAT, their pseudotransformations and modifications.

(BR.2) For all morphisms a_1, a_2: $A_0 \to A$ and every morphism τ: $Ta_1(t_0) \to Ta_2(t_0)$ in the category TA there is a unique $\mathbf{A}$-cell α: $a_1 \to a_2$ such that $\tau = T\alpha(t_0)$.

Thus a birepresentation (A_0, t_0) is a *biinitial* object of $[1{\downarrow}T]$ and is determined up to an equivalence of $\mathbf{A}$, unique up to isomorphism.

However, biinitiality in $[1,T]$ just means satisfaction of (BR.1) and of the restriction of (BR.2) to *isocells*; indeed, it is not possible to express in full generality the condition (BR.2) by means of the 2-category $[1,T]$, which gives no information on the morphisms of the categories TA. As already remarked, this fact will become possible under convenient hypotheses on the bicotensors of $\mathbf{A}$ (1.9).

1.4. A *biuniversal arrow* from the object B_0 of $\mathbf{B}$ to the 2-functor U: $\mathbf{A} \to \mathbf{B}$ is an object A_0 of $\mathbf{A}$ provided with a family of equivalences of categories, natural for A in $\mathbf{A}$:

$$\lambda A\colon \mathbf{A}(A_0,A) \simeq \mathbf{B}(B_0,UA). \tag{1}$$

In other words, by Yoneda, it is a pair $(A_0,\ b_0\colon B_0 \to UA_0)$ verifying:

(BA.1) For each pair $(A,\ b\colon B_0 \to UA)$ there is some $a\colon A_0 \to A$ such that $b \simeq Ua.b_0$.

(BA.2) For all morphisms a_1, a_2: $A_0 \to A$ and each cell

$$\beta\colon Ua_1.b_0 \to Ua_2.b_0\colon B_0 \to UA$$

there is a unique cell $\alpha\colon a_1 \to a_2$ such that $\beta = U\alpha.b_0$.

The solution, if existing, is determined up to an equivalence of $\mathbf{A}$, unique up to isomorphism.

Again, the biuniversal arrow (A_0, b_0) is *biinitial* in $[B_0{\downarrow}U]$, which means that it verifies (BA.1) and the restriction of (BA.2) to isocells:

(BA.I) For all morphisms a_1, a_2: $A_0 \to A$ and each *isocell*

$$\sigma\colon Ua_1.b_0 \to Ua_2.b_0\colon B_0 \to UA$$

there is a unique isocell $\rho\colon a_1 \to a_2$ such that $\sigma = U\rho.b_0$.

Also here the converse is not true: the 2-category $[B_0{\downarrow}U]$ gives no information on the $\mathbf{B}$-cells $\beta\colon b_1 \to b_2\colon B_0 \to UA$, without suitable assumptions on the bicotensors of $\mathbf{A}$ (1.8).

Last we notice that, in case $b_0\colon B_0 \to UA_0$ is a biuniversal arrow, for each morphism $b\colon B_0 \to UA$ the morphism a verifying (BA.1) is a right Kan extension from b_0 to b, in the 2-category $\mathbf{A}_{B_0}$ formed by adding to $\mathbf{A}$ the object B_0, and a cell $\beta\colon B_0 \Rightarrow A$ for each cell $\beta\colon B_0 \Rightarrow UA$ of $\mathbf{B}$. The construction of $\mathbf{A}_{B_0}$ will be used for theories, and more explicitly described in 4.1.

The strict notion of 2-universal arrow, determined up to a unique isomorphism, will be considered in 1.11.

1.5. Thus a birepresentation of $T\colon \mathbf{A} \to \mathbf{CAT}$ is just a biuniversal arrow from $\mathbf{1}$ to T. Conversely a biuniversal arrow from the object B_0 to the 2-functor $U\colon \mathbf{A} \to \mathbf{B}$ is precisely a birepresentation of the 2-functor:

(1) $$\mathbf{B}(B_0,U-)\colon \mathbf{A} \to \mathbf{CAT}.$$

In particular, applying both results, a biuniversal arrow from B_0 to U is the same as a biuniversal arrow from $\mathbf{1}$ to the above 2-functor (1).

1.6. Biuniversal arrows *compose* in the usual way. To get a biuniversal arrow from B_0 to the composite 2-functor

(1) $$\mathbf{X} \xrightarrow{V} \mathbf{A} \xrightarrow{U} \mathbf{B}$$

assume we have biuniversal arrows

(2) $$(A_0,\ b_0\colon B_0 \to UA_0), \qquad (X_0,\ a_0\colon A_0 \to VX_0),$$

respectively from B_0 to U and from the former solution object A_0 to V; then

(3) $$(X_0,\ Ua_0.b_0\colon B_0 \to UVX_0)$$

is biuniversal from B_0 to UV.

Analogously, if (A_0,t_0) is a *birepresentation* of $T\colon \mathbf{A} \to \mathbf{CAT}$ and $(X_0,\ a_0\colon A_0 \to VX_0)$ is a *biuniversal arrow* from A_0 to V, then $(X_0,\ Ta_0(t_0))$ is a birepresentation of $TV\colon \mathbf{X} \to \mathbf{CAT}$. Similar results hold for the strict notions.

1.7. The *bicotensor product* of the $\mathbf{A}$-object A with the arrow-category $\underline{2}$ is a birepresentation of the 2-functor

(1) $$\mathbf{CAT}(\underline{2}, \mathbf{A}(-,A)): \mathbf{A}^{op} \to \mathbf{CAT},$$

i.e., an object $[\underline{2}\pitchfork A]$ of $\mathbf{A}$ together with a family of equivalences of categories, natural for X in $\mathbf{A}$:

(2) $$\lambda X: \mathbf{A}(X,[\underline{2}\pitchfork A]) \simeq \mathbf{CAT}(\underline{2},\mathbf{A}(X,A)).$$

More explicitly, by the conditions (BR1, 2) expressing birepresentations (1.3), this bicotensor is an object $[\underline{2}\pitchfork A]$ with a cell

(3) $$\delta: d_1 \to d_2: [\underline{2}\pitchfork A] \to A$$

satisfying

(BC.1) for any cell $\alpha: a_1 \to a_2: X \to A$ there is some morphism a: $X \to [\underline{2}\pitchfork A]$ such that $\alpha \simeq \delta \circ a$, i.e., there are isocells $\rho_i: a_i \simeq d_i.a$: $X \to A$ $(i = 1,2)$ such that the pasting of the following diagram is α:

(4)
$$\begin{tikzcd}
 & & A \arrow[dd, equal] \\
X \arrow[urr, "a_1"] \arrow[r, "a"] \arrow[drr, "a_2"'] & {[\underline{2}\pitchfork A]} \arrow[ur] \arrow[dr] \arrow[u, phantom, "\rho_1\Downarrow"] \arrow[d, phantom, "\rho_2\Uparrow"] \arrow[r, phantom, "\Downarrow\delta"] & \\
 & & A
\end{tikzcd}
\qquad \alpha = \rho_2^{-1}(\delta.a)\rho_1$$

(BC.2) for all morphisms $a_r: X \to [\underline{2}\pitchfork A]$ $(r = 1,2)$ and all cells $\alpha_i: d_ia_1 \to d_ia_2: X \to A$ $(i = 1,2)$ verifying:

(5) $$(\delta.a_2).\alpha_1 = \alpha_2(\delta.a_1): d_1a_1 \to d_2a_2: X \to A,$$

there exists a unique $\alpha: a_1 \to a_2: X \to [\underline{2}\pitchfork A]$ such that $\alpha_i = d_i.\alpha$ $(i = 1,2)$.

The corresponding strict notion of *cotensor product* $\underline{2}\pitchfork A$ is well known: it amounts to a 2-representation of the 2-functor (1); explicitly, to a cell $\delta: \underline{2}\pitchfork A \to A$ satisfying the universal property (UC.1) below and (BC.2):

(UC.1) for any cell $\alpha: a_1 \to a_2: X \to A$ there is a unique morphism a: $X \to \underline{2}\pitchfork A$ such that $\alpha = \delta.a$.

1.8. THEOREM: Biuniversal arrows and biinitial objects. *If* $\mathbf{A}$ *has bicotensors* (1) *with* $\underline{2}$, *preserved by the 2-functor* U, *the object* (A_0, b_0) *is biinitial in* $[B_0 \downarrow U]$ *iff it is a biuniversal arrow from* B_0 *to* U.

PROOF. The pair (A_0, b_0) satisfies (BA.1, I). The proof of (BA.2) is based on the existence of bicotensors δ: $[\underline{2} \pitchfork A] \Rightarrow A$ (1.7) and their preservation by U: the cell $U\delta$: $U[\underline{2} \pitchfork A] \Rightarrow UA$ is a bicotensor product of UA with $\underline{2}$ in $\mathbf{B}$. Let be given the $\mathbf{A}$-cell:

(1) β: $Ua_1.b_0 \to Ua_2.b_0$: $B_0 \to UA$ $\quad (a_i$: $A_0 \to A)$;

by (BC.1) it factors as:

(2) $\beta = \sigma_2^{-1}(U\delta.b)\sigma_1$, $\quad \sigma_i$: $Ua_i.b_0 \simeq Ud_i.b$: $B_0 \to UA$ $\quad (i = 1,2)$,

for some b: $B_0 \to U[\underline{2} \pitchfork A]$:

(3)
$$\begin{tikzcd}
 & & UA \arrow[dd, equal] \\
B_0 \arrow[urr, "Ua_1 b_0"] \arrow[r, "b"] \arrow[drr, "Ua_2 b_0"'] \arrow[r, phantom, shift left=3, "\sigma_1\Downarrow"] \arrow[r, phantom, shift right=3, "\sigma_2\Uparrow"'] & U[\underline{2} \pitchfork A] \arrow[ur] \arrow[dr] \arrow[r, phantom, "\Downarrow U\delta"] & \\
 & & UA
\end{tikzcd}$$

Since (A_0, b_0) verifies (BA.1) there is some a: $A_0 \to [\underline{2} \pitchfork A]$ with $\mathbf{B}$-isocells:

(4) σ: $b \simeq Ua.b_0$: $B_0 \to U[\underline{2} \pitchfork A]$,

(5) $\sigma'_i = (Ud_i.\sigma)\sigma_i$: $Ua_i.b_0 \simeq U(d_i a).b_0$: $B_0 \to UA$ $\quad (i = 1,2)$.

Now, by (BA.I), there are unique isocells ρ_i of $\mathbf{A}$ verifying:

(6) ρ_i: $a_i \simeq d_i a$: $A_0 \to A$, $\quad U\rho_i.b_0 = \sigma'_i$ $\quad (i = 1,2)$.

Last, define the cell α of $\mathbf{A}$ as the vertical composition:

(7) $\alpha = (a_1 \xrightarrow{\rho_1} d_1 a \xrightarrow{\delta.a} d_2 a \xrightarrow{\rho_2^{-1}} a_2)$: $a_1 \to a_2$: $A_0 \to A$;

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(1) Assuming the stronger hypothesis of cotensor products (which indeed exist in all the examples we shall consider), the proof can be somewhat simplified; just replace the following isocells with identities: $\sigma_i, \sigma, \sigma'_i, \rho_{ir}, \sigma_{ir}, \sigma'$.

this solves our problem since:

$$(8)\qquad \begin{aligned} U\alpha.b_0 &= (U\rho_2^{-1})(U\delta.Ua)(U\rho_1).b_0 = (U\rho_2^{-1}.b_0)(U\delta.Ua.b_0)(U\rho_1.b_0) \\ &= \sigma'_2{}^{-1}(U\delta.Ua.b_0)\sigma'_1 = \sigma_2^{-1}(Ud_2.\sigma^{-1})(U\delta.Ua.b_0)(Ud_1.\sigma)\sigma_1 \\ &= \sigma_2^{-1}(Ud_2.\sigma^{-1})(Ud_2.\sigma)(U\delta.b)\sigma_1 \qquad \text{(exchange property in } \mathbf{B}) \\ &= \sigma_2^{-1}(U\delta.b)\sigma_1 = \beta. \end{aligned}$$

As to uniqueness, assume we have two $\mathbf{A}$-cells α_1, α_2 solving the same problem:

$$(9)\qquad \alpha_r\colon a_1 \to a_2\colon A_0 \to A, \qquad U\alpha_r.b_0 = \beta \qquad (r = 1,2).$$

By the biuniversal property of $\delta\colon [\underline{2}\Phi A] \Rightarrow A$ there are morphisms a'_r and isocells ρ_{ir} $(i,r = 1,2)$ such that

$$(10)\quad a'_r\colon A_0 \to [\underline{2}\Phi A], \quad \alpha_r = \rho_{2r}^{-1}(\delta.a'_r)\rho_{1r}, \quad \rho_{ir}\colon a_i \simeq d_i a'_r\colon A_0 \to A.$$

Thus

$$(11)\qquad \beta = U\alpha_r.b_0 = \sigma_{2r}^{-1}(U\delta.Ua'_r.b_0)\sigma_{1r},$$

$$(12)\qquad \sigma_{ir} = U\rho_{ir}.b_0\colon Ua_i.b_0 \simeq U(d_i a'_r).b_0\colon B_0 \to UA,$$

and, by (BC.2) for $U\delta$, there is a (unique) $\mathbf{B}$-isocell σ' such that:

$$(13)\qquad \sigma'\colon Ua'_1.b_0 \simeq Ua_2.b_0\colon B_0 \to UA, \qquad Ud_i.\sigma' = \sigma_{i2}\sigma_{i1}^{-1}.$$

By the biinitial property (BA.I) of (A_0,b_0), there is a unique $\mathbf{A}$-isomorphism ρ such that:

$$(14)\qquad \rho\colon a'_1 \to a'_2, \qquad U\rho.b_0 = \sigma'.$$

Now, by (14), (13) and (12):

$$(15)\qquad U(d_i.\rho).b_0 = Ud_i.\sigma' = \sigma_{i2}\sigma_{i1}^{-1} = U(\rho_{i2}\rho_{i1}^{-1}).b_0 \ ;$$

since the cells $d_i.\rho\colon a'_1 \to a'_2$ and $\rho_{i2}\rho_{i1}^{-1}\colon a'_1 \to a'_2$ are (vertically) parallel, again by (BA.I):

$$(16)\qquad d_i.\rho = \rho_{i2}\rho_{i1}^{-1} \qquad (i = 1,2).$$

Finally, by using the exchange axiom in $\mathbf{A}$, (16) and (10), we conclude that $\alpha_1 = \alpha_2$:

$$(17)\qquad \delta.\rho = (d_2.\rho)(\delta.a'_1) = (\rho_{22}\rho_{21}^{-1})(\rho_{21}\alpha_1\rho_{11}^{-1}) = \rho_{22}\alpha_1\rho_{11}^{-1},$$

$$(18)\qquad \delta.\rho = (\delta.a'_2)(d_1.\rho) = (\rho_{22}\alpha_2\rho_{12}^{-1})(\rho_{12}\rho_{11}^{-1}) = \rho_{22}\alpha_2\rho_{11}^{-1}.$$

1.9. THEOREM: birepresentations and biinitial objects. *If* $\mathbf{A}$ *has bicotensors with* $\mathbf{2}$, *preserved by the 2-functor* $T: \mathbf{A} \to \mathbf{CAT}$, *the object* (A_0, t_0) *is biinitial in* $[\mathbf{1}{\downarrow}T]$ *iff it is a birepresentation of* T.

