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ONTO GELFAND TRANSFORMATIONS

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RÉSUMÉ. Cet article clarifie et généralise certains résultats sur les dualités de Binz-Hong-Nel. La méthode utilisée ramène l'étude de l'application de Gelfand appropriée au cas compact classique. D'autres exemples seront publiés ailleurs.

INTRODUCTION.

Since the original Binz duality [1] and the ensuing works of Porst-Wischnewsky [8] and Hong-Nel [4], much has been done to study surjectivity of the Gelfand transformation in topological situations. The aim of the author's article [2] was to develop a "non-topological" framework for spectral duality derived in this manner and, in this sequel to [2] (of which a brief relevant summary is made in Section 1), we describe a method of upgrading the original Binz duality in the spirit of Nel [6], while not depending on *ad hoc* topological constructions for contexts in which to make this study. This independence is also used in studying more general versions of duality in which the single dualising object (K) needs to be replaced by a functor from a (small) category.

The upgrading process is described in Section 2, where its seeming practical dependence on the classical Stone-Weierstrass Theorem becomes apparent. We produce a typical application in Section 3, where the reader will see how this relates to the other recent works in this direction. The main aim of Section 3, however, is to give a *unified* treatment of the dualities for limitspace rings (dualising object K = **R**), limitspace rings with involution (K = **C**), real pointed limitspace vector lattices (K = **R**), and complex such with involution (K = **C**). This is done by methodical reduction to the classical compact case; see Theorem 3.2 (cf. [4]).

In Section 4, the Gelfand transformation for zero-dimensional Hausdorff spaces is examined (and a correction to [2] Section 3 results). This section is mainly concerned with straight-out dualities for limitspace algebras over commutative Hausdorff topological rings with no idempotents (except 0 and 1) and no attempt is made to upgrade this result here.

It should be noted that the third limitspace axiom, governing finite intersections of convergent filters, is used *nowhere* in this article, so that the applications could equally well be based on "prelimitspaces" (just two axioms) or, equivalently, "net spaces" (Bourbaki v. Moore-Smith).

1. REVIEW OF RELEVANT SPECTRAL-DUALITY THEORY.

Let $\mathbf{V} = (\mathbf{V}_0, I, \otimes, [-,-], \ldots)$ be a complete symmetric monoidal closed base category and suppose that, unless otherwise indicated, the category theory is relative to this **V**. As part of completeness, let **V** admit canonical (epic, strong-monic)-factorisations and also the intersection of any class of strong monics with a common codomain.

Let C be a category (enriched over **V**) which admits canonical (epic, strong-monic)-factorisations, equalisers of pairs, and the intersection of any class of strong monics with a common codomain. Moreover, let K be a fixed "dualising" object in C such that all the cotensor powers (that is, exponents) of the form $[X,K]$, $X \in \mathbf{V}$, exist in C.

THEOREM 1.1. *There exists a full reflective subcategory* C" *of* C *containing all the exponents* $[X,K]$, $X \in \mathbf{V}$, *and such that* K *strongly* **V**-*cogenerates* C". *Moreover, if the canonical Gelfand transformation*

$$\epsilon_X \colon X \to C([X,K],K)$$

is epic, then the inclusion Spec $\subset$ **V** *is epireflective where* Spec *consists of the* K-*embeddable objects of* **V** *and there is a canonical* **V**-*equivalence between* Spec^{op} *and* C". •

For the proof of this result, we refer the reader to [2] Section 1. Recall that, in the proof, C" is constructed as the (**V**-enriched) epireflective hull of K in C', where C' is the (**V**-enriched) epireflective hull of K in C.

Let us call the objects of C' the K-*separated* objects of **C** and call an object C of **C** *point-separating* in the exponent [X,K] when ϵ_X is epic, C is a strong subobject of [X,K] in **C** (hence lies in **C'**), and the composite map

$$\mathbf{V}_o(I,X) \longrightarrow \mathbf{C}_o([X,K],K) \longrightarrow \mathbf{C}_o(C,K)$$

is an injection. Then, if I is a projective generator of $\mathbf{V}_o$, we have that the inclusion of C into [X,K] is an epic in C' since K **V**-cogenerators C'. Thus this inclusion is what we call a K-*dense* morphism, which is essentially the content of Nel's Stone-Weierstrass Theorem [7].

2. UPGRADING THE BINZ-HONG-NEL DUALITIES.

In this section we shall formulate conditions on **V** and **C** which allow "upgrading", or "enrichment", of some of the more prominent spectral dualities [1,4]. It is emphasised again that this upgrading process generally does *not* depend on special "topological" constructions; cartesian closed categories suffice.

