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FINITE OBJECTS AND EXTENSIONAL RELATIONS
BY Osvaldo ACUÑA-ORTEGA

RÉSUMÉ. Dans cet article, on caractérise les cardinaux finis dans un topos arbitraire au moyen des relations extensionnelles (Corollaire 11 du Théorème 10). On y démontre aussi que l'objet N des nombres naturels est bien ordonné dans le sens suivant:

$$\models \forall_{x \in \Omega N} (X \in 2^N \wedge \exists_{a \in N} a \in X \\ \Rightarrow \exists_{b \in N} b \in X \wedge \forall_{n \in N} (n \in X \Rightarrow b \leq n))$$

(Corollaire 4 du Théorème 3).

This paper presents a characterization of the finite cardinals in any topos by means of extensional relations. In the process we prove that N the natural numbers object is well ordered in the following sense:

$$\models \forall_{x \in \Omega N} (X \in 2^N \wedge \exists_{a \in N} a \in X \\ \Rightarrow \exists_{b \in N} \forall_{n \in N} (n \in X \wedge n \leq b \Rightarrow n = b) \wedge b \in X).$$

In the boolean case this was proved in [5].

If E is an arbitrary topos, $Y \in |E|$, $K(Y)$ is the smallest subobject of Ω^Y that contains

$$\{.\}_Y: Y \rightarrow \Omega^Y, \quad '0': 1 \rightarrow \Omega^Y$$

and is closed under binary unions. $K^+(Y)$ is the smallest subobject of Ω^Y that contains $\{.\}_Y: Y \rightarrow \Omega^Y$ and is closed under binary unions.

OBSERVATIONS.

- (i) $K^+(Y) = \{W \in K(Y) \mid \exists_{x \in Y} x \in W\}.$
- (ii) $K^+(Y) \subset K(Y) \xleftarrow{'0'}} 1$ is a coproduct diagram: $K(Y) = K^+(Y) + 1.$
- (iii) If Y is decidable

$$(\models \forall_{x, y \in Y} (x = y \vee x \neq y))$$

then $K(Y)$ is decidable. $K(Y)$ is a boolean subring of 2^Y , and moreover $K(Y)$ is an ideal of 2^Y .

The first is obvious, the second is Proposition 3.8 in [1], and the third is the union of Corollary 3.7 and Corollary 3.9 of [1].

DEFINITION. Let E be an arbitrary topos, $Y \in |E|$, $\leq \subset Y \times Y$ a partial order. If $\leq = \leq \cap (\neg \Delta_Y)$, then:

(a) $(Y, \leq)$ is *extensional* if $r: Y \rightarrow \Omega^Y$ is monic, where

$$r(x) = \{ y \in Y \mid y \leq x \}.$$

(b) $(Y, \leq)$ is *strongly inductive* if:

$$\models \forall x_1 \in \Omega^Y (\forall x \in Y (\{y \in Y \mid y \leq x\} \subset X_1 \Rightarrow x \in X_1) \Rightarrow X_1 = \ulcorner Y \urcorner).$$

(c) $(Y, \leq)$ is *strongly transitive* if it is extensional and strongly inductive.

(d) If X_1 is a variable of type Ω^Y and x is a variable of type Y , then " x minimal X_1 " denotes the formula

$$\forall x \in Y (y \leq x \wedge y \in X_1 \Rightarrow y = x) \wedge x \in X_1,$$

analogously define " x maximal X_1 ".

(e) $L(Y)$ and $L^{op}(Y)$ denote

$$\{X \in 2^Y \mid \forall x_1 \in \Omega^Y (X_1 \subset X \wedge X_1 \in K^+(Y) \Rightarrow \exists x \in Y \text{ } x \text{ minimal } X_1)\}$$

and

$$\{X \in 2^Y \mid \forall x_1 \in \Omega^Y (X_1 \subset X \wedge X_1 \in K^+(Y) \Rightarrow \exists x \in Y \text{ } x \text{ maximal } X_1)\}$$

respectively.