1.10. In the last two sections of this chapter we shortly present the strict version of universal properties in 2-categories. A *2-representation* of the 2-functor $T: \mathbf{A} \to \mathbf{CAT}$ is an object A_0 of $\mathbf{A}$ provided with an *isomorphism* of 2-functors:

$$(1) \qquad \lambda: \mathbf{A}(A_0,-) \cong T: \mathbf{A} \to \mathbf{CAT},$$

in other words, by Yoneda, a pair (A_0, t_0) in $(1{\downarrow}T)$ ($t_0 = \lambda(\mathrm{id}A_0) \in \mathrm{Ob}\, TA_0$) such that:

(UR) for every $\mathbf{A}$-object A and every morphism $\tau: t_1 \to t_2$ in the category TA there is a unique $\mathbf{A}$-cell $\alpha: A_0 \Rightarrow A$ such that $\tau = T\alpha(t_0)$.

Equivalently, separating the aspects of dimension one and two, (A_0, t_0) has to verify (UR.1) below and (BR.2) of 1.3:

(UR.1) for each pair (A, t) in $(\mathbf{1}{\downarrow}T)$ there is a unique $a: A_0 \to A$ such that $t = Ta(t_0)$.

Any 2-representation (A_0, t_0) is a 2-initial object (0.5) of $(1{\downarrow}T)$. If $\mathbf{A}$ has cotensors with $\mathbf{2}$, preserved by T, these two facts are equivalent.

When existing, the 2-representation is determined up to (a unique) isomorphism of $\mathbf{A}$. Clearly any 2-representation is a birepresentation; as we shall see, the latter may exist when the former does not.

1.11. Analogously a *2-universal arrow* from the object B_0 of $\mathbf{B}$ to the 2-functor U is a pair $(A_0,\ b_0: B_0 \to UA_0)$ verifying:

(UA) for each $\mathbf{A}$-object A and each $\mathbf{B}$-cell $\beta: b_1 \to b_2: B_0 \to UA$ there is a unique $\mathbf{A}$-cell $\alpha: a_1 \to a_2: A_0 \to A$ such that $\beta = U\alpha.b_0$;

or equivalently, (UA.1) below and (BA.2) of 1.4:

(UA.1) for each pair $(A,\ b: B_0 \to UA)$ there is a unique $a: A_0 \to A$ such that $b = Ua.b_0$.

Any 2-universal arrow (A_0, b_0) from B_0 to U is a 2-initial object of $(B_0{\downarrow}U)$. If $\mathbf{A}$ has cotensors with $\mathbf{2}$, preserved by U, these two facts are equivalent: the proof is a much simplified version of 1.8, all isocells becoming identities.

A 2-representation of T: $\mathbf{A} \to \mathbf{CAT}$ is precisely a 2-universal arrow from the category $\underline{1}$ to T. Conversely, a 2-universal arrow from the object B_0 to the 2-functor U: $\mathbf{A} \to \mathbf{B}$ is just a 2-representation of the 2-functor:

(1) $$\mathbf{B}(B_0, U-): \mathbf{A} \to \mathbf{CAT}.$$

2. Bicompleteness.

Limits in 2-categories appear in various forms, depending on the laxification one allows on cones or, on the other hand, on the universal property.

The strict notion (2-universal strict cone), which we call here *2-limit*, has been studied in Street [S1]. We shall consider here the relaxed notion of *bilimit* (biuniversal pseudocone) and also the "mixed" notion of pseudolimit (2-universal pseudocone), studied e.g. in [S2], for which we give construction theorems (2.6 and 2.8). The other mixed notion, biuniversal strict cone, will be used only in two particular cases (2.7-8) where it happens to yield an equivalent, simpler formulation of a bilimit.

The "homogeneous" notions, 2-limits and bilimits, appear to yield the best results, e.g. as concerns construction theorems (see 2.1, 2.6, 2.8) or the existence of universal solutions (see Chap. 3). Pseudolimits, however, are useful as an intermediate step to bilimits.

Notice that the bilimits and pseudolimits we use here are *conical*, apart from particular cases as bicotensors products [$\underline{2}\pitchfork$A] (see 1.7). General *indexed* bilimits and pseudolimits are considered in [S2].

F: $\mathbf{\Delta} \to \mathbf{A}$ is always a 2-functor from a small 2-category $\mathbf{\Delta}$ into $\mathbf{A}$. A cell of $\mathbf{\Delta}$ will be typically written δ: $d \to d'$: $i \to j$.

2.1. For what regards conical and indexed 2-limits, we just recall here some results from [S1]. 2-products and 2-equalizers are defined by the 2-dimensional version (for cells) of the usual universal property. A basic example of conical 2-limit, having no one-dimensional analogue, is the *identifier*, that is the 2-limit of a single-cell diagram α: A $\Rightarrow$ A'. The basic non conical 2-limit is the cotensor product $\underline{2}\pitchfork$A (1.7).

The 2-category $\mathbf{A}$ is *conically 2-complete* (has all small conical 2-limits) whenever it has: small 2-products, 2-equalizers and identifiers (actually the proof of 2.6 shows that identifiers of *endo*cells

$\alpha: a \to a$ are sufficient). $\mathbf{A}$ is *2-complete* (has all small indexed 2-limits) iff it has small 2-products, 2-equalizers and cotensor products with $\mathbf{2}$. Analogous results hold for the preservation of 2-limits by 2-functors.

Another indexed 2-limit we shall use in the construction of pseudolimits is the *isoinserter* of parallel morphisms a_1, a_2: A $\to$ A'. This is a triple (X, x, χ) with x: X $\to$ A and $\chi: a_1 x \simeq a_2 x$: X $\to$ A', which is 2-universal with respect to this property.

2.2. We recall now the definition of conical bilimit and pseudolimit. Consider the diagonal 2-functor

(1) $$K: \mathbf{A} \to [\mathbf{\Delta}, \mathbf{A}],$$

into the 2-category $[\mathbf{\Delta}, \mathbf{A}]$ of 2-functors $\mathbf{\Delta} \to \mathbf{A}$, their *pseudo*transformations and modifications: KA is the constant functor at A.

A (conical) *bilimit* of the 2-functor F: $\mathbf{\Delta} \to \mathbf{A}$ is a birepresentation of the 2-functor

(2) $$[\mathbf{\Delta}, \mathbf{A}](K, F): \mathbf{A}^{op} \to \mathbf{CAT},$$

i.e., a family of equivalences of categories, natural for A in $\mathbf{A}^{op}$:

(3) $$\lambda A: \mathbf{A}(A, \text{bilim } F) \simeq [\mathbf{\Delta}, \mathbf{A}](KA, F).$$

Bilimits, when existing, are determined up to equivalence in $\mathbf{A}$. It may happen that the object bilim F can be chosen so that the equivalences (3) are in fact isomorphisms (hence a 2-representation of (2)): in this case we use the term *pseudolimit* and the notation pslim F; pseudolimits are determined up to a unique isomorphisms.

2.3. More explicitly, a *pseudocone* of the 2-functor F: $\mathbf{\Delta} \to \mathbf{A}$ is an object of the 2-category $(K{\downarrow}F)$:

(1) $(X,\ x^{\sim}: KX \to F: \mathbf{\Delta} \to \mathbf{A})$ ($x^{\sim}$ is a pseudotransformation),

i.e., (0.6) a system $(X, (x_i), (\chi_d))$ indexed on the objects i and the arrows $d: i \to j$ of $\mathbf{\Delta}$:

(2) $$x_i: X \to Fi, \qquad \chi_d: x_j \simeq Fd.X_i: X \to Fj,$$

verifying the coherence conditions (PT.1-3) in 0.6, concerning respectively the identical arrows, the pairs of composable arrows and the cells of $\mathbf{\Delta}$.

The bilimit of the 2-functor F: $\mathbf{\Delta} \to \mathbf{A}$ is any biuniversal arrow from K to the object F of $[\mathbf{\Delta},\mathbf{A}]$; in particular, it is a biinitial object of $(K\downarrow F)$.

Therefore it is characterized as a pseudocone bilim F = $(X,x^{\sim})$ of F such that:

(BL.1) for each pseudocone $(A, a^{\sim})$, $a^{\sim}$: KA $\to$ F, there exists *some* morphism a: A $\to$ X such that $a^{\sim} \simeq x^{\sim}.Ka$,

(BL.2) for all a_1,a_2: A $\to$ X and each modification $\alpha^{\sim}$: $a_1^{\sim} \to a_2^{\sim}$: KA $\to$ F where $a_r^{\sim} = x^{\sim}.Ka_r$, there exists a unique cell α: $a_1 \to a_2$ such that $\alpha^{\sim} = x^{\sim}.K\alpha$.

The pseudocone $(X,x^{\sim})$ is a pseudolimit of F if it verifies (PL.1) below and (BL.2):

(PL.1) for each pseudocone $(A, a^{\sim})$, $a^{\sim}$: KA $\to$ F, there exists a *unique* morphism a: A $\to$ X such that $a^{\sim} = x^{\sim}.Ka$.

2.4. The 2-category $\mathbf{A}$ is said to be *conically bicomplete* whenever it has all (small) conical bilimits. We say it is *arrow-bicomplete* (or simply *bicomplete*) if moreover it admits bicotensor products with $\underline{2}$. We recall ([S2], 1.24) that $\mathbf{A}$ has all small indexed bilimits if it is conically bicomplete and has bicotensors $\underline{C}\pitchfork A$ for *each* small category $\underline{C}$; this stronger notion of bicompleteness will not be used here.

A 2-functor U: $\mathbf{A} \to \mathbf{B}$ will be said to be *conically bicontinuous (arrow-bicontinuous*, or simply *bicontinuous*) if it preserves conical bilimits (plus bicotensor products with $\underline{2}$) in so far as they exist in $\mathbf{A}$.

Basic conical bilimits, generating the others, will be considered in the sections 2.7-8.

2.5. Analogously we consider *(conical* or *arrow) pseudocompleteness* and *pseudocontinuity*. *Pseudoproducts* coincide with 2-products and *pseudocotensors* with cotensors; instead *pseudoequalizers* and *pseudo-identifiers* are generally distinct from 2-equalizers and identifiers (e.g., in **CAT**).

We give below (2.6) a construction theorem of conical pseudo-limits from *2-products, isoinserters* and *identifiers of endocells* (which will be shown to exist in various cases lacking equalizers). It should be noticed that isoinserters and identifiers are *not* conical pseudolimits (while they are indexed 2-limits). Thus our sufficient condition *seems* not to be necessary. However, if in our construction isoinserters are replaced by pseudoequalizers and identifiers by pseudoidentifiers, one gets an object which, even for **A** = **CAT**, is *equivalent*, generally *not* isomorphic, to the pseudolimit. Loosely speaking, this happens because pseudoequalizers and pseudoidentifiers *introduce too many isocells*. Thus the above pseudolimits *seem* not sufficient to build (together with 2-products = pseudoproducts) all conical pseudolimits.

Notice also that the construction of 2-limits given in Street [S1] from 2-products, 2-equalizers and identifiers cannot be transferred to pseudolimits (or bilimits): e.g., in **CAT** it would yield a category not even equivalent to the pseudolimit, as soon as the 2-category **Δ** has different cells having the same vertical domain (or the same vertical codomain).

2.6. THEOREM: construction of conical pseudolimits. *If the 2-category* **A** *has (small) 2-products, isoinserters and identifiers of endocells then it is conically pseudocomplete. In such a case a 2-functor* U: **A** → **B** *which preserves the above limits is pseudocontinuous.*

PROOF. We just prove the assertion concerning the existence of pseudolimits, as the preservation property follows straightforwardly from the construction argument. Let be given the 2-functor F: **Δ** → **A**.

a) *Preliminary constructions.* Consider the 2-products:

$$(1) \qquad p_i\colon \mathrm{P} = \prod_i \mathrm{F}i \to \mathrm{F}i, \quad q_d\colon \mathrm{Q} = \prod_d \mathrm{F}j \to \mathrm{F}j,$$

$$(2) \qquad m_\delta\colon \mathrm{M} = \prod_\delta \mathrm{F}j \to \mathrm{F}j, \quad n_{d,d^-}\colon \mathrm{N} = \prod_{d,d^-} \mathrm{F}k \to \mathrm{F}k,$$

where (as always in the following) i varies over the objects of **Δ**, $d\colon i \to j$ varies over the arrows of **Δ**, $\delta\colon d \to d'\colon i \to j$ varies over the cells of **Δ** and the pair (d,d^-) over the set of composable arrows $d\colon i \to j$, $d^-\colon j \to k$ of **Δ**; in the last case we write $d^\sim = d^-d\colon i \to k$ the composite arrow.