Let $\mathbf{V} = (\mathbf{V}_o, I, \times, [-,-], \ldots)$ be a cartesian closed category (complete as in Section 1), and let $\mathbf{Lim} = (\mathbf{Lim}, 1, \times, [-,-], \ldots)$ denote the cartesian closed category of limitspaces and continuous maps between them. Suppose, furthermore, that there is a faithful, strong-monic preserving symmetric monoidal closed functor

$$U = (u, u^{\sim}, u^{\wedge}, u_o) : \mathbf{V} \to \mathbf{Lim}$$

(in the notation of [3]). Also let **Fin** denote the category of finite sets and let **C** be the (complete) category of universal algebras in **V**, defined by actions of the form

$$\chi_C : \Sigma_n \Omega(n) \times C^n \to C$$

where C is in **V** and Ω: **Fin** $\to \mathbf{V}_o$ is a fixed functor with the following properties:

(a) Each object of **C** is an abelian group object in **V** (under Ω);
(b) There is a distinguished ring object K in C;

(c) The function $u^{\hat{}}$: $u[X,K] \to [uX, uX]$ induces a dense subspace for all X in **V**.

We then obtain:

LEMMA 2.1. *The monoidal functor* U *induces a category* **Alg** *of universal algebras in* **Lim**, *by means of the canonical derived actions*

$$\Sigma_n u(n) \times (uC)^n \to u(\Sigma_n \Omega(n) \times C_n) \xrightarrow{u\alpha} uC$$

and there is an induced functor U: C $\to$ **Alg** *with a subspace map, also denoted* $u^{\hat{}}$, *from* uC(C,K) *to* **Alg**(uC, uK). •

THEOREM 2.2. *Given* C *as above (satisfying hypotheses* (a), (b), (c)), *the canonical Gelfand transformation* ϵ *is epic provided the canonical map*

$$uX \to \mathbf{Alg}([uX, uK], uK)$$

is a surjection in **Lim** *for all* X *in* **V**.

PROOF. It is required to show that the canonical Gelfand transformation

$$\epsilon_X: X \to C([X,K],K)$$

is an epic in **V** for all X in **V**. Apply U (which is assumed to be faithful) and obtain

$$u\epsilon: uX \to uC([X,K],K)$$

an epic, since

$$u^{\hat{}}: uC([X,K],K) \to \mathbf{Alg}(u[X,K], uK)$$

is an epic, while **Alg**$(u^{\hat{}}, uK)$ is a bijection by hypotheses (a) and (c) on U (draw a square!). •

REMARKS. In the examples, K = $\mathbf{R}^d$ as a limitspace abelian group, and hypothesis (c) is usually guaranteed by the *classical* Stone-Weierstrass Theorem applied at the level of uX being a compact Hausdorff space. The reason for this will become apparent in the following section, when we prove Lemma 3.1; see also Example 3.3. •

3. LIMITSPACE AND RELATED DUALITIES.

In order to give upgraded dualities, it is first desirable to reprove the original limitspace dualities in a generalisable manner.

Let m: X → D denote the dense inclusion of a Tychonoff space X into its Stone-Čech compactification D. Denote $\mathbf{R}^d$ by R, and suppose each algebra in **Alg** is (at least) a pointed abelian group with base point denoted by 1 (*not* equal to the identity 0 of the group). The reader should keep in mind the original example of Binz [1] in which **Alg** is the category of real limitspace algebras, $d = 1$, and R is given the structure of the usual ring of real numbers.

LEMMA 3.1. *The induced map* $[m,\mathrm{R}]$ *is dense monic in* **Lim**.

PROOF. It suffices to take $d = 1$ since dense maps between Hausdorff limitspaces are precisely the epics, hence are closed under cartesian product. Next, given any f in [X,R], we must show that it can be approximated to by continuous maps of bounded variation. To do this, take any finite subset $\mathrm{F} = \{x_1,\ldots, x_n\}$ of X and let f_F in [X,R] be such that:

(a) f_F is bounded, and
(b) $f_\mathrm{F}|\mathrm{F} = f|\mathrm{F}$;

this is possible since X is a Tychonoff space. Now, for each F and each real $\delta > 0$, let

$$\mathrm{W}(\mathrm{F};\delta) = \mathrm{W}(x_1,\ldots,x_n;\,\delta)$$
$$= \{g \in [\mathrm{X},\mathrm{R}];\ g \text{ bounded on X},\ g|\mathrm{F} = f|\mathrm{F}, \text{ and } |f_\mathrm{F}(x) - f(x_i)| < \delta \text{ implies } |g(x) - f(x_i)| < \delta \text{ for all } x_i \in \mathrm{F}\}.$$

Since

$$\mathrm{W}(\mathrm{F};\delta)\cap\mathrm{W}(\mathrm{G};\lambda) \supset \mathrm{W}(\mathrm{F}\cup\mathrm{G};\ \min(\delta,\lambda)),$$

we obtain a filter base on [X,R] which is easily seen to converge to f, as required. •

THEOREM 3.2. *Suppose each* **Alg**-*morphism from* [D,R] *to* R *is a point-evaluation map when* D *is compact Hausdorff. Then the canonical Gelfand transformation*

$$\epsilon_\mathrm{X}:\ \mathrm{X} \to \mathbf{Alg}([\mathrm{X},\mathrm{R}],\mathrm{R})$$

is epic in **Lim**.