(f) " X_1 is l.or." denotes the formula:

$$\forall x, y \in Y (x, y \in X_1 \Rightarrow x = y \vee x < y \vee y < x).$$

LEMMA 1. Let E be an arbitrary topos, $(Y, \leq)$ a partially ordered object, Y decidable. Then $K(Y) \subset L(Y) \cap L^{op}(Y)$.

PROOF. It suffices to prove that $L(Y)$ is closed under binary unions and contains

$$\ulcorner \emptyset \urcorner: 1 \rightarrow \Omega^Y \quad \text{and} \quad \{.\}_Y: Y \rightarrow \Omega^Y.$$

(i) It is clear that $\langle \delta \rangle: 1 \rightarrow \Omega^Y$ factorizes through $L(Y)$.

(ii) $\langle \cdot \rangle_Y: Y \rightarrow \Omega^Y$ factorizes through $L(Y)$:

$$\begin{aligned} \models x \in \langle Y \rangle \wedge X_1 \subset \{x\} \wedge X_1 \in K^+(Y) &\Rightarrow X_1 = \{x\} \\ &\Rightarrow (\forall_{y \in Y} y \in X_1 \wedge y \leq x \Rightarrow y = x) \wedge x \in X_1 \\ &\Rightarrow \exists_{y \in Y} y \text{ minimal } X_1. \end{aligned}$$

Therefore

$$\models x \in Y \Rightarrow \forall_{X_1 \in \Omega Y} (X_1 \subset \{x\} \wedge X_1 \in K^+(Y) \Rightarrow \exists_{y \in Y} y \text{ minimal } X_1).$$

(Y is decidable: $\models \{x\} \in 2^Y \Rightarrow \{x\} \in L(Y)$).

(iii) $L(Y)$ is closed under binary unions:

$$\models a, b \in L(Y) \wedge I \subset a \cup b \wedge I \in K^+(Y)$$

$$\begin{aligned} &\Rightarrow a, b \in L(Y) \wedge a \cap I \subset a \wedge b \cap I \subset b \wedge I \in K^+(Y) \wedge I \subset a \cup b \\ &\Rightarrow a, b \in L(Y) \wedge (a \cap I \in K^+(Y) \vee a \cap I = \langle \delta \rangle) \wedge a \cap I \subset a \wedge b \cap I \\ &\quad \subset b \wedge I \in K^+(Y) \wedge I \subset a \cup b \\ &\Rightarrow ((a, b \in L(Y) \wedge a \cap I \in K^+(Y) \wedge a \cap I \subset a \wedge b \cap I \subset b) \\ &\quad \vee (I \subset b \wedge I \in K^+(Y) \wedge b \in L(Y))) \wedge I \subset a \cup b \\ &\Rightarrow ((a, b \in L(Y) \wedge I, a \cap I \in K^+(Y) \wedge a \cap I \subset a \wedge b \cap I \subset b) \\ &\quad \vee \exists_{z \in Y} z \text{ minimal } I) \wedge I \subset a \cup b \\ &\Rightarrow (\exists_{x \in Y} x \text{ minimal } a \cap I \wedge \exists_{y \in Y} y \text{ minimal } b \cap I \wedge I \subset a \cup b) \\ &\quad \vee \exists_{z \in Y} z \text{ minimal } I \\ &\Rightarrow (\exists_{x \in Y} \exists_{y \in Y} (x \text{ minimal } a \cap I \wedge y \text{ minimal } b \cap I \wedge I \subset a \cup b) \\ &\quad \vee \exists_{z \in Y} z \text{ minimal } I \\ &\Rightarrow (\exists_{x, y \in Y} x \text{ minimal } a \cap I \wedge y \text{ minimal } b \cap I \wedge (\{s \in a \cap I \mid s < y\} = \langle \delta \rangle \\ &\quad \vee \{s \in a \cap I \mid s < y\} \in K^+(Y)) \wedge I \subset a \cup b) \vee \exists_{z \in Y} z \text{ minimal } I \\ &\Rightarrow (\exists_{x, y \in Y} y \text{ minimal } I \vee \exists_{x, y \in Y} x \text{ minimal } a \cap I \wedge y \text{ minimal } b \cap I \\ &\quad \wedge (\{s \in a \cap I \mid s \leq y\} \in K^+(Y)) \wedge I \subset a \cup b) \vee \exists_{z \in Y} z \text{ minimal } I \\ &\Rightarrow (\exists_{x, y, w \in Y} x \text{ minimal } a \cap I \wedge y \text{ minimal } b \cap I \\ &\quad \wedge w \text{ minimal } \{s \in a \cap I \mid s \leq y\} \wedge I \subset a \cup b) \vee \exists_{z \in Y} z \text{ minimal } I. \end{aligned}$$