As a first approach to the pseudolimit of F, consider the isoinserter (X,x,χ):

$$(3) \qquad \mathrm{X} \xrightarrow{x} \mathrm{P} \underset{}{\overset{a,b}{\rightrightarrows}} \mathrm{Q}, \quad \chi\colon ax \simeq bx\colon \mathrm{X} \to \mathrm{Q}$$

of the morphisms a,b having the following d-components:

(4) $$q_d a = p_j\colon \Pi \mathrm{F}i \to \mathrm{F}j, \qquad q_d b = \mathrm{F}d.p_i\colon \Pi\mathrm{F}i \to \mathrm{F}j.$$

Now, the object X has a system of maps $p_i x\colon \mathrm{X} \to \mathrm{F}i$ and isocells $q_d\chi\colon p_j x \sim \mathrm{F}d.p_i x\colon\ \mathrm{X} \to \mathrm{F}j$ which, generally, is *not* a pseudocone of F. In order to force the three conditions (PT.1-3) for the coherence of the pseudocones of F (2.3, 0.6), with respect to the identical morphisms, the composition of morphisms, and the cells of $\mathbf{\Delta}$, we introduce three endocells $\rho\colon \mathrm{X} \Rightarrow \mathrm{P}$, $\sigma\colon \mathrm{X} \Rightarrow \mathrm{N}$, $\tau\colon \mathrm{X} \Rightarrow \mathrm{M}$:

(5) $$\rho\colon x \to x\colon \mathrm{X} \to \mathrm{P},$$

(6) $$p_i\rho = q_{1_i}\chi, \qquad\qquad \text{for all } i \text{ in } \mathbf{\Delta},$$

$$q_{1_i}\chi\colon p_i x = q_{1_i} ax \to q_{1_i} bx = \mathrm{F}1_i.p_i x = p_i x,$$

(7) $$\sigma\colon s \to s\colon \mathrm{X} \to \mathrm{N},$$

(8) $$n_{d,d^-}s = p_k x \qquad (\text{for all } d\colon i \to j \text{ and } d^-\colon j \to k \text{ in } \mathbf{\Delta},\ d^\sim = d^- d)$$

(9) $$n_{d,d^-}\sigma = (q_{d^\sim}\chi)^{-1}(\mathrm{F}d^-.q_d\chi)(q_{d^-}\chi),$$

(10)
$$\begin{tikzcd}
n_{d,d^-}s = p_k x = q_{d^\sim}ax \arrow[r, "q_{d^-}\chi"] \arrow[d, "n_{d,d^-}\sigma"] & q_{d^-}bx = \mathrm{F}d^-.p_j x = \mathrm{F}d^-.q_d ax \arrow[d, "\mathrm{F}d^-.q_d\chi"] \\
n_{d,d^-}s = p_k x = q_{d^\sim}ax \arrow[r, "q_{d^\sim}\chi"] & q_{d^\sim}bx = \mathrm{F}d^\sim.p_i x = \mathrm{F}d^-\mathrm{F}d.p_i x = \mathrm{F}d^-.q_d bx
\end{tikzcd}$$

(11) $$\tau\colon t \to t\colon \mathrm{X} \to \mathrm{M},$$

(12) $$m_\delta t = p_j x \qquad (\text{for all } \delta\colon d \to d'\colon i \to j \text{ in } \mathbf{\Delta}),$$

(13) $$m_\delta\tau = (q_{d'}\chi)^{-1}(\mathrm{F}\delta.p_i x)(q_d\chi),$$

(14)
$$\begin{tikzcd}
m_\delta t = p_j x = q_d ax \arrow[r, "q_d\chi"] \arrow[d, "m_\delta\tau"] & q_d bx = \mathrm{F}d.p_i x \arrow[d, "\mathrm{F}_\delta.p_i x"'] \\
m_\delta t = p_j x = q_{d'}ax \arrow[r, "q_{d'}\chi"] & q_d bx = \mathrm{F}d'.p_i x
\end{tikzcd}$$

b) *The pseudolimit.* Let (Y,y) be the *identifier* of these three endocells ρ, σ, τ:

(15) $$y\colon \mathrm{Y} \to \mathrm{X}, \quad \rho y = 1, \quad \sigma y = 1, \quad \tau y = 1,$$

i.e., the identifier of the endocell $(\rho,\sigma,\tau)\colon \mathrm{X} \Rightarrow \mathrm{P}\times\mathrm{N}\times\mathrm{M}$.

We want to prove that the pseudolimit of F is Y with pseudocone

(16) $$y_i = p_i xy\colon \mathrm{Y} \to \mathrm{F}i,$$

(17) $$\gamma_d = q_d\chi y\colon\ y_j \simeq Fd.y_i,$$

(18) $$y_j = p_j xy = q_d axy \xrightarrow{q_d\chi y} q_d bxy = Fd.p_i xy = Fd.y_i,$$

where the coherence of the isocells γ_d comes from (15) and from the definition of the cells ρ, σ, τ; e.g., we check the property (PT.3) of coherence, with respect to the cell δ of Δ:

(19) $$(F\delta.y_i)\gamma_d = (F\delta.p_i xy)(q_d\chi y) = ((F\delta.p_i x)(q_d\chi)).y = ((q_{d'}\chi)(m_\delta.\tau)).y = (q_{d'}\chi y)(m_\delta.\tau y) = (q_{d'}\chi y)1 = q_{d'}\chi y = \gamma_{d'}\ .$$

c) In order to prove the *1-dimensional universal property*, let be given a pseudocone $(Z,(z_i),(\omega_d))$ of F:

(20) $$z_i\colon Z \to Fi, \qquad \omega_d\colon z_j \simeq Fd.z_i\colon Z \to Fj\ ;$$

we have to verify that it factors uniquely through $(Y,(y_i),(\gamma_d))$. As to the *existence*, the map

(21) $$z\colon Z \to P = \Pi Fi, \qquad p_i z = z_i,$$

"inserts an isocell ω between a and b ", defined by:

(22) $$\omega\colon az \simeq bz\colon Z \to Q, \qquad q_d\omega = \omega_d\ ,$$
$$q_d az = p_j z = z_j \longrightarrow Fd.z_i = Fd.p_i z = q_d bz,$$

therefore (z,ω) factors uniquely through the isoinserter (X,x,χ) of a and b:

(23) $$z'\colon Z \to X, \qquad z = xz', \qquad \omega = \chi z'.$$

Now the map $z'\colon Z \to X$ "identifies" the endocells ρ, σ, τ; e.g., $\tau z' = 1$ because for any δ in Δ:

(24) $$m_\delta.\tau z' = ((q_{d'}\chi)^{-1}(F\delta.p_i x)(q_d\chi))z'$$
$$= (q_{d'}\chi z')^{-1}(F\delta.p_i xz')(q_d\chi z') = (q_{d'}\omega)^{-1}(F\delta.p_i z)(q_d\omega) = \omega_{d'}^{-1}(F\delta.z_i)\omega_d =$$
$$1,$$

the last equality being the coherence of the pseudocone $(Z,(z_i),(\omega_d))$ of F with respect to the cell δ. Thus z' factors uniquely through the identifier (Y,y):

(25) $$z''\colon Z \to Y, \qquad z' = yz'',$$

and this map z'' solves our problem, i.e., composed with $(Y,(y_i),(\gamma_d))$ gives back the pseudocone $(Z,(z_i),(\omega_d))$:

(26)
$$y_i z'' = p_i x y z'' = p_i x z' = p_i z = z_i,$$
$$\gamma_d z'' = q_d \chi y z'' = q_d \chi z' = q_d z = \omega_d.$$

d) As to the *uniqueness* of the factorization, if also $z^{\sim}$ solves the problem:

(27)
$$z^{\sim}\colon Z \to Y, \qquad z_i = y_i z^{\sim}, \qquad \omega_d = \gamma_d z^{\sim},$$

then

$$p_i(xyz^{\sim}) = y_i z^{\sim} = z_i = p_i(xyz'')$$

for all i, hence $xyz^{\sim} = xyz''$; moreover $\chi y z^{\sim} = \chi y z''$, since for all d :

$$q_d(\chi y z^{\sim}) = q_d \chi y . z^{\sim} = \gamma_d z^{\sim} = \omega_d = q_d(\chi y z'') \;;$$

thus, by the isoinserter property, $yz^{\sim} = yz''$. By the identifier property, $z^{\sim} = z''$.

e) Finally we have to check the *factorization property for cells*. Let:

(28)
$$f,g\colon Z \to Y, \qquad \pi_i\colon y_i f \to y_i g\colon Z \to Fi,$$

be a modification from the pseudocone $(Z,(y_i f),(\gamma_d f))$ into $(Z,(y_i g),(\gamma_d g))$. By the product property of P there is a unique $\pi\colon xyf \to xyg\colon Z \to P$ such that $p_i \pi = \pi_i$ for all i. This π is coherent with the isoinserter (X,x,χ), i.e., the following square of **A**-cells commutes:

(29)
$$\begin{tikzcd}
axyf \arrow[r, "a\pi"] \arrow[d, "\chi yf"] & axyg \arrow[d, "\chi yg"'] \\
bxyf \arrow[r, "b\pi"] & bxyg
\end{tikzcd}$$

as it follows by composing with the projections q_d :

(30) $$q_d((\chi y g)(a\pi)) = (q_d \chi y g)(q_d a\pi) = (\gamma_d\, g)(p_j \pi) = (\gamma_d g)\pi_j \;,$$
(31) $$q_d((b\pi)(\chi y f)) = (q_d b\pi)(q_d \chi y f) = (Fd.p_i\pi)(\gamma_d f) = (Fd.\pi_i)(\gamma_d f),$$

and applying the coherence of (π_i). Thus there is a unique cell ψ such that:

(32) $$\psi\colon yf \to yg\colon Z \to X, \qquad x\psi = \pi.$$

Now, by the identifier property, there is a unique cell ϕ verifying

(33) $$\phi\colon f \to g\colon Z \to Y, \qquad y\phi = \psi.$$

This ϕ solves our problem.

(34) $$y_i\phi = p_i xy\phi = p_i x\psi = p_i\pi = \pi_i,$$

and it is easy to see that it is the unique solution.

2.7. Much in the same way as in the strict case (2.1), we shall see below (2.8) that the *basic* conical bilimits, generating the others, are *biproducts* (the bilimit of a discrete diagram), *biequalizers* (the bilimit of two parallel arrows) and *bi-endoidentifiers* (the bilimit of an endocell, in the sense specified below).

Given a pair of parallel morphisms a_1, a_2: A → A', it is easy to see that the problem of their biequalizer $(X;x,x';\chi_1,\chi_2)$:

(1) $$x\colon X \to A, \quad x'\colon X \to A', \quad \chi_i\colon x' \simeq a_i x\colon X \to A' \qquad (i = 1,2),$$

is equivalent to the simpler problem of "inserting biuniversally an isocell", i.e., finding a *biuniversal (strict) isoinserter*

(2) $$x\colon X \to A, \quad \chi\colon a_1 x \simeq a_2 x\colon X \to A' \qquad (i = 1,2).$$

Indeed, if (1) is a biequalizer then $(X;x;\chi_2\chi_1^{-1})$ solves the simpler problem, while if (2) does so then $(X;x,a_1x;1,\chi)$ is a biequalizer.

As concerns the bi-endoidentifier of the endocell:

(3) $$\alpha\colon a \to a\colon A \to A',$$

this is defined to be the bilimit of the diagram (3) on the 2-graph $\mathbf{\Delta}$:

(4) $$\delta\colon d \to d\colon 0 \to 1,$$

(i.e., of the 2-functor F: $\mathbf{\Delta} \to \mathbf{A}$ generated by this diagram). This amounts to finding a biuniversal pseudocone $(X;x,x';\chi)$:

(5) $$x\colon X \to A, \quad x'\colon X \to A', \quad \chi\colon x' \simeq ax\colon A \to A', \quad (\alpha x)\chi = \chi,$$

where the last condition (coherence of the pseudocone) is clearly equivalent to $\alpha x = 1_{ax}$. Also here there is an equivalent, simpler formulation: to find a *biuniversal strict identifier*

(6) $$x\colon X \to A, \quad \alpha x = 1.$$

Last, it should be noticed that the *bi-identifier* of the cell (3), i.e., the bilimit of the endocell α *as* a diagram on the 2-graph

(7) $$\delta\colon d_1 \to d_2\colon 0 \to 1,$$

is *not* equivalent to the above bilimit, and not suitable for the following construction theorem.

2.8. THEOREM: construction of conical bilimits. *The 2-category* $\mathbf{A}$ *is conically bicomplete iff it has: (small) biproducts, biequalizers, bi-endoidentifiers (2.7). In such a case a 2-functor* U: $\mathbf{A} \to \mathbf{B}$ *is conically bicontinuous iff it preserves the above bilimits.*

PROOF. By the preceding arguments (2.7) the proof can be obtained from the one concerning pseudolimits (2.6), by replacing some equalities of arrows in the formulae (4),(6),(8),... with isocells and working out the new coherence conditions.

2.9. COROLLARY: construction of arrow-bilimits. *The 2-category* $\mathbf{A}$ *is (arrow) bicomplete iff it has: (small) biproducts, biequalizers, bi-endoidentifiers (2.7), bicotensors with* $\underline{2}$. *In such a case a 2-functor* U: $\mathbf{A} \to \mathbf{B}$ *is (arrow) bicontinuous iff it preserves the above bilimits.*

3. Solution Sets for biuniversal problems.

In this section we extend the Initial Object Theorem to the 2-dimensional case and derive existence theorems for biuniversal arrows and birepresentations.

3.1. Biinitial Object THEOREM. *A conically bicomplete 2-category* $\mathbf{A}$ *has a biinitial object iff:*

Solution Set Condition: there exists a small bifull (0.5) sub-2-category $\mathbf{H}$ *co-initial* in $\mathbf{A}$ (i.e., for each A in $\mathbf{A}$ there is some morphism a: H $\to$ A with H in $\mathbf{H}$).

PROOF. Necessity: if I is biinitial the 2-subcategory $\mathbf{H}$ of $\mathbf{A}$ whose unique cell is $1\colon 1 \to 1\colon \mathrm{I} \to \mathrm{I}$ satisfies our condition, since any endomorphism $i\colon 1 \to \mathrm{I}$ has a unique isocell $\rho\colon i \to 1$. Notice that the *2-full* subcategory generated by I may be large.

Conversely, assume that the Solution Set Condition holds and let (I,p) be the (conical) bilimit of the inclusion $\mathrm{F}\colon \mathbf{H} \to \mathbf{A}$. We shall prove that I is biinitial in $\mathbf{A}$.

a) The pseudocone $p\colon \mathrm{KI} \to \mathrm{F}$ will be written

$$(1) \qquad p\mathrm{H}\colon \mathrm{I} \to \mathrm{H} \quad (\text{for H in } \mathbf{H}), \qquad ph\colon p\mathrm{H}' \simeq h.p\mathrm{H}\colon \mathrm{I} \to \mathrm{H}' \quad (\text{for } h\colon \mathrm{H} \to \mathrm{H}' \text{ in } \mathbf{H}),$$

where the isocells ph satisfy the coherence conditions (PT.1-3) (2.3, 06).