PROOF. It suffices to show that ϵ_X is a surjection for each Tychonoff space X (otherwise, first take the Tychonoff reflection of X). Let m: X → D be the dense embedding of X into its Stone-Čech compactification D and let h from [D,R] to R be an **Alg**-morphism. By hypothesis, we have that $h.[m,1]$ is evaluation-at-p for some point p in D. Suppose p is not in X; then we obtain a contradiction from the fact that the filter on [D,R] generated by the base of sets of the form

$$B(V) = \{g \in [D,R];\ g(p) = 1,\ g(V^{\sim}) = 0\},$$

where V ⊂ D is an open neighborhood of p in D and $V^{\sim}$ is the complement of V in D, converges to 0 while evalutation-at-p maps this filter base to a base generating the principal filter ⟨1⟩. Thus p lies in X and Lemma 3.1 tells us that h is then evalutation-at-p. •

EXAMPLE 3.3. Let **V** be the cartesian closed category of bornological limitspaces, and let **C** be the category of real algebras in **V**, with K = **R** and the closed unit interval I having their usual bornologies. Let D be the closure of the image of the canonical map from X (in **V**) to the power of I indexed by the set $\mathbf{V}_o(X,I)$ in **V**. Then there are enough maps from D to I to separate points so the map $u^{\hat{}}_D$ from $u[D,R]$ to $[uD,R]$ is dense by the classical Stone-Weierstrass Theorem. It then follows from Lemma 3.1, and the fact that uD lies between uX and the usual Stone-Čech compactification of uX, that $u^{\hat{}}_X$ from $u[X,R]$ is dense, as required to satisfy hypothesis (c). Since hypotheses (a) and (b) already hold, we have a spectral duality with K = **R**, by Theorem 2.2. •

Note that the other examples of Nel [6] can be treated in a similar manner since there are, in each case, enough functions into the unit interval to separate points. Analogous results hold for bornological limitspace rings with involution (K = C), and pointed real bornological limitspace vector lattices (K = R).

4. ZERO-DIMENSIONAL SPECTRAL SPACES.

The dualities presented in this section concern limitspace K-algebras and continuous K-algebra homomorphisms (here called *morphisms*) over a commutative Hausdorff topological ring K with no idempotents except 0 and 1. We show that if **Alg** denotes such a category, then:

PROPOSITION 4.1. *If X is a zero-dimensional Hausdorff topological space, then the Gelfand transformation*

$$X \to \mathbf{Alg}([X,K],K)$$

is a surjection (hence a homeomorphism).

PROOF. Let h from $[X,K]$ to K be a morphism which is not a point-evaluation map. Then, for each x in X, there exists f_x in $\ker(h)$ with $f_x(x) \neq 0$. Let ϕ_x denote the filter on $[X,K]$ generated by the base of sets of the form

$$V^{\wedge} = \{e_W{}^* \ ; \ x \in W \subset V, \ W \text{ clopen}\},$$

where V is a clopen set containing x and $e_V{}^* = 1 - e_V$ where e_V is the idempotent

$$e_V(w) = 1 \text{ if } w \in V \text{ and } e_V(w) = 0 \text{ otherwise.}$$

Using the facts that X is zero-dimensional Hausdorff, K is Hausdorff, and f_x is continuous, we easily verify that the filter generated by

$$(f_x(x) - f_x)\ \phi_x$$

converges to the function $f_x(x) - f_x$ in $[X,K]$. Since $f_x(x) \neq 0$ and h is a morphism with $h(f_x) = 0$, we deduce that $h(e_V{}^*) \neq 0$ whence $h(e_V) = 0$ since 0 and 1 are the only idempotents of K. Thus our assumption gives, for each point x in X, a clopen $V(x)$ containing x and being such that $h(e_{V(x)}) = 0$. We then obtain a contradiction from the fact that the filter generated on $[X,K]$ by the base of sets of the form

$$B(x_1,\ldots,x_n) = \{e_{\cup_F V(x)} \ ; \ \{x_1,\ldots,x_n\} \subset F \subset X, \ F \text{ finite}\}$$

converges to 1 in $[X,K]$, while $h(e_{\cup_F V(x)}) = 0$ since $h(e_{V_i}) = 0$ for $i = 1,\ldots n$ (by induction on n) implies $h(e_{\cup V_i}) = 0$ since h is a (continuous) K-algebra morphism. •

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