On the other hand:

$$\begin{aligned} &\models x \text{ minimal } a \cap I \wedge y \text{ minimal } b \cap I \wedge w \text{ minimal } \{s \in a \cap I \mid s \leq y\} \\ &\quad \wedge I \subset a \cap b \wedge r \leq w \wedge r \in I \\ &\Rightarrow x \text{ minimal } a \cap I \wedge y \text{ minimal } b \cap I \wedge w \text{ minimal } \{s \in a \cap I \mid s \leq y\} \\ &\quad \wedge r \leq w \wedge (r \in a \cap I \vee r \in b \cap I) \\ &\Rightarrow r = w \vee (x \text{ minimal } a \cap I \wedge y \text{ minimal } b \cap I \\ &\quad \wedge w \text{ minimal } \{s \in a \cap I \mid s \leq y\} \wedge r \leq w \wedge r \in b \cap I) \\ &\Rightarrow r = w \vee (r \leq w \wedge w \leq y \wedge r = y) \\ &\Rightarrow r = w \vee (r = w) \qquad \qquad \qquad \Rightarrow r = w. \end{aligned}$$

Then we have

$$\models x \text{ minimal } a \cap b \wedge y \text{ minimal } b \cap I \wedge w \text{ minimal } \{s \in a \cap I \mid s \leq y\} \\ \Rightarrow w \text{ minimal } I.$$

Therefore:

$$\models a, b \in L(Y) \wedge I \subset a \cup b \wedge I \in K^+(Y) \Rightarrow \exists_{w \in Y} w \text{ minimal } I.$$

Then:

$$\models a, b \in L(Y) \Rightarrow \forall_{I \in \Omega^Y} (I \subset a \cup b \wedge I \in K^+(Y) \Rightarrow \exists_{w \in Y} w \text{ minimal } I). \\ \Rightarrow a \cup b \in L(Y).$$

We have proved that $K(Y) \subset L(Y)$. Replacing $\leq$ by

$$\leq^{\text{op}} (= \{(x, y) \mid (y, x) \in \leq\})$$

and applying the previous argument we have

$$K(Y) \subset L^{\text{op}}(Y), \quad \text{and so } K(Y) \subset L(Y) \cap L^{\text{op}}(Y).$$

COROLLARY 2. *If E is an arbitrary topos, $Y \in |E|$ decidable, $\leq \subset Y \times Y$ a partial order. Then:*

(a) *If $(Y, \leq)$ is linearly ordered then there exist $f, g: K^+(Y) \rightarrow Y$ such that:*

$$\models X_1 \in K^+(Y) \Rightarrow (f(X_1) \in X_1 \wedge g(X_1) \in X_1 \\ \wedge \forall_{x \in Y} (x \in X_1 \Rightarrow f(X_1) \leq x \wedge x \leq g(X_1))).$$

(b) *If $r: Y \rightarrow \Omega^Y$ is such that*

$$\models r(y) = \{x \in Y \mid x < y\}$$

and r factors through $K(Y)$ then:

$$\models Y_1 \in (2^Y)^+ \Rightarrow \exists_{x \in Y} x \text{ minimal } Y_1$$

where

$$\models (2^Y)^+ = \{X_1 \in 2^Y \mid \exists_{x \in Y} x \in X_1\}.$$

PROOF. (a) It follows immediately from Lemma 1.