It can be proved, by a tedious pasting argument, that the pseudocone (I,p) may be extended to the (possibly non-small) 2-full 2-subcategory $\mathbf{H}'$ generated by $\mathbf{H}$, remaining the bilimit of the inclusion of $\mathbf{H}'$ in $\mathbf{A}$. Therefore *we assume from now on that* $\mathbf{H}$ *is 2-full in* $\mathbf{A}$.

b) For every A there is some map $\mathrm{I} \to \mathrm{A}$, namely the composition $a.p\mathrm{H}$ where $a\colon \mathrm{H} \to \mathrm{A}$ is a morphism of $\mathbf{A}$ starting from some H in $\mathbf{H}$, existing by hypothesis. We want to prove that the maps $\mathrm{I} \to \mathrm{A}$ are determined "up to a unique isomorphism".

c) If $i = a^{-}.p\mathrm{H}^{-}\colon \mathrm{I} \to \mathrm{I}$ is obtained as above, then $i \simeq 1$. Indeed, consider the pseudocone $p' = p.\,\mathrm{K}i$ given by:

$$(2) \qquad p'\mathrm{H} = p\mathrm{H}.i\colon \mathrm{I} \to \mathrm{H}, \quad p'h = ph.i\colon p\mathrm{H}'i \simeq h.p\mathrm{H}.i\colon \mathrm{I} \to \mathrm{H}'.$$

There is an isomodification $\pi\colon p \to p'$ of components:

$$(3) \qquad \pi\mathrm{H} = ph^{-}\colon p\mathrm{H} \simeq p\mathrm{H}.i\colon \mathrm{I} \to \mathrm{H} \qquad (\text{where } h^{-} = p\mathrm{H}\cdot a^{-}\colon \mathrm{H}^{-} \to \mathrm{H})$$

$$(4) \qquad \begin{tikzcd} \mathrm{I} \arrow[r, "p\mathrm{H}^{-}"] \arrow[d, equal] \arrow[dr, phantom, "\overset{ph^{-}}{\to}"] & \mathrm{H}^{-} \arrow[d, "h^{-}"] \\ \mathrm{I} \arrow[r, "p\mathrm{H}"'] & \mathrm{H} \end{tikzcd} \qquad (h^{-}.p\mathrm{H}^{-} = p\mathrm{H}.a^{-}.p\mathrm{H}^{-} = p\mathrm{H}.i)$$

whose coherence we check in d). Thus, by (BA.I), there exists a unique isomorphism $\rho\colon 1 \to i$ such that $\pi = p.\mathrm{K}\rho$.

d) Coherence of π. Let $h\colon \mathrm{H} \to \mathrm{H}'$; according to (MD) in 0.6, we must check that

(5) $$(h.\pi H)ph = (ph.i)\pi H' ;$$
since
$$ph.i = ph.a^{-}.pH^{-} \quad \text{and} \quad \pi H' = ph'^{-},$$

where $h'^{-} = pH'.a^{-}$: $H^{-} \to H'$, the conclusion follows from the following application of the coherence of p (where $\theta = ph.a^{-}$: $pH'.a^{-} \simeq h.pH.a^{-}$, i.e., θ: $h'^{-} \simeq hh^{-}$ in $\mathbf{H}$):

(6) $$\begin{tikzcd} I \arrow[r, "pH^{-}"] \arrow[d, equal] \arrow[dr, phantom, "ph^{-}\ \to"] & H^{-} \arrow[d, "h^{-}"] \\ I \arrow[r, "pH"] \arrow[d, equal] \arrow[dr, phantom, "ph\ \to"] & H \arrow[d, "h"] \\ I \arrow[r, "pH'"] & H' \end{tikzcd} \quad = p(hh^{-}) = \quad \begin{tikzcd} I \arrow[r, "pH^{-}"] \arrow[d, equal] \arrow[dr, phantom, "ph'^{-}\ \to"] & H^{-} \arrow[d, bend right, "h'^{-}"'] \arrow[d, bend left, "hh^{-}"] \arrow[d, phantom, "\theta\ \to"] \\ I \arrow[r, "pH'"] & H' \end{tikzcd}$$

e) Any two morphisms a_r: $I \to A$ $(r = 1,2)$ are isomorphic. Take the biequalizer e: $L \to I$ of a_1 and a_2 in $\mathbf{A}$ (hence $a_1 e \simeq a_2 e$) and insert some map $a = (I \to H \to L)$:

(7) $$I \longrightarrow H \longrightarrow L \longrightarrow I \rightrightarrows A''$$

since $i = ae \simeq 1$ by d,

$$a_1 \simeq a_1.i = a_1 ea \simeq a_2 ea = a_2.i \simeq a_2.$$

f) Any two parallel cells from I:

(8) $$\alpha,\ \alpha'\colon a_1 \to a_2\colon I \to A$$

coincide. Indeed, consider the bilimit L of the (conical) diagram (8) of $\mathbf{A}$

(9) e: $L \to I$, e': $L \to A$, ρ_r: $e' \simeq a_r e$, $\rho_2 = (\alpha.e)\rho_1$, $\rho_2 = (\alpha'.e)\rho_1$.

Thus $\alpha.e = \rho_2\rho_1^{-1} = \alpha'.e$. Choose now some morphism a: $I \to L$:

(10) $$I \xrightarrow{a} L \xrightarrow{e} I \overset{\alpha,\alpha'}{\rightrightarrows} A ,$$

by e there exists an isomorphism ρ: $1 \simeq i$ $(i = ea$: $I \to I)$: since $\alpha.i = \alpha'.i$ the conclusion $\alpha = \alpha'$ follows from the following lemma.

3.2. LEMMA. In the 2-category $\mathbf{A}$ let be given the cells

(1) $\bullet \xRightarrow{\rho} \bullet \xRightarrow{\alpha,\beta} \bullet$ $\qquad \rho\colon a \to b, \quad \alpha,\beta\colon c \to d.$

If ρ is iso and $\alpha.a = \beta.a$ then $\alpha.b = \beta.b$.

PROOF. By the exchange axiom

(2) $\alpha.\rho = (d\alpha).(\rho a) = (d.\rho)(\alpha.a) = (d.\rho)(\beta.a) = (d\beta).(\rho a) = \beta.\rho$;

analogously $\alpha.\rho^{-1} = \beta.\rho^{-1}$. Thus:

(3) $\alpha.b = \alpha.(\rho\rho^{-1}) = (\alpha\rho).(\alpha\rho^{-1}) = (\beta\rho).(\beta\rho^{-1}) = \beta.b.$

3.3. THEOREM. *If* $1\colon \mathbf{A} \to \mathbf{A}$ *has a conical bilimit* (I, p) *then the object* I *is biinitial in* $\mathbf{A}$.

PROOF. Consider the proof of the previous theorem (3.1) and remark that, if the embedding $\mathbf{H} \to \mathbf{A}$ is assumed to have a bilimit, both the hypotheses of smallness of $\mathbf{H}$ and of conical bicompleteness of $\mathbf{A}$ can be obviously dropped.

3.4. Biuniversal Arrow THEOREM. *If* $\mathbf{A}$ *is arrow-bicomplete and the 2-functor* U *is arrow-bicontinuous, the biuniversal arrow from* B_0 *to* U *exists iff:*

Solution Set Condition: there exists a small bifull co-initial sub-2-category of $[\mathrm{B}_0{\downarrow}\mathrm{T}]$.

PROOF. One proves in the standard way that $[\mathrm{B}_0{\downarrow}\mathrm{U}]$ is conically bicomplete, deduces the existence of the biinitial object from 3.1, and the existence of the biuniversal arrow from 1.8.

3.5. Birepresentation THEOREM. *If* $\mathbf{A}$ *is arrow-bicomplete,* T *is birepresentable iff it is arrow-bicontinuous and:*

Solution Set Condition: there exists a small bifull co-initial sub-2-category of $[\mathbf{1}{\downarrow}\mathrm{T}]$.

PROOF. From 3.4, as a birepresentation of T is just a biuniversal arrow from $\mathbf{1}$ to T.

3.6. The strict version of these results is obvious. Let $\mathbf{A}$ be *2-complete.*

a) $\mathbf{A}$ has a 2-initial object iff it has a small 2-full co-initial sub-2-category.

b) If the 2-functor $U\colon \mathbf{A} \to \mathbf{B}$ is 2-continuous, there is a 2-universal arrow from B_0 to U iff there exists a small 2-full co-initial sub-2-category of $(B_0{\downarrow}U)$.

c) The 2-functor $T\colon \mathbf{A} \to \mathbf{CAT}$ is 2-representable iff it is 2-continuous and there exists a small 2-full co-initial sub-2-category of $(\underline{1}{\downarrow}T)$.

3.7. Notice that *there is no pseudo-version of the above statements*, in the sense that pseudolimits yield no stronger results than bi-limits.

Indeed, consider the 2-category **AB** of abelian categories, exact functors and natural transformations. **AB** is clearly arrow-pseudo-complete, hence arrow-bicomplete, and not 2-complete (it lacks equalizers). **AB** has a biinitial object, the category $\underline{1}$, but *no* pseudoinitial (= 2-initial) or even initial one, although the 2-full sub-2-category of **AB** generated by $\underline{1}$ provides a solution set (is small and co-initial).

Part II. THEORIES

4. Concrete theories and universal models.

From now on, Δ is a small graph and $\mathbf{A}$ is a concrete 2-category with structural functor $|\ |\colon \mathbf{A} \to \mathbf{CAT}$; a cell in $\mathbf{A}$ is typically written $\alpha\colon a_1 \to a_2\colon \underline{A} \to \underline{A}'$ or also $\alpha\colon \underline{A} \Rightarrow \underline{A}'$.

4.1. The graph Δ determines a canonical extension $\mathbf{A}_\Delta$ of the concrete 2-category $\mathbf{A}$. Add one *object*, Δ itself; the following *morphisms*

$$(1) \qquad 1\colon \Delta \to \Delta, \qquad t\colon \Delta \to \underline{A}$$

where $\underline{A}$ is in $\mathbf{A}$ and t is any graph morphism $\Delta \to |\underline{A}|$ [1]; the following cells

----

[1] More precisely, t can be thought of as the pair $(\underline{A}, |t|\colon \Delta \to |\underline{A}|)$ where $|t|$ is a graph morphism; analogously $\tau = (\underline{A}, |\tau|)$.

(2) $$1\colon 1 \to 1\colon \Delta \to \Delta, \qquad \tau\colon t_1 \to t_2\colon \Delta \to \underline{A}\ ,$$

where τ is any natural transformation of graph morphisms [1]. The horizontal and vertical compositions obviously extend (even if transformations of graph morphisms do *not* have a general horizontal composition).

The inclusion

(3) $$U\colon \mathbf{A} \to \mathbf{A}_\Delta$$

is a 2-full 2-functor. The 2-functor $|\ |$ of $\mathbf{A}$ extends to

(4) $$|\ |\colon \mathbf{A}_\Delta \to \mathbf{CAT}_\Delta$$

sending Δ into itself and acting on the new morphisms t and the new cells τ as indicated in footnote [1] page 26.

Thus, the graph Δ determines the 2-functor

(5) $$T_\Delta = \mathbf{A}_\Delta(\Delta,-)\colon \mathbf{A} \to \mathbf{CAT},$$

to be called the *total* $\mathbf{A}$*-theory* on Δ. This is a first example of an $\mathbf{A}$-theory on Δ, in the sense of the following definition.

4.2. DEFINITION. A *(concrete) theory* T on the graph Δ, with values in the concrete 2-category $\mathbf{A}$ (briefly: an $\mathbf{A}$-theory on Δ) will be any 2-subfunctor $T\colon \mathbf{A} \to \mathbf{CAT}$ of the total theory T_Δ such that, for any $\underline{A}$ in $\mathbf{A}$, $T\underline{A}$ is a full subcategory of $T_\Delta(\underline{A}) = \mathbf{A}_\Delta(\Delta,\underline{A})$.

In other words T associates to every object $\underline{A}$ of $\mathbf{A}$ a (generally non small) set $T\underline{A}$ of morphisms $t\colon \Delta \to \underline{A}$ in $\mathbf{A}_\Delta$ (the *models* of T in $\underline{A}$), so that

(T.0) if $t \in T\underline{A}$ and $a\colon \underline{A} \to \underline{A}'$ is in $\mathbf{A}$, then $t' = a.t \in T\underline{A}'$.

A cell $\tau\colon t_1 \to t_2\colon \Delta \to \underline{A}$ of $\mathbf{A}_\Delta$, with $t_i \in T\underline{A}$, will be called a *transformation of models*. The theory T will be said to be *replete* if every morphism $t'\colon \Delta \to \underline{A}$ isomorphic to a model $t \in T(\underline{A})$ is a model.

Two theories $T_1,T_2\colon \mathbf{A} \to \mathbf{CAT}$, possibly based on different graphs, will be said to be *isomorphic* whenever they are so in the category $(\mathbf{A},\mathbf{CAT})$, and *equivalent* if they are so in the 2-category $[\mathbf{A},\mathbf{CAT}]$.

4.3. Each theory T determines a 2-category $\mathbf{A}_T$ verifying $\mathbf{A} \subset \mathbf{A}_T \subset \mathbf{A}_\Delta$, containing the object Δ and locally full in $\mathbf{A}_\Delta$: take $\mathbf{A}_T(\Delta,\underline{A}) = T(\underline{A})$; conversely, any such 2-category determines the theory $T = \mathbf{A}_T(\Delta,-)\colon \mathbf{A} \to \mathbf{CAT}$. The 2-functors 4.1-4.3 restrict to

(1) $U_T\colon \mathbf{A} \to \mathbf{A}_T, \quad |\ |\colon \mathbf{A}_T \to \mathbf{CAT}_\Delta.$

The theory T determines also the 2-category of models

$$\mathbf{Mod}(T) = (\underline{1}{\downarrow}T) = (\Delta{\downarrow}U_T):$$

the *objects* are all the models t (in any $\mathbf{A}$-object); a *cell* $\alpha\colon a \to a'\colon t_1 \to t_2$ between models $t_i \in T(\mathbb{A}_i)$ is given by any $\mathbf{A}$-cell $\alpha\colon \mathbb{A}_1 \Rightarrow \mathbb{A}_2$ such that $\alpha.t_1 = t_2$. This category is provided with the forgetful structural 2-functor $\mathbf{Mod}(T) \to \mathbf{A}$ taking $t\colon \Delta \to \mathbb{A}$ into $\mathbb{A}$.

Analogously (see 1.1), we have the *2-category of models and pseudotransformations*

$$\mathbf{Mod}[T] = [\underline{1}{\downarrow}T] = [\Delta{\downarrow}U_T]:$$

the *objects* are as above, a *morphism* $(a,\sigma)\colon t_1 \to t_2$ between models $t_i \in T(\mathbb{A}_i)$ is given by an $\mathbf{A}$-morphism $a\colon \mathbb{A}_1 \to \mathbb{A}_2$ together with an isocell $\sigma\colon t_2 \simeq at_1\colon \Delta \to \mathbb{A}_2$; a *cell* $\alpha\colon (a,\sigma) \to (a',\sigma')\colon t_1 \to t_2$ is an $\mathbf{A}$-cell $\alpha\colon a \to a'\colon \mathbf{A}_1 \to \mathbf{A}_2$ such that

$$\sigma' = (\alpha.t_1)\sigma\colon t_2 \to a.t_1\colon B_0 \to UA'.$$

Notice that the 2-categories $\mathbf{Mod}(T)$ and $\mathbf{Mod}[T]$ do not contain the transformations of models.

4.4. The concrete $\mathbf{A}$-theories on Δ are *ordered* by pointwise inclusion $T \leqslant T'$ whenever $T(\mathbb{A}) \subset T'(\mathbb{A})$ for all $\mathbb{A}$ in $\mathbf{A}$; equivalent conditions are: T is a subfunctor of T'; $\mathbf{A}_T \subset \mathbf{A}_{T'}$; $\mathbf{Mod}(T) \subset \mathbf{Mod}(T')$; $\mathbf{Mod}[T] \subset \mathbf{Mod}[T']$.

The smallest theory on Δ is the empty one; the largest is the total $\mathbf{A}$-theory T_Δ, whose models in $\mathbb{A}$ are all the morphisms $\Delta \to \mathbb{A}$.

4.5. DEFINITION. The *2-universal* (resp. *biuniversal*) model $t_0\colon \Delta \to \mathbb{A}_0$ of the theory T will be given by any 2-universal (resp. biuniversal) arrow $(\mathbb{A}_0, t_0)$ from the object $\underline{1}$ to the 2-functor $T\colon \mathbf{A} \to \mathbf{CAT}$.

