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NOTE ON THE GLUING THEOREM FOR GROUPOIDS AND VAN KAMPEN'S THEOREM

by Klaus Heiner KAMPS

RÉSUMÉ. Le but de cette note est de montrer que le Théorème de van Kampen classique pour le *groupe* fondamental d'une union d'espaces peut être déduit du Théorème de van Kampen général pour le *groupoïde* fondamental dans [2] (voir aussi [6]) en appliquant le théorème de recollement pour les groupoïdes qui est un cas particulier d'un théorème général de recollement en théorie de l'homotopie abstraite (voir [7, 8]).

The following classical van Kampen Theorem determines under suitable local conditions the fundamental *group* $\pi_1(X,x_0)$ of a topological space X at $x_0 \in X$, when X is the union of two subspaces.

THEOREM 1. *Let* X_1, X_2 *be subspaces of a topological space* X *such that* X *is the union of the interiors of* X_1, X_2, let $X_0 = X_1 \cap X_2$ *and* $x_0 \in X_0$. *Then, if* X_0, X_1, X_2 *are path-connected, the diagram*

$$\begin{array}{ccc} \pi_1(X_0,x_0) & \longrightarrow & \pi_1(X_2,x_0) \\ \downarrow & & \downarrow \\ \pi_1(X_1,x_0) & \longrightarrow & \pi_1(X,x_0), \end{array}$$

where the arrows are induced by the inclusions of subspaces, is a pushout in the category G of groups.

It has been shown in [2] (see also [6], Ch. 17) that Theorem 1 can be deduced by a retraction argument from the following general van Kampen type theorem for the fundamental *groupoid* πX of a topological space X.

THEOREM 2. *Let* X_1, X_2 *be subspaces of a topological space* X *such that* X *is the union of the interiors of* X_1, X_2, let $X_0 = X_1 \cap X_2$. *Then the diagram*

$$\begin{array}{ccc} \pi X_0 & \longrightarrow & \pi X_2 \\ \downarrow & & \downarrow \\ \pi X_1 & \longrightarrow & \pi X \end{array},$$

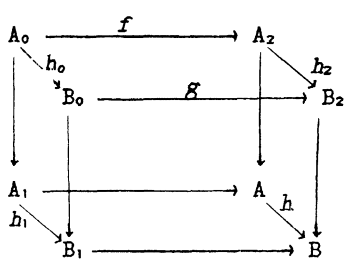
where the arrows are induced by the inclusions of subspaces, is a pushout in the category **Gd** *of groupoids.*

The aim of this note is to show that, alternatively, Theorem 1 can be deduced from Theorem 2 in a more conceptual manner by an application of the gluing theorem for groupoids which is a special case of a general gluing theorem in abstract homotopy theory.

This note was suggested by a talk given by M. Tierney.

In the category ***Top*** of topological spaces we have the following classical gluing theorem (see [3], 7.5.7).

THEOREM 3. *Let*



be a commutative diagram of maps of topological spaces, such that the front and the back squares are pushouts and f *and* g *are closed* *cofibrations. Then if h_0, h_1, h_2 are homotopy equivalences, the map h is also a homotopy equivalence.*

This theorem has been generalized to abstract homotopy theory in [7], (8.2), the category ***Top*** being replaced by an arbitrary category with pushouts C equipped with a homotopy system (I, j_0, j_1, q), i.e., a functor I: $C \to C$ called *cylinder functor* and natural transformations

$$j_0, j_1 \colon 1_C \to I, \quad q \colon I \to 1_C \text{ such that } qj_0 = qj_1 = 1.$$

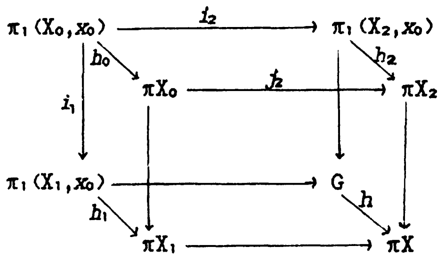
The cylinder functor has to be assumed to have a right adjoint and to satisfy suitable Kan conditions in low dimensions (see [7], 2,3). There are obvious notions of homotopy and homotopy equivalence in this abstract setting (see [7], (1.3), (1.6)). There is also a notion of cofibration defined by the homotopy extension property and a weaker notion of h-cofibration (see [7], (1.8)). Then the abstract version of the gluing theorem generalizes in two ways from closed cofibrations in the topological case to h-cofibrations in the abstract case.

In [8], 2,5 it has been shown that the category ***Gd*** of groupoids carries the structure of an abstract homotopy theory in the above sense. Here, the cylinder functor I is given by the product with the groupoid I consisting of two objects 0, 1 and only two non-identity morphisms $\iota \colon 0 \to 1$ and its inverse $\iota^{-1} \colon 1 \to 0$. In ***Gd*** a homotopy is a natural equivalence of functors (see also [3], 6.5, [4], 1). The notion of cofibration has been characterized in [5], 5.8.

PROPOSITION 4. *A functor of groupoids is a cofibration if and only if it is injective on objects.*

Note that this notion of cofibration for groupoids has also proved adequate for the construction of a model structure in the sense of Quillen [9] for the category of groupoids (see for example [1]).

In order to deduce Theorem 1 from Theorem 2 consider the following diagram, in which the front square is that of Theorem 2 and hence a pushout in the category ***Gd*** of groupoids by Theorem 2.



Let i_1, i_2 be the arrows induced by the inclusion of subspaces and let the back square be chosen as a pushout of i_1, i_2 in the category G of groups. Then the back square is also a pushout in *Gd*. Let h_0, h_1, h_2 denote the inclusions of a fundamental group into the corresponding fundamental groupoid and let h be the induced morphism from the back square pushout. Then i_2 and j_2 are cofibrations of groupoids by Proposition 4 and h_0, h_1, h_2 are homotopy equivalences of groupoids since in Theorem 1 X_0, X_1, X_2 are assumed to be path-connected. By an application of the gluing theorem for groupoids it follows that h is a homotopy equivalence of groupoids. Hence the induced homomorphism of groups $G \to \pi_1(X,x_0)$ is an isomorphism and Theorem 1 is proved.

References.

1. D.W. ANDERSON, Fibrations and geometric realizations, *Bull. A.M.S.* 84 (1978), 765-788.
2. R. BROWN, Groupoids and van Kampen's Theorem, *Proc. London Math. Soc.* (3) 17 (1967), 385-401.
3. R. BROWN, *Elements of Modern Topology*, McGraw Hill, 1968.
4. R. BROWN, Fibrations of groupoids, *J. Algebra* 15 (1970), 103-132.
5. P.R. HEATH & K.H. KAMPS, Induced homotopy in structured categories, *Rend. di Matem.* 9 (1976), 71-84.
6. P.J. HIGGINS, *Categories and Groupoids*, van Nostrand Reinhold Math. Studies 32, 1971.
7. K.H. KAMPS, Kan-Bedingungen und abstrakte Homotopietheorie, *Math. Z.* 124 (1972), 215-236.
8. K.H. KAMPS, Zur Homotopietheorie von Gruppoiden, *Arch. Math.* 23 (1972), 610-618.
9. D.G. QUILLEN, Homotopical Algebra, *Lecture Notes in Math.* 43, Springer, 1967.

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