$$\begin{aligned} \text{(b)} \quad & \models Y_1 \in (2^Y)^+ \Rightarrow \exists_{y \in Y} Y_1 \in 2^Y \wedge y \in Y_1 \\ & \Rightarrow \exists_{y \in Y} Y_1 \in 2^Y \wedge \{x \mid x \leq y\} \in K(Y) \wedge y \in Y_1 \\ & \Rightarrow \exists_{y \in Y} y \in Y_1 \cap \{x \mid x \leq y\} \wedge Y_1 \cap \{x \mid x \leq y\} \in K(Y) \\ & \Rightarrow Y_1 \cap \{x \mid x \leq y\} \in K^+(Y) \wedge y \in Y_1 \\ & \Rightarrow \exists_{x \in Y} x \text{ minimal } Y_1 \cap \{x \mid x \leq y\} \wedge y \in Y_1 \\ & \Rightarrow \exists_{x \in Y} x \text{ minimal } Y_1. \end{aligned}$$

THEOREM 3. If E is an arbitrary topos, $Y \in |E|$ decidable, $\leq \subset Y \times Y$ a linear order and if $r: Y \rightarrow \Omega^Y$ factors through $K(Y)$. Then there exists $f: (2^Y)^+ \rightarrow Y$ such that

$$\models Y_1 \in (2^Y)^+ \Rightarrow f(Y_1) \in Y_1 \wedge \forall_{x \in Y} (x \in Y_1 \Rightarrow f(Y_1) \leq x).$$

Proof. Immediate from Corollary 2.

COROLLARY 4. If E is an arbitrary topos and $X \in |E|$ is a finite cardinal or the natural numbers object and $\leq \subset X \times X$ is the canonical order. Then there exists $f: (2^X)^+ \rightarrow X$ such that:

$$\models A \in (2^X)^+ \Rightarrow f(A) \in A \wedge \forall_{n \in X} (n \in A \Rightarrow f(A) \leq n).$$

PROOF. Obvious.

Corollary 4 shows that the natural numbers object is well ordered in the classical sense when E is boolean, this result was proved in [5].

PROPOSITION 5. Let E be an arbitrary topos, $Y \in |E|$, K -finite, decidable and $\leq \subset Y \times Y$ a partial order. Then:

$$\models \forall_{X_1 \in 2^Y} (\forall_{x \in Y} (\{y \in Y \mid y < x\} \subset X_1 \Rightarrow x \in X_1) \Rightarrow X_1 = 'Y').$$

PROOF.

$$\begin{aligned} & \models X_1 \in 2^Y \wedge \forall_{x \in Y} (\{y \in Y \mid y < x\} \subset X_1 \Rightarrow x \in X_1) \\ & \Rightarrow X_1 \in 2^Y \wedge \forall_{x \in Y} (\{y \in Y \mid y < x\} \subset X_1 \Rightarrow x \in X_1) \\ & \qquad \qquad \qquad \wedge (X_1 = 'Y' \vee 'Y' - X_1 \neq \emptyset) \\ & \Rightarrow X_1 = 'Y' \wedge \forall_{x \in Y} (\{y \in Y \mid y < x\} \subset X_1 \Rightarrow x \in X_1) \\ & \qquad \qquad \qquad \wedge \exists_{z \in Y} z \text{ minimal } ('Y' - X_1) \\ & \Rightarrow X_1 = 'Y' \vee \forall_{x \in Y} (\{y \in Y \mid y < x\} \subset X_1 \Rightarrow x \in X_1) \\ & \qquad \qquad \qquad \wedge \exists_{z \in Y} \{y \in Y \mid y < z\} \subset X_1 \wedge z \notin X_1 \\ & \Rightarrow X_1 = 'Y' \vee z \in X_1 \wedge z \notin X_1 \Rightarrow X_1 = 'Y' \vee \text{false} \\ & \Rightarrow X_1 = 'Y'. \end{aligned}$$

COROLLARY 6. Let E be boolean, $Y \in |E|$, K -finite and $\leq \subset Y \times Y$ a partial order. Then $(Y, \leq)$ is strongly inductive.