Equivalently, it is a 2-representation $T(\mathbb{A}) \simeq \mathbf{A}(\mathbb{A}_0,\mathbb{A})$ (resp. a birepresentation $T(\mathbb{A}) \simeq \mathbf{A}(\mathbb{A}_0,\mathbb{A})$) of the 2-functor T. Or also, by a remark in 1.5, it is a 2-universal (resp. biuniversal) arrow $(\mathbb{A}_0,\ t_0\colon \Delta \to U_T\mathbb{A}_0)$ from the object Δ of $\mathbf{A}_T$ to the inclusion 2-functor $U_T\colon \mathbf{A} \to \mathbf{A}_T$.

When existing, this model is determined up to isomorphism (resp. to equivalence): if also $t_1\colon \Delta \to \mathbb{A}_1$ is so, there is exactly one

$\mathbf{A}$-isomorphism $A_0 \to A_1$ commuting with t_0 and t_1 and unique up to isomorphism).

The object A_0 will be called the *2-classifying* (resp. *biclassifying) object* of T and written $\mathbf{A}(T)$ (resp. $\mathbf{A}[T]$). For the total $\mathbf{A}$-theory $T = T_\Delta$, the object A_0 will also be called the *2-free* (resp. *bifree*) $\mathbf{A}$-object on Δ, and written $\mathbf{A}(\Delta)$ (resp. $\mathbf{A}[\Delta]$).

Two theories T_1, T_2: $\mathbf{A} \to \mathbf{CAT}$, possibly based on different graphs and both having a 2/bi-universal model are isomorphic/equivalent iff their 2/bi-classifying objects are so.

4.6. Thus a model $t_0: \Delta \to A_0$ of T is *2*-universal iff

(UM) for each transformation τ: $t_1 \to t_2$: $\Delta \to A$ there is exactly one cell α: $a_1 \to a_2$: $A_0 \to A$ of $\mathbf{A}$ such that $\tau = \alpha.t_0$.

Or, equivalently, iff

(UM.1) for each model t: $\Delta \to A$ there is exactly one morphism a: $A_0 \to A$ of $\mathbf{A}$ such that $t = a.t_0$.

(UM.2) for all a_1, a_2: $A_0 \to A$ and all τ: $a_1 t_0 \to a_2 t_0$: $\Delta \to A$ there is exactly one cell α: $a_1 \to a_2$ of $\mathbf{A}$ such that $\tau = \alpha.t_0$.

The first condition means that t_0 is an *initial object* of $\mathbf{Mod}(T)$; the pair implies that t_0 is *2-initial*. Moreover t_0 *2-generates* A_0, in the sense that:

(1) for all cells α,α': $A_0 \rightrightarrows A$, if $\alpha t_0 = \alpha' t_0$ then $\alpha = \alpha'$.

The more general notion of *bigeneration* (requiring that the above cancellation property holds whenever α and α' are vertically parallel) will be used in the next section ([1]).

4.7. More generally, the model t_0 is *biuniversal* iff it verifies (BM.1) below and (UM.2).

(BM.1) For each model t: $\Delta \to A$ there is some morphism a: $A_0 \to A$ such that $t \simeq a.t_0$.

In such a case t_0 bigenerates A_0 (4.6) and it is a *biinitial* object for the 2-category $\mathbf{Mod}[T]$, which means that it verifies (BM.1) and the following consequence of (UM.2):

([1]) In other words one could say that t_0 is a *2-epimorphism (bi-epimorphism)* in the 2-category $\mathbf{A}_\Delta$.

(BM.I) For all morphisms a_1, a_2: $\mathbb{A}_0 \to \mathbb{A}$ and each natural isomorphism σ: $a_1.t_0 \simeq a_2.t_0$: $\Delta \to \mathbb{A}$, there is a unique $\mathbf{A}$-isocell ρ: $a_1 \to a_2$ such that $\sigma = \rho.t_0$.

4.8. If T and T' are theories and t_0 is a 2-universal model of T, then T $\leqslant$ T' iff t_0 is a model of T'. The same holds for t_0 biuniversal provided that T is replete (4.2).

4.9. We treat now the transfer of theories, or *change of base*. Assume, for the rest of this chapter, that $| \; |$: $\mathbf{A} \to \mathbf{CAT}$ and $| \; |$: $\mathbf{X} \to \mathbf{CAT}$ are concrete 2-categories, and V: $\mathbf{X} \to \mathbf{A}$ is a *concrete* 2-functor (i.e., commutes with the functors $| \; |$ of $\mathbf{X}$ and $\mathbf{A}$).

This situation produces an isomorphism of 2-functors (recall that $|V\mathbb{X}| = |\mathbb{X}|$ by the hypothesis on V):

$$\lambda\colon \mathbf{X}_\Delta(\Delta,-) \to \mathbf{A}_\Delta(\Delta,V-)\colon \mathbf{X} \to \mathbf{CAT}, \tag{1}$$

$$\lambda\mathbb{X}(t) = (V\mathbb{X}, |t|\colon \Delta \to |V\mathbb{X}|)\colon \Delta \to V\mathbb{X} \tag{2}$$

$$\text{for } t = (\mathbb{X}, |t|\colon \Delta \to |\mathbb{X}|).$$

Now, the $\mathbf{A}$-theory T defines an $\mathbf{X}$-theory $T^* = V^*(T)$ on the same graph Δ, to be called the *counterimage* of T along V: for every $\mathbb{X}$ in $\mathbf{X}$ the morphism:

$$t = (\mathbb{X}, |t|\colon \Delta \to |\mathbb{X}|)\colon \Delta \to \mathbb{X} \tag{3}$$

is a model of T^* iff the *associated morphism* $\lambda\mathbb{X}(t)$ is a model of T in $V\mathbb{X}$.

Indeed the axiom (T.0) holds for T^*: if in the above case x: $\mathbb{X} \to \mathbb{X}'$ is in $\mathbf{X}$, the composition

$$\Delta \xrightarrow{|t|} |V\mathbb{X}| \xrightarrow{|Vx|} |V\mathbb{X}'|, \tag{4}$$

is a model of T in $V\mathbb{X}'$; as $|Vx| = |x|$, it follows that xt is a model of T^* in $\mathbb{X}'$.

Notice that the functor T^*: $\mathbf{X} \to \mathbf{CAT}$ we have defined is isomorphic to TV (which is not an $\mathbf{X}$-theory), via the restriction of $\lambda\mathbb{X}$:

$$\lambda'\mathbb{X}\colon T^*\mathbb{X} \to TV\mathbb{X}, \qquad t \mapsto \lambda\mathbb{X}(t). \tag{5}$$

4.10. THEOREM. *In the hypotheses of 4.9, let* t_0: $\Delta \to \mathbb{A}_0$ *be a 2/biuniv-*

ersal model of T *and assume we have a 2/bi-universal arrow from* $\mathbb{A}_0$ *to* V:

(1) $$(\mathbb{X}_0, a\colon \mathbb{A}_0 \to V\mathbb{X}_0).$$

Then a 2/bi-universal model of T^* *is given by:*

(2) $$t_0^* = (\mathbb{X}_0, |a|.|t_0|\colon \Delta \to |\mathbb{X}_0|).$$

PROOF. By 1.6, the T-model $t'_0 = a.t_0$

(3) $$\underline{1} \xrightarrow{t_0} T\mathbb{A}_0 \xrightarrow{Ta} TV\mathbb{X}_0$$

is a 2/bi-universal arrow from $\underline{1}$ to TV: $\mathbf{X} \to \mathbf{CAT}$. As $\lambda(t'_0) = t_0^*$, the conclusion follows.

5. Complete theories and universal models.

2-complete, pseudocomplete and bicomplete theories are introduced and described via the construction theorems for 2-limits and pseudolimits. For these theories solution set conditions for the existence of the 2-universal or biuniversal model are derived from the analogous ones for 2-universal and biuniversal arrows (3.6 and 3.4). For brevity, the term *arrow-bicomplete* (referring to the existence of conical bilimits and bicotensors with $\underline{2}$, see 2.4) will always be shortened to *bicomplete*; analogously for the related notions considered in 2.4.

T is always an $\mathbf{A}$-theory on the small graph Δ.

5.1. DEFINITION. The $\mathbf{A}$-theory T on Δ will be said to be *2-complete* (resp. *pseudocomplete, bicomplete*) whenever the 2-category $\mathbf{A}$ is 2-complete (resp. pseudocomplete, bicomplete) and the 2-functor T: $\mathbf{A} \to \mathbf{CAT}$ is 2-continuous (resp. psudocontinuous, bicontinuous); it is easy to see that the second condition is equivalent to the 2-continuity (resp. pseudocontinuity, bicontinuity) of $U_T\colon \mathbf{A} \to \mathbf{A}_T$.

5.2. Clearly, if $\mathbf{A}$ is 2-complete (resp. pseudocomplete) and $|\ |$: $\mathbf{A} \to \mathbf{CAT}$ is 2-continuous (resp. pseudocontinuous) then the theory T_Δ is 2-complete (resp. pseudocomplete).

5.3. LEMMA. The theory T: $\mathcal{A} \to$ **CAT** is 2-complete provided that:

(T.1) $\mathcal{A}$ has small 2-products, preserved by $|\ |$; for each small family t_i: $\Delta \to \mathbf{A}_i$ ($i \in I$) of models, the morphism $t = (t_i)$: $\Delta \to \Pi\mathbf{A}_i$ is a model.

(T.2) for each pair a,b: $\mathbf{A} \to \mathbf{A}'$ of parallel morphisms, $\mathcal{A}$ has a 2-equalizer e: $\mathbf{A}_0 \to \mathbf{A}$ preserved by $|\ |$; moreover, if the model $t \in \mathrm{T}(\mathbf{A})$ equalizes a,b ($at = bt$):

(1)
$$\begin{tikzcd}
\mathbf{A}_0 \arrow[r, "e"] & \mathbf{A} \arrow[r, shift left, "a"] \arrow[r, shift right, "b"'] & \mathbf{A}' \\
\Delta \arrow[u, "t_0"] \arrow[r, equal] & \Delta \arrow[u, "t"'] \arrow[r, equal] & \Delta \arrow[u]
\end{tikzcd}$$

then the unique graph-morphism t_0: $\Delta \to \mathbf{A}_0$ such that $et_0 = t$ is a model of T.

(T.3) for every object $\mathbf{A}$, $\mathcal{A}$ has a cotensor product δ: $2\pitchfork\mathbf{A} \Rightarrow \mathbf{A}$ preserved by $|\ |$; moreover if τ: $t_1 \to t_2$: $\Delta \to \mathbf{A}$ is a natural transformation of models of T in $\mathbf{A}$

(2)
$$\begin{tikzcd}
 & & \mathbf{A} \arrow[dd, equal] \\
\Delta \arrow[urr, "t_1"] \arrow[drr, "t_2"'] \arrow[r, phantom, "\tau\downarrow"] & 2\pitchfork\mathbf{A} \arrow[ur, "d_1"] \arrow[dr, "d_2"'] \arrow[r, phantom, "\delta\downarrow"] & \\
 & & \mathbf{A}
\end{tikzcd}$$

and t: $\Delta \to \mathbf{A}_1$ is the graph-morphism such that $\delta.t = \tau$, then t is a model of T.

The conditions (T.1,2) yield the 1-completeness of **Mod**(T) together with the 1-continuity of the forgetful functor **Mod**(T) $\to \mathcal{A}$.

5.4. LEMMA. T is pseudocomplete provided that it verifies the conditions (T.1,3) of the previous Lemma 5.3, together with:

(T.2') for each pair of parallel morphisms a,b: $\mathbf{A} \to \mathbf{A}'$, $\mathcal{A}$ has an isoinserter (e: $\mathbf{A}_0 \to \mathbf{A}$, ϵ: $ae \simeq be$) preserved by $|\ |$; moreover, if the model $t \in \mathrm{T}(\mathbf{A})$ "inserts an isomorphism of graphs τ between a and b " (τ: $at \simeq bt$), then the unique morphism t_0: $\Delta \to \mathbf{A}_0$ such that $t = et_0$, $\tau = \epsilon t_0$ is a model of T.

(T.2") for each endocell α: $a \to a$: $\mathbf{A} \to \mathbf{A}'$, $\mathcal{A}$ has an identifier e: $\mathbf{A}_0 \to \mathbf{A}$ preserved by $|\ |$; moreover, if the model $t \in \mathrm{T}(\mathbf{A})$ "identifies α"

($\alpha t = 1$), then the unique morphism t_0: $\Delta \to \mathbb{A}_0$ such that $t = et_0$ is a model of T.

5.5. THEOREM: existence of 2-universal models. *Let* T *be a 2-complete* A*-theory on* Δ. T *has a 2-universal model iff:*

Solution Set Condition: there exists a small 2-full coinitial sub-2-category of **Mod**(T).

5.6. THEOREM: existence of biuniversal models. *Let* T *be a bicomplete* A*-theory on* Δ. T *has a biuniversal model iff:*

Solution Set Condition: there exists a small bifull coinitial sub-2-category of **Mod**[T].

6. Reflective theories and well-adapted 2-categories.

Reflective theories in well adapted 2-categories are introduced and shown to have always a biuniversal model. All the examples of Part III will be of this kind.

6.1. DEFINITION. The A-theory T on Δ will be said to be *reflective* if it verifies (T.1) and the following property of reflection of models:

(T.R) for every morphism t: $\Delta \to \mathbb{A}$ in A_Δ and every a: $\mathbb{A} \to \mathbb{A}'$ in A, if

i) the associated functor $|a|$ is faithful and reflects the isomorphisms ([1]),

ii) $a.t$ is a model of T in $\mathbb{A}'$,

then the morphism t is a model in $\mathbb{A}$.

6.2. LEMMA. If A is 2-complete and $|\ |$ is 2-continuous, every reflective theory in A is 2-complete.

PROOF. By the Lemma 5.3 we just need to check the conditions (T.2,3). The first one is trivially satisfied: in the situation described in 5.

([1]) Since faithful functors always reflect monics and epis, the second part of this condition follows from the first one whenever all the categories $|\mathbb{A}|$ are balanced; for instance in the 2-category of abelian categories or of toposes.

3, the functor $|e|$ is the equalizer of the functors $|a|$ and $|b|$ in **CAT**, hence it is a full embedding: by (T.R) this proves that t_0 is a model. As regards (T.3), again with the notations of 5.3, consider the $\mathbf{A}$-morphism

$$(1) \qquad J: 2\Phi\mathbb{A} \to \mathbb{A}\times\mathbb{A}, \qquad p_1J = d_1, \quad p_2J = d_2,$$

whose underlying functor, by the 2-continuity of $|\ |$, is:

$$(2) \qquad |J|: 2\Phi\mathbb{A} \to |\mathbb{A}|\times|\mathbb{A}|,$$

$$(3) \qquad |J|(A,A',a:A \to A') = (A,A'), \qquad |J|(a_1,a_2) = (a_1,a_2);$$

clearly $|J|$ is faithful and reflects the isomorphisms ((a_1,a_2) is iso in $2\Phi|\mathbb{A}|$ iff both a_1 and a_2 are iso in $|\mathbb{A}|$). Now the transformation of models $\tau: t_1 \to t_2: \Delta \to \mathbb{A}$ defines one *morphism* (of $\mathbf{A}_\Delta$) $t: \Delta \to 2\Phi\mathbb{A}$ such that $\delta t = \tau$; it also defines, by (T.1), a *model* $t' = (t_1,t_2): \Delta \to \mathbb{A}\times\mathbb{A}$ verifying $Jt = t'$. By (T.R), t is a model.

6.3. LEMMA. If $\mathbf{A}$ has 2-products, isoinserters, identifiers of endo-cells and cotensors with $\underline{2}$, preserved by $|\ |$, every reflective theory in $\mathbf{A}$ is pseudocomplete.

PROOF. By the Lemma 5.4 it is sufficient to check (T.2',2",3). The proof of (T.2') is similar to the proof of (T.2) in the above Lemma 6.2: it depends on the fact that, in **CAT**, the isoinserter $(\mathbb{X},x,\chi)$ of two parallel functors yields a functor x which is faithful and reflects the isomorphisms. Analogously for (T.2"): the identifier of an (endo)cell is even a full embedding. The proof of (T.3) is the same as in 6.2.

6.4. Now say that the concrete 2-category $\mathbf{A}$ is *well adapted* (for theories) if it satisfies the following conditions (always true in the applications which follow):

(WA.1) *Limits.* $\mathbf{A}$ has products, isoinserters, identifiers of endocells and cotensors with $\underline{2}$, all preserved by $|\ |$.

(WA.2) *Small fibers.* $|\ |: \mathbf{A} \to \mathbf{CAT}$ is 2-faithful and each small category has a small counterimage in $\mathbf{A}$.

(WA.3) *Isomorphism lifting.* If $\mathbb{A}$ is in $\mathbf{A}$ and $f: |\mathbb{A}| \to \underline{C}$ is an isomorphism of $\mathcal{U}$-categories, there exists an $\mathbf{A}$-isomorphism $a: \mathbb{A} \to \mathbb{A}'$ lifting f ($|a| = f$).

(WA.4) *Bounded factorization.* For each small graph Δ there exists a small cardinal $\omega(\Delta)$ such that every morphism $t\colon \Delta \to \underline{A}$ in $\mathbf{A}_\Delta$ factors (in $\mathbf{A}_\Delta$) as:

$$(1) \qquad \Delta \xrightarrow{t_1} \underline{A}_1 \xrightarrow{a} \underline{A}, \qquad t = at_1 ,$$

where:

i) t_1 is a bigenerating morphism (4.6) and $\mathrm{card}|\underline{A}_1| \leqslant \omega(\Delta)$,

ii) the functor $|a|$ is faithful and reflects the isomorphisms.

We say also that $\mathbf{A}$ is *strictly adapted* when, moreover, $\mathbf{A}$ is 2-complete, $|\ |$ is 2-continuous and in (WA.4) the morphism t_1 may be chosen to be 2-generating.

A function ω satisfying (WA.4) will be called a *bounding function* for $\mathbf{A}$.

6.5. THEOREM: existence of 2-universal models, II. *If* $\mathbf{A}$ *is strictly adapted, every reflective* $\mathbf{A}$*-theory* T *on* Δ *is 2-complete and has a 2-universal model. The 2-classifying category* $\mathbf{A}(\mathrm{T})$ *is bounded by the cardinal* $\omega(\Delta)$.

PROOF. T is 2-complete by 6.2; we want to prove that it has a solution set for the existence of the 2-universal model (5.5). Consider the small set $\mathbf{C}$ of all categories $\underline{C}$ where $\mathrm{Ob}\underline{C}$ and $\mathrm{Mor}\underline{C}$ *are* cardinals $\leqslant \omega = \omega(\Delta)$; it follows that the set of graph morphisms $\Delta \to \underline{C}$ ($\underline{C} \in \mathbf{C}$) is small; by the small-fiber property (WA.2) (6.4), also the set of T-models $t = (\mathbf{A}, |t|\colon \Delta \to |\underline{A}|)$ with $|\underline{A}| \in \mathbf{C}$ is small. By (WA.2-4) and the reflective property of T, the 2-full subcategory of $\mathbf{Mod}(\mathrm{T})$ generated by these models is a solution set for T.

Now, if $t_0\colon \Delta \to \underline{A}_0$ is a 2-universal model of T, consider its bounded factorization $t_0 = a.t_1$ (WA.4); if $\tau\colon \Delta \Rightarrow \underline{A}$ is a transformation of models, then τ factors through t_1 ($\tau = \alpha t_0 = \alpha a.t_1$), uniquely because of the 2-generating property of t_1. Thus also t_1 is 2-universal and $\mathrm{card}|\underline{A}_1| \leqslant \omega$.

6.6. THEOREM: existence of biuniversal models, II. *If* $\mathbf{A}$ *is well adapted, every reflective* $\mathbf{A}$*-theory* T *on* Δ *is pseudo-complete and has a biuniversal model. The biclassifying category* $\mathbf{A}(\mathrm{T})$ *is bounded by the cardinal* $\omega(\Delta)$.

Part III. APPLICATIONS

We consider now reflective theories in various well adapted concrete 2-categories. In all the examples of Chapters 7-9 the 2-functor $|\ |$: $\mathbf{A} \to \mathbf{CAT}$ is the inclusion or the obvious forgetful functor, and verifies trivially the small-fiber and isomorphism-lifting properties (WA.2,3) of 6.4.

Δ is always a small graph; its (small) *cardinal*, card Δ, is the greatest between the cardinals of its object-set and its arrow-set.

7. Theories and sketches.

We treat here theories with values in **CAT**, in the 2-category of finitely complete categories and, more generally, in the 2-category of categories having specified $\mathcal{F}$-limits and $\mathcal{F}'$-colimits. The well known result of Bastiani-Ehresmann [BE] on the existence of the generic model of a "sketched theory" is thus given a proof "from above".

7.1. Take $\mathbf{A}$ = **CAT**, the 2-complete 2-category of $\mathcal{U}$-categories, where $|\ |$ is the identity 2-functor. **CAT** is strictly adapted with bounding function

$$(1) \qquad \omega(\Delta) = \max(\mathrm{card}\ \Delta, \aleph_0).$$

Actually each graph morphism t: $\Delta \to \mathbb{A}$ into a category factors through its codomain-restriction t_1: $\Delta \to \mathbb{A}_1$, where $\mathbb{A}_1$ is the invariant (¹) subcategory of $\mathbb{A}$ generated by the subgraph $t(\Delta)$; clearly we have card $\mathbb{A}_1 \leqslant \omega(\Delta)$. Moreover $\mathbb{A}_1$ is 2-generated by t_1. Indeed, a functor a: $\mathbb{A}_1 \to \mathbb{A}$ is determined by at_1; a transformation α: $a_1 \to a_2$: $\mathbb{A}_1 \to \mathbb{A}$ is determined by a_1, a_2 and its values on the objects of $\mathbb{A}_1$, coinciding with those of $t_1(\Delta)$; therefore α is determined by αt_1.

Therefore, by 6.5, *each* reflective **CAT**-theory T on Δ is 2-complete and has a 2-universal model bounded by $\omega(\Delta)$.

In particular this holds for the total **CAT**-theory T_Δ, whose models are all the graph morphisms t: $\Delta \to \mathbb{A}$ with values in some $\mathcal{U}$-category. It follows the (well konwn) existence of the *2-free category* CAT(Δ) *generated by* Δ, as well as the estimate:

(¹) Say that a subcategory is invariant whenever the embedding functor reflects the isomorphisms.

$$\text{card } \mathbf{CAT}(\Delta) \leqslant \max(\text{card } \Delta, \aleph_0).$$

7.2. Syntactically, each (small) set K of *commutativity conditions* on Δ

(1) $$u_m \dots u_2 u_1 = v_n \dots v_2 v_1 \; ,$$

(2) $$u'_m \dots u'_2 u'_1 = 1_k \quad [1]$$

determines a **CAT**-theory $T(\Delta,K) \leqslant T_\Delta$: the models are those morphisms $t\colon \Delta \to \mathbf{A}$ which satisfy all conditions in K, in the obvious sense. Such a theory is clearly reflective; thus the 2-universal models supplies the *2-free category generated by* Δ *under the conditions of* K, $\mathbf{CAT}(\Delta,K)$, still bounded by $\omega(\Delta)$.

Conversely *each* reflective theory T in **CAT** is isomorphic (4.2, 4.5) to some theory $T(\Delta,K)$: indeed, if $\mathbf{A}_0$ is the (small) 2-classifying category of T, choose some subgraph Δ of $\mathbf{A}_0$ which generates $\mathbf{A}_0$ under a suitable set K of commutativity conditions. For instance, the following (non economical!) choice is always possible: Δ is the whole graph underlying $\mathbf{A}_0$ and K is the set of all commutativity conditions on Δ which hold true in $\mathbf{A}_0$ (or, more simply, the set of conditions $vu = w$, $u = 1$ holding true in $\mathbf{A}_0$).

7.3. Consider now the 2-category $\mathcal{A}$ = **FLM** of finitely complete categories ($\mathcal{U}$-categories $\mathbf{A}$ having all finite limits), together with the finitely continuous functors and their natural transformations. **FLM** is easily seen to satisfy (VA.1), hence to be pseudocomplete with pseudocontinuous $|\;|$; it is not 2-complete as it lacks equalizers. We prove now that it is well adapted. Analogously one treats the 2-category **FP** of $\mathcal{U}$-categories with finite products (and functors preserving them), the 2-category of $\mathcal{U}$-categories with equalizers and so on.

7.4. THEOREM. *The concrete 2-category* **FLM** *is well adapted, with bounding function* $\omega(\Delta) = \max(\text{card } \Delta, \aleph_0)$. *Each reflective theory in* **FLM** *is pseudocomplete and has a biuniversal model, bounded as above.*

PROOF. Fix a graph morphism $t\colon \Delta \to \mathbf{A}$; we want to prove that it factors through the embedding $\mathbf{A}_1 \to \mathbf{A}$ of a suitable invariant finitely complete subcategory $\mathbf{A}_1$ bounded by $\omega = \omega(\Delta)$.

----

[1] The u_i, v_i and u'_i are consecutive arrows of Δ while k is an object. Moreover:

$$\text{Dom } u_1 = \text{Dom } v_1, \; \text{Cod } u_m = \text{Cod } v_n, \; \text{Dom } u'_1 = \text{Cod } u'_m = k.$$

Let F be the (countable) set of all graphs Φ whose sets $\mathrm{Ob}\Phi$ and $\mathrm{Mor}\Phi$ are finite cardinals. For any diagram $F\colon \Phi \to \mathbf{A}$ ($\Phi \in F$), choose one limit cone $f\colon A^\wedge \to F$ of F in $\mathbf{A}$, where $A^\wedge$ denotes the Φ-diagram constant at A.

Now form the subcategory $\mathbf{A}_1 = \cup\Delta_n$ of $\mathbf{A}$, where Δ_n is a subgraph of $\mathbf{A}$ defined by the following inductive procedure:

a) $\Delta_0 = t(\Delta)$,

b) Δ_{n+1} contains Δ_n, together with the *identity* 1_A of each A in Δ_n,

c) for all consecutive u,u' in Δ_n, the *composition* $u'.u$ is in Δ_{n+1},

d) for all u in Δ_n, if u is iso in $\mathbf{A}$, the *inverse* u^{-1} is in Δ_{n+1},

e) for each diagram $F\colon \Phi \to \Delta_n$ ($\Phi \in F$), the *chosen limit cone* $f\colon A^\wedge \to F$ is "contained" in Δ_{n+1} (i.e., its vertex A and its morphisms $f_i\colon A \to Fi$ all belong to Δ_{n+1}),

f) for each cone $g\colon B^\wedge \to F\colon \Phi \to \mathbf{A}$ "contained" in Δ_n ($\Phi \in F$), the *limit morphism* $u\colon B \to A$ in $\mathbf{A}$ belongs to Δ_{n+1}.

Thus $\mathbf{A}_1$ is an invariant, finitely complete sunbcategory of $\mathbf{A}$.

Its cardinal is bounded by ω, since by induction each graph Δ_n is so. Indeed, notice first that the set $\mathbf{F}$ of F-diagrams in Δ_n is bounded by ω: for each $\Phi \in F$ the set of diagrams $F\colon \Phi \to \Delta_n$ is bounded by $\omega^{\mathrm{card}\Phi} \leqslant \omega$; hence also the sum of these sets, for Φ varying in the countable set F, is bounded by ω. Now, with respect to the rule e, each chosen limit $f\colon A^\wedge \to F$ ($i \in \mathrm{Ob}\Phi$) is a finite family (f_i), so that the union for $F \in \mathbf{F}$ of all these families is bounded by ω. Similarly, as concerns f, for each $F \in \mathbf{F}$ the set of cones $g = (g_i)\colon B^\wedge \to F$ "contained" in Δ_n is bounded by $\omega.\omega^{\mathrm{card\ Ob}\Phi} = \omega$; hence also their sum for $F \in \mathbf{F}$ is bounded by ω.

Last, the codomain-restriction $t_1\colon \Delta \to \mathbf{A}_1$ of t *bigenerates* $\mathbf{A}_1$. Indeed, assume that $\alpha' t_1 = \alpha'' t_1$ for parallel cells $\alpha',\alpha''\colon a \to a'\colon \mathbf{A}_1 \to \mathbf{A}_2$. Then α' and α'' coincide on the objects of Δ_0. Suppose by induction that they coincide on the objects of Δ_n.

An object A may be added in Δ_{n+1} only according to the rule e: hence A appears as the vertex of the chosen limit $f\colon A^\wedge \to F$, with $F\colon \Phi \to \Delta_n$; thus $\alpha'(Fi) = \alpha''(Fi)$ for all $i \in \mathrm{Ob}\Phi$ (the vertexes Fi belong to Δ_n); by naturality, both the morphisms $u = \alpha' A$ and $v = \alpha'' A$ make the following diagram commutative (for each i):

(1)
$$\begin{tikzcd}
a(Fi) \arrow[r, "u_i"] & a'(Fi) \\
a(A) \arrow[u, "a(f_i)"] \arrow[r, "{u,v}"] & a'(A) \arrow[u, "a'(f_i)"']
\end{tikzcd}$$

where $u_i = \alpha'(Fi) = \alpha''(Fi)$; as a' preserves finite limits, $\alpha'A = \alpha''A$.

7.5. More generally, let $\mathcal{F}$ and $\mathcal{F}'$ be small sets of small graphs and consider $\mathbf{A} = \mathcal{F}\mathcal{F}'\mathbf{LM}$ the 2-category of $\mathcal{F}$-complete, $\mathcal{F}'$-cocomplete categories ($\mathcal{U}$-categories $\mathbb{A}$ having all $\mathcal{F}$-limits [1] and $\mathcal{F}'$-colimits), together with the $\mathcal{F}$-continuous, $\mathcal{F}'$-cocontinuous functors (preserving the above limits and colimits) and their natural transformations.