PROOF. Apply Proposition 5.

PROPOSITION 7. Let X be decidable, $\leq \subset X \times X$ a partial order. If $r: X \rightarrow \Omega^*$ factors through $K(X)$ and $(X, \leq)$ is extensional then

$$\models x \in X \wedge y \in 'X' \wedge \{z \mid z \leq x\} \text{ is l.or.} \wedge \{z \mid z \leq y\} \text{ is l.or.} \\ \Rightarrow x < y \vee y < x \vee y = x.$$

PROOF. Consider

$$\models x \in 'X' \wedge p \text{ minimal } 'X' \Rightarrow x \in 'X' \wedge \{y \mid y < p\} = '\emptyset' \\ \Rightarrow (\{y \mid y < x\} = '\emptyset' \vee \{y \mid y < x\} \in K^+(X) \wedge \{y \mid y < p\} = '\emptyset') \\ \Rightarrow x = p \vee (\exists_{q \in X} q \text{ minimal } \{y \mid y < x\} \wedge \{y \mid y < p\} = '\emptyset') \\ \Rightarrow x = p \vee (\{y \mid y < q\} = '\emptyset' \wedge \{y \mid y < p\} = '\emptyset' \wedge q < x) \\ \Rightarrow x = p \vee (q = p \wedge q < x) \Rightarrow x = p \vee p < x.$$

Therefore

$$\models x, y \in X \wedge p \text{ minimal } 'X' \Rightarrow (p < x \vee p = x) \wedge (p < y \vee p = y) \\ \Rightarrow (p < x \wedge p < y) \vee y < x \vee x < y \vee x = y \\ \Rightarrow (\exists_{p \in X} p < x \wedge p < y) \vee y < x \vee x < y \vee x = y \\ \Rightarrow \{z \mid z < x \wedge z < y\} \in K^+(X) \vee y < x \vee x < y \vee x = y \\ \Rightarrow (\exists_{r \in X} r \text{ maximal } \{z \mid z < x \wedge z < y\}) \vee y < x \vee x < y \vee x = y.$$

On the other hand

$$\models a \text{ minimal } \{z \mid r < z \wedge z \leq x\} \wedge b \text{ minimal } \{z \mid r < z \wedge z \leq y\} \\ \wedge \{z \mid z \leq x\} \text{ is l.or.} \wedge \{z \mid z \leq y\} \text{ is l.or.} \wedge h < a \\ \Rightarrow a \text{ minimal } \{z \mid r < z \wedge z \leq x\} \wedge r < b \wedge h < a \wedge r < a \\ \Rightarrow a \text{ minimal } \{z \mid r < z \wedge z \leq x\} \wedge r < b \\ \wedge (h \leq r \vee r < h) \wedge h \leq a \wedge h \neq a \\ \Rightarrow h < b \vee (a \text{ minimal } \{z \mid r < z \wedge z \leq x\} \\ \wedge r < h \wedge h \leq x \wedge h \leq a \wedge h \neq a \\ \Rightarrow h < b \vee (h = a \wedge h \neq a) \Rightarrow h < b \vee \text{false} \Rightarrow h < b.$$

By symmetry we obtain that

$$\models a \text{ minimal } \{z \mid r < z \wedge z \leq x\} \wedge b \text{ minimal } \{z \mid r < z \wedge z \leq y\} \\ \wedge \{z \mid z \leq x\} \text{ is l.or.} \wedge \{z \mid z \leq y\} \text{ is l.or.} \\ \Rightarrow \{h \mid h < b\} = \{h \mid h < a\} \Rightarrow a = b.$$

Then:

$$\models x \in X \wedge y \in Y \wedge p \text{ minimal } 'X' \wedge \{z \mid z \leq x\} \text{ is l.or.} \\ \wedge \{z \mid z \leq y\} \text{ is l.or.} \\ \Rightarrow (\exists_{r \in X} r \text{ maximal } \{z \mid z < x \wedge z < y\}) \\ \wedge \exists_{