7.6. THEOREM. *The concrete 2-category* $\mathcal{F}\mathcal{F}'\mathbf{LM}$ *is well-adapted; each reflective theory in* $\mathcal{F}\mathcal{F}'\mathbf{LM}$ *is pseudocomplete and has a biuniversal model. An upper bound* $\omega = \omega(\Delta)$ *may be obtained by taking:*
(1) β: any regular infinite small cardinal such that $\beta >$ card Φ, for all $\Phi \in \mathcal{F} \cup \mathcal{F}'$ [2],
(2) $\gamma = \max(\text{card } \mathcal{F}, \text{card } \mathcal{F}')$, $\delta = \max(\text{card } \Delta, \beta, \gamma)$, $\omega = 2^{\delta}$.

PROOF. In order to verify the bounded factorization property (WA.4), we fix a graph morphism $t\colon \Delta \to \mathbb{A}$ and prove that it is contained in a suitable invariant $\mathcal{F}$-complete, $\mathcal{F}'$-cocomplete subcategory $\mathbb{A}_1$, bounded by ω.

a) As in the previous case (7.4), choose one limit cone $f\colon \hat{A} \to F$ of F in $\mathbb{A}$, for any diagram $F\colon \Phi \to \mathbb{A}$ $(\Phi \in \mathcal{F})$. Analogously choose one colimit cocone $f'\colon F \to \hat{A'}$ of F in $\mathbb{A}$, for any diagram $F\colon \Phi \to \mathbb{A}$ $(\Phi \in \mathcal{F}')$.

Then form the subcategory $\mathbb{A}_1 = \Delta_\beta$ of $\mathbb{A}$, defining a subgraph Δ_n by transfinite induction on all the ordinals $n \leq \beta$. Δ_0 is $t(\Delta)$; if $n < \beta$, Δ_{n+1} is given by the rules b-f of 7.4, together with two more rules e', f' concerning the colimits of $\mathcal{F}'$-diagrams; if $n' \leq \beta$ is a limit ordinal, then $\Delta_{n'} = \cup_{n<n'} \Delta_n$.

b) Now $\mathbb{A}_1$ is an invariant $\mathcal{F}$-complete, $\mathcal{F}'$-cocomplete subcategory of $\mathbb{A}$: we just check the stability with respect to $\mathcal{F}$-limits. Take an $\mathcal{F}$-diagram $F\colon \Phi \to \mathbb{A}_1$ and consider

(3) $\quad H = \{n \in \beta \mid \Delta_n \text{ contains some object or some arrow of } F\}$,

(4) $$m = \sup H = \cup H;$$

----

(1) Limits of diagrams $F\colon \Phi \to \mathbb{A}$ whose domain Φ belongs to $\mathcal{F}$.

(2) Since the cardinal successor of any infinite cardinal is regular ([Je], p.40), one can always take for β the cardinal successor of

$$\alpha = \max(\aleph_0, \sup\{\text{card } \Phi \mid \Phi \in \mathcal{F} \cup \mathcal{F}'\}).$$

However α itself can suffice sometimes, e.g., when all the graphs Φ are finite and $\alpha = \aleph_0$. In the proof below we follow the terminology of Jech [Je] on cardinals and ordinals; in particular, any cardinal is assumed to be an ordinal.

since β is regular and card $H \leqslant$ card $\Phi < \beta$, it follows that $m < \beta$ ([Je], Lemma 3.6); therefore F is contained in Δ_m and its chosen limit cone $f\colon A^\wedge \to F$ is contained in $\Delta_{m+1} \subset \mathbb{A}_1$. In the same way one proves that each cone $g\colon B^\wedge \to F\colon \Phi \to \mathbb{A}_1$ is contained in $\Delta_{m'}$ for some ordinal $m' < \beta$ $(m' \geqslant m)$, so that the limit morphism $u\colon B \to A$ belongs to $\Delta_{m'+1} \subset \mathbb{A}_1$.

c) In order to prove our estimate on cardinality, let us prove by transfinite induction on the ordinal n that card $\Delta_n \leqslant \omega$ (for all $n \leqslant \beta$).

This is true for $n = 0$. Assume the property for $n < \beta$ and verify it for $n+1$; clearly we just need to control the additions of morphisms to Δ_n caused by the rules e,f,e' and f'.

First notice that, for each $\Phi \in \mathcal{F}$, the set of diagrams $F\colon \Phi \to \Delta_n$ is bounded by

$$(5) \qquad \omega^\beta = (2^\delta)^\beta = 2^{\delta.\beta} = 2^\delta = \omega \;;$$

hence the set $\mathbf{F}$ of all these diagrams, for Φ varying in the set $\mathcal{F}$, is bounded by $\omega.\gamma \leqslant \omega$.

Now, with respect to the rule e, for each chosen limit $f\colon A^\wedge \to F$ the family $f = (f_i)$ $(i \in \mathrm{Ob}\Phi)$ we have to add to Δ_n is bounded by β, so that the union of these families for $F \in \mathbf{F}$ is bounded by $\beta.\omega = \omega$. Similarly, as concerns the rule f, for each $F \in \mathbf{F}$ the set of cones $g = (g_i)\colon B^\wedge \to F$ "contained" in Δ_n is bounded by $\omega.\omega^\beta = \omega$; hence their sum for $F \in \mathbf{F}$ is bounded by $\omega.\omega = \omega$. Analogously for colimits. Thus card $\Delta_{n+1} \leqslant \omega$.

Last, if $n' \leqslant \beta$ is a limit ordinal and card $\Delta_n \leqslant \omega$ for all $n < n'$, then

$$\mathrm{card}\ \Delta_{n'} = \mathrm{card}(\cup_{n<n'}\Delta_n) \leqslant \Sigma_{n<n'}\mathrm{card}\ \Delta_n \leqslant \omega.\beta = \omega.$$

d) Finally, the codomain-restriction $t_1\colon \Delta \to \mathbb{A}_1$ of t *bigenerates* $\mathbb{A}_1$, as shown in 7.4.

7.7. Also here a reflective theory can be described by syntactic conditions. Say *$\mathcal{F}$-limit condition* on Δ any transformation

$$(1) \qquad g\colon k^\wedge \to F\colon \Phi \to \Delta, \qquad g = (g_i\colon A \to Fi)_{i\in\mathrm{Ob}\Phi}\,,$$

from the constant diagram at some Δ-object k to some diagram $F\colon \Phi \to \Delta$ $(\Phi \in \mathcal{F})$. Analogously an *$\mathcal{F}'$-colimit* condition on Δ is a transformation from a diagram $F\colon \Phi' \to \Delta$ $(\Phi' \in \mathcal{F}')$ to the constant diagram at an object k of Δ.

Now a *sketch* with respect to the pair (F,F') will be a system $\mathbf{\Delta} = (\Delta,K,\Gamma,\Gamma')$ where K is a small set of commutativity conditions on Δ (7.2), Γ is a small set of F-limit conditions and Γ' a small set of F'-colimit conditions.

This system defines a *sketched theory* $T = T(\Delta,K,\Gamma,\Gamma')$ in $\mathbf{FF'LX}$, whose models in $\mathbf{A}$ are the graph morphisms $t\colon \Delta \to \mathbf{A}$ satisfying the commutativity conditions of K, the limit conditions of Γ and the colimit conditions of Γ'; e.g., for each F-condition $g\colon k^\wedge \to F$ in Γ, $tg\colon tk^\wedge \to tF$ is a limit cone (of $tF\colon \Phi \to \mathbf{A}$) in $\mathbf{A}$ (in particular it is a *natural* transformation).

Since in $\mathbf{FF'LX}$ a faithful functor which reflects the isomorphisms reflects all these conditions, T is reflective and has a biuniversal model satisfying the given bound.

For such theories Bastiani-Ehresmann [BE] give a constructive proof, by transfinite induction, of the existence of the biuniversal model ([BE], Prop. 4 and 15 [1]). Simple "one-step" constructive proofs can be given in the particular cases of *projective* sketches ($F' = \emptyset$, $\Gamma' = \emptyset$), as in Kelly [K2]. It is known that algebraic (or essentially algebraic) objects can be described as the models of a suitable sketch in a category with finite products (or finite limits): in this case the biclassifying category is the theory in the sense of Lawvere's functorial semantics [La,KR].

7.8. Conversely every reflective $\mathbf{FF'LX}$-theory may be presented by some sketch $(\Delta,K,\Gamma,\Gamma')$: e.g., let Δ be the graph underlying the biclassifying category $\mathbf{A}_0$ and let K, Γ and Γ' be respectively the set of all commutativity, F-limit and F'-colimit conditions on Δ which hold true in $\mathbf{A}_0$ (more precisely, suitable small realizations of these condition sets).

8. Linear, exact, abelian and regular theories.

8.1. Let be given a small ring Λ. Consider the *2-complete* 2-category Λ-**CAT** of Λ-*linear* categories (i.e., U-categories enriched on the monoidal closed category Λ-Mod of small left Λ-modules), Λ-*linear* functors (preserving linear combinations of parallel maps) and natural transformations. In particular for $\Lambda = \mathbf{Z}$ one gets the 2-category of Ab-categories, also called semiadditive.

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[1] Actually the commutativity conditions were given through a graph with partial composition.

By an argument similar to 7.1 one proves that Λ-**CAT** is strictly adapted, with bounding function $\omega(\Delta) = \max(\text{card } \Delta, \text{card } \Lambda, \aleph_0)$. Thus each reflective theory in Λ-**CAT** has a 2-universal model, bounded by $\omega(\Delta)$.

Syntactically, let be given in Δ a set K of Λ-*linearity* conditions

(1) $$\Sigma_i \lambda_i (u_{in_i} . \; ... \; . u_{i2} . u_{i1}) = 0,$$

where $\lambda_i \in \Lambda$, the arrows in parenthesis are consecutive in Δ and the Δ-objects $h = \text{Dom } u_{i1}$, $k = \text{Cod } u_{in_i}$ do not depend on i.

These data define a *reflective* theory $T = T(\Delta,K)$ whose models are the diagrams $t: \Delta \to \mathbb{A}$ ($\mathbb{A}$ in Λ-**CAT**) preserving the commutativity conditions of K:

(2) $\Sigma_i \lambda_i (tu_{in_i} . \; ... \; . tu_{i2} . tu_{i1}) = 0$ in the Λ-module $\mathbb{A}(th,tk)$.

Assume now, in particular, that the set K of linearity conditions can be expressed via a set K' of commutativity conditions (7.1). Consider the **CAT**-theory $T' = T(\Delta,K')$ and the forgetful 2-functor $V = | \; |: \Lambda\text{-}\mathbf{CAT} \to \mathbf{CAT}$: clearly $T = V^*(T')$. Since V has a left 2-adjoint (the free Λ-linear category generated by a $\mathcal{U}$-category), the universal model of T can be obtained by composing the universal model $t_0: \Delta \to \mathbf{CAT}(\Delta,K')$ of T' with the embedding of the former classifying category into its free A-linear category.

8.2. Let **EX** be the pseudocomplete 2-category of *exact* categories in the sense of Puppe-Mitchell [Pu; Mi] (1), *exact* functors and natural transformations.

By a proceeding similar to the one of 7.4, one proves that **EX** is well adapted with bounding function $\omega(\Delta) = \max(\text{card } \Delta, \aleph_0)$. Thus each reflective theory in **EX** has a biuniversal model bounded by $\omega(\Delta)$.

To describe **EX**-theories by syntactic means, assign in Δ a set K of commutativity conditions, a set Z of *annihilation* conditions (a subset of Δ) and a set E of *exactness* conditions (a set of sequences of Δ). Consider the pseudocomplete theory $T = T(\Delta,K,Z,E)$ whose models are the diagrams $t: \Delta \to \mathbb{A}$ ($\mathbb{A}$ in **EX**) preserving the commutativity conditions of K, the annihilation conditions of Z (every object or morphism of Z is taken by t into a zero object or a zero morphism of $\mathbb{A}$) and the exactness conditions of E (each sequence of E is trans-

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(1) An exact category is a well-powered $\mathcal{U}$-category with zero object, kernels and cokernels, in which every map factors through a conormal epi and a normal mono. An exact functor preserves exact sequences (equivalently: kernels and cokernels).

formed by t into an exact sequence of $\mathbb{A}$). Clearly such a theory is reflective; conversely, by the usual argument, each reflective theory is equivalent to a theory of this type.

As shown in [G3], many interesting homological theories (e.g., the filtered complex and the double complex) may be studied as **EX**-theories; their biuniversal model "is" the Zeeman diagram of the associated spectral sequence.

8.3. Analogously the pseudocomplete 2-category **AB** of abelian categories, exact functors and natural transformations (¹) is well adapted, with bounding function $\omega(\Delta) = \max(\text{card } \Delta, \aleph_0)$.

Syntactically, assign in Δ a set K of Z-linearity conditions (8.1), a set Γ of finite limit conditions (7.7), a set Γ' of finite colimit conditions (7.7), and a set E of exactness conditions (8.2).

Notice that the annihilation conditions (8.2) can be given in K. In the contrary, we prefer to keep the exactness conditions because to assign them by limit and colimit conditions would often require to complicate the graph Δ.

Consider the pseudocomplete theory $T = T(\Delta,K,\Gamma,\Gamma',E)$ whose models are the diagrams $t\colon \Delta \to \mathbb{A}$ ($\mathbb{A}$ in **AB**) preserving the linearity conditions of K, the limit and colimit conditions of Γ and Γ', the exactness conditions of E. The theory is reflective and its biclassifying category $\mathbf{AB}[\Delta,K,\Gamma,\Gamma',E]$ is bounded by $\max(\text{card } \Delta, \aleph_0)$.

8.4. The embedding 2-functor $V\colon \mathbf{AB} \to \mathbf{EX}$ produces, for every **EX**-theory T, the associated **AB**-theory $T^* = V^*T$: just consider only the T-models $t\colon \Delta \to \mathbb{A}$ with $\mathbb{A}$ abelian.

Since, for each small exact category $\mathbb{E}$, the existence of the biuniversal arrow $(\mathbb{A}, f\colon \mathbb{E} \to V\mathbb{A})$ can be proved by our results (just consider the **AB**-theory on $\mathbb{E}$ whose models are the exact functors $\mathbb{E} \to \mathbb{A}$), 4.10 proves that the biuniversal model of T^* can be obtained by composing the biuniversal model $t_0\colon \Delta \to \mathbf{EX}[T]$ of T (if existing) with the biuniversal arrow $\mathbf{EX}[T] \to \mathbb{A}$ from $\mathbf{EX}[T]$ to V.

Of course not all **AB**-theories can be obtained as T^* from some **EX**-theory T.

8.5. Consider now the pseudocomplete 2-category RG of *regular U*-cat-

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(¹) An abelian category may be defined to be an exact category with finite limits and colimits; it will be provided with its unique additive structure. An exact functor between abelian categories necessarily preserves finite limits, finite colimits and the additive structure.

egories, *regular* functors and natural transformations [1].

Also here, by a proceeding similar to 7.4, one shows that **RG** is well adapted, with the same bounding function $\max(\mathrm{card}\,\Delta, \aleph_0)$. A reflective theory $T = T(\Delta,K,\Gamma,R)$ may be syntactically defined on Δ through a set K of commutativity conditions, a set Γ of finite limit conditions and a set R of *coregularity conditions* (a subset R of $\mathrm{Mor}\,\Delta$): the models are the diagrams $t\colon \Delta \to \mathbf{A}$ ($\mathbf{A}$ in **RG**) preserving the above conditions of R (in particular, each arrow in R is to be transformed by t in a coregular epi of $\mathbf{A}$).

9. Elementary toposes and logical morphisms.

In this example we consider the 2-category $\mathcal{A}$ = TPL of *elementary toposes* (always assumed to be $\mathcal{U}$-categories), *logical morphisms* (i.e., functors which preserve, up to isomorphism, finite limits and colimits, exponentiation, the classifier of subobjects) and *natural transformations*.

9.1. THEOREM. TPL *is well adapted, with bounding function* $\omega(\Delta) = \max(\mathrm{card}\,\Delta, \aleph_0)$.

PROOF. It is easy to see that TPL satisfies the limit, small-fiber and isomorphism-lifting properties (WA.1-3) in 6.4. As to (WA.4), fix a graph morphism $t\colon \Delta \to \mathbf{A}$ into some elementary topos $\mathbf{A}$: we shall prove that it factors through a bigenerating morphism $t_1\colon \Delta \to \mathbf{A}_1$ into an invariant subtopos $\mathbf{A}_1$ bounded by $\omega(\Delta)$.

The subcategory $\mathbf{A}_1 = \cup\Delta_n$ of $\mathbf{A}$ may be constructed by an inductive proceeding similar to the one in 7.4. Take Δ_0 to be $t(\Delta)$ "together with" the subobject classifier Ω, the terminal object 1 and the "true" morphism $1 \to \Omega$. In the inductive step from Δ_n to Δ_{n+1}, besides the objects and morphisms to be added in order to get an invariant finitely complete subcategory (rules b-f of 7.4), add the following ones:

g) for all objects A,B,C in Δ_n, the object B^A of $\mathbf{A}$ together with all the $\mathbf{A}$-morphisms $C \to B^A$ corresponding to arrows $A\times C \to B$ of Δ_n,

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[1] A category $\mathbf{A}$ is regular (Grillet [Gr]) if: it is finitely complete, each map factors by a coregular epi and monic, the pullback-axiom holds for coregular epis. It should be noticed that, in this case, any coregular epi (i.e., coequalizer of some pair) is the coequalizer of its kernel-pair. A regular functor (between regular categories) has to preserve the above structure.

h) for each morphism $A' \to A$ in Δ_n, monic in $\mathbb{A}$, its characteristic map $\chi: A \to \Omega$ in $\mathbb{A}$.

Thus $\mathbb{A}_1$ is a subtopos of $\mathbb{A}$ [1], bounded by ω. The codomain restriction $t_1: \Delta \to \mathbb{A}_1$ of t is a model and it is not difficult to check that it bigenerates $\mathbb{A}_1$.

9.2. Thus any reflective theory in **TPL** is pseudocomplete and has a biclassifying topos, bounded as specified. In particular the total **TPL**-theory T_Δ supplies the *bifree topos over the given graph*, **TPL**(Δ).

9.3. Consider now a small category $\mathbb{C}$. A **TPL**-theory on $\mathbb{C}$ (more precisely, on the underlying graph) is obtained by taking as models in the elementary topos $\mathbb{A}$ all functors $t: \mathbb{C} \to \mathbb{A}$. This theory $T_{\mathbb{C}}$ is again a reflective theory (a faithful functor "reflects functors").

The biuniversal model $t_0: \mathbb{C} \to \mathbb{A}_0$ provides in this case the bifree topos over the category $\mathbb{C}$, i.e., an equivalence of categories

$$\mathbf{CAT}(\mathbb{C},\mathbb{A}) \simeq \mathbf{TPL}(\mathbb{A}_0,\mathbb{A}), \tag{1}$$

natural for $\mathbb{A}$ in **TPL**; we also have the estimate

$$\text{card } \mathbb{A} \leqslant \max(\text{card } \mathbb{C}, \aleph_0).$$

By the "change of base" Theorem (4.10), if Δ is a small graph and $\mathbb{C}$ the 2-free category on Δ, the bifree topos on Δ coincides with the bifree topos on the category $\mathbb{C}$.

Analogously one proves the existence of the bifree topos on a small finitely complete category $\mathbb{C}$ (the models $t: \mathbb{C} \to \mathbb{A}$ being the finitely continuous functors) or on a cartesian closed category; all "intermediate steps" can be chained to get the global one.

9.4. As a conclusion of our considerations so far, the bifree object over a small graph can be obtained for many "categorical structures", as categories (7.1), finitely complete categories (7.4), categories with finite products, categories with equalizers (7.6, with suitable F and $F' = \emptyset$), abelian categories (8.3), elementary toposes (9.1), carte-

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[1] Notice that the embedding $\mathbb{A}_1 \to \mathbb{A}$ preserves finite limits, hence kernel-pairs and monomorphisms.

sian closed categories (a slight modification of the proceeding in 9.1). In particular one finds results of Burroni [Bu] and Mac Donald-Stone [MS], obtained by syntactic means, under the assumption that "morphisms" strictly preserve limits and so on.

In each of these cases we deal with a total theory, but relative bifree structures are also available, as in 8.4 and 9.3, and intermediate steps can be chained.

10. Theories with values in involutive ordered categories.

These theories were introduced in [G2] for the 2-complete 2-category **RE**, whose objects generalize the categories of relations over exact categories. The *strict* 2-completeness of **RE** allowed to deduce the existence of the 2-universal model from the Freyd's Initial Object Theorem, and to derive the existence of the biuniversal model for reflective **EX**-theories. We treat here a more general case, **RO**.

Notice that the theories we consider now are still 2-functors T: $\mathbf{A} \to$ **CAT**, but the 2-category $\mathbf{A}$ is just 1-concrete (a 2-category of categories, functors and possibly non-natural transformations): this requires some slight modifications on the terminology of Part II; however, the definition of completeness and of universal model just concerns the 2-functor T and needs no adaptation.

10.1. An *RO-category* is a $\mathcal{U}$-category $\underline{A}$ with an involution $\tilde{}: \underline{A} \to \underline{A}$ (a contravariant involutory endofunctor, identical on objects) and with a consistent order on parallel morphisms; moreover we assume that the involution is *regular* ($a = a\tilde{a}a$ for any morphism a) and that for each object A the set Prj(A) of *projections* of A (endomorphisms e: A $\to$ A such that $e = ee = e\tilde{}$) is small. A morphism u in $\underline{A}$ is said to be *proper* whenever $u\tilde{}u \geq 1$ and $uu\tilde{} \leq 1$.

An *RO-functor* F: $\underline{A} \to \underline{B}$ preserves involution and order. An *RO-transformation* (or *proper-lax* transformation of RO-functors) ϕ: F $\to$ G: $\underline{A} \to \underline{B}$ is a collection (ϕA), with A $\in$ Ob$\underline{A}$, such that (1):

(1) for each $\underline{A}$-object A, ϕA: FA $\to$ GA is a proper morphism of $\underline{B}$,
(2) for each a: A $\to$ A' in $\underline{A}$, ϕA'.F$a \leq$ Ga.ϕA.

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(1) Indeed the "symmetrization" Rel(ϕ) of a natural transformation ϕ between exact functors is of such kind, generally non natural [G1].

These data form naturally a 2-category ([G1], Ch. 2): the definition of the horizontal composition of cells depends on the fact that the restriction of the order $\leqslant$ to proper morphisms is easily seen to be trivial; if u, v are proper and $u \leqslant v$ then $u = v$. **RO** is 2-complete (see the analogous proof in [G1], Ch. 9, for its 2-subcategory **RE**).

Moreover we have a 2-continuous 2-functor

$$\text{Prp}: \mathbf{RO} \to \mathbf{CAT}, \tag{3}$$

associating to each RO-category its subcategory of proper morphisms, and acting similarly on arrows and cells. However, we are not going to use this 2-functor to introduce RO-theories, because we do not want to confine their models $\Delta \to \mathbb{A}$ to take values in Prp $\mathbb{A}$. Thus we do *not* consider **RO** as a concrete 2-category and we must get out of the frame of Chapter 4; we only use the 1-functor

$$|\ |: \mathbf{RO}_1 \to \mathbf{CAT}_1 \tag{4}$$

between the associated categories, assigning to each RO-category its underlying category; the latter cannot be extended to cells.

10.2. For a small graph Δ consider the 2-extension $\mathbf{RO}_\Delta$ of **RO** obtained by a proceeding analogous to the construction of $\mathbf{A}_\Delta$ in 4.1: a morphism $t: \Delta \to \mathbb{A}$ is a pair $(\mathbb{A}, |t|: \Delta \to |\mathbb{A}|)$ where $|t|$ is a graph morphism; a cell $\tau: t_1 \to t_2: \Delta \to \mathbb{A}$ is a *proper-lax* transformation of graph morphisms (as in 10.1), possibly non-natural.

Thus the *total RO-theory* on Δ will be the 2-functor

$$T_\Delta = \mathbf{RO}_\Delta(\Delta,-): \mathbf{RO} \to \mathbf{CAT} \tag{1}$$

assigning to each RO-category $\mathbb{A}$ the category of all diagrams $\Delta \to \mathbb{A}$ with their proper-lax transformations. An *RO-theory* on Δ will be any sub-2-functor $T: \mathbf{RO} \to \mathbf{CAT}$ of T_Δ such that, for any $\mathbb{A}$ in **RO**, $T\mathbb{A}$ is a full subcategory of $T_\Delta\mathbb{A}$.

10.3. The theory T is *2-complete* (i.e., T is 2-continuous) iff it verifies three conditions (ROT.1-3) analogous to (T.1-3) in 5.3. T is said to be *reflective* if it verifies (ROT.1) and

(ROT.R) if $t: \Delta \to \mathbb{A}$ is a graph morphism, $F: \mathbb{A} \to \mathbb{A}'$ is a faithful RO-functor which reflects the order $\leqslant$ between parallel morphisms and

$Ft \in T(\mathbb{A}')$, then $t \in T(\mathbb{A})$ (1).

It is easy to prove, as in 6.5, that each reflective RO-theory is 2-complete and has a 2-classifying RO-category bounded by $\max(\mathrm{card}\ \Delta, \aleph_0)$. Actually, each model $t\colon \Delta \to \mathbb{A}$ factors through its restriction $t_1\colon \Delta \to \mathbb{A}_1$ where $\mathbb{A}_1$ is the involutive subcategory of $\mathbb{A}$ generated by the subgraph $t(\Delta)$, provided with the induced order; notice that $\mathbb{A}_1$ has the same objects as $t(\Delta)$. Since $\mathbb{A}_1$ is bounded by ω and clearly t_1 2-generates $\mathbb{A}_1$, the conclusion follows.

10.4. Syntactically, consider on Δ a set K of *RO-conditions*

(1) $$u \leqslant v$$

where u and v are parallel morphisms of the free involutive category $I(\Delta)$ generated by Δ (2).

These data define a reflective RO-theory $T = T(\Delta, K)$ on Δ: its models $t\colon \Delta \to \mathbb{A}$ are the graph morphisms satisfying the conditions of K:

(2) $$t^{-}u \leqslant t^{-}v,$$

where $t^{-}\colon I(\Delta) \to \mathbb{A}$ is the unique involution-preserving functor extending t.

Thus T has a 2-universal model, bounded as above. Conversely each reflective RO-theory can be presented in such a way.

10.5. Analogously one defines and treats theories with values in the 2-category **RE** [G1-3]; in particular the existence theorem for 2-universal models of **RE**-theories ([G2], Thm. 2.3) can be deduced from the present results. The relations between **RE**-theories and **EX**-theories where considered in [G2], Ch. 7.

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(1) Since each RO-category is balanced, any faithful RO-functor reflects the isos.

(2) Equivalently an RO-condition can be written as $u_m \ldots u_2 u_1 \leqslant v_n \ldots v_2 v_1$, where the u_i, v_i are arrows of Δ or *formal* involutes of such, and appropriate consecutiveness conditions (corresponding to composability in $I(\Delta)$) are imposed.

REFERENCES.

BE. A. (BASTIANI-)EHRESMANN & C. EHRESMANN, Categories of sketched structures, *Cahiers Top. et Géom. Diff.* XIII (1972), 105-214; reprinted in: *Charles Ehresmann: Oeuvres complètes et commentées*, Part IV-2, Amiens, 1983.

Bu. A. BURRONI, Algèbres graphiques, *Cahiers Top. et Géom. Diff.* XXII-3 (1981), 249-265.

G1. M. GRANDIS, On distributive homological algebra, I. RE-categories, *Cahiers Top. et Géom. Diff.* XXV-3 (1984), 259-301.

G2. M. GRANDIS, On distributive homological algebra, II. Theories and models, *Id.* XXV-4, 353-379.

G3. M. GRANDIS, On distributive homological algebra, III. Homological theories, *Id.* XXVI-2 (1985), 169-213.

Gr. P.A. GRILLET, Regular categories, *Lecture Notes in Math.* 236, Springer (1971), 121-222.

Je. T. JECH, *Set Theory*, Academic Press, 1978.

K1. G.M. KELLY, *Basic concepts of enriched category theory*, Cambridge Univ. Press, 1982.

K2. G.M. KELLY, On the essentially-algebraic theory generated by a sketch, *Bull. Austral. Math. Soc.* 26 (1982), 44-56.

KS. G.M. KELLY & R. STREET, Review of the elements of 2-categories, *Lecture Notes in Math.* 611, Springer (1977).

KR. A. KOCK & G.E. REYES, Doctrines in categorical logic, in: *Handbook of mathematical logic*, North Holland 1977, 283-313.

La. F.W. LAWVERE, Functorial semantics of algebraic theories, *Proc. Nat. Acad. Sci.* U.S.A., 50 (1963), 869-872.

Ma. S. MAC LANE, *Categories for the working mathematician*, Springer 1971.

Mi. B. MITCHELL, *Theory of categories*, Academic Press, 1965.

MS. J. MAC DONALD & A. STONE, Topoi over graphs, *Cahiers Top. et Géom. Diff.* XXV-1 (1984), 51-63.

Pu. D. PUPPE, Korrrespondenzen in abelschen Kategorien, *Math. Annalen* 148 (1962), 1-30.

S1. R. STREET, Limits indexed by category valued 2-functors, *J. Pure Appl. Alg.* 8 (1976), 149-181.

S2. R. STREET, Fibrations in bicategories, *Cahiers Top. et Géom. Diff.* XXI-2 (1980), 111-160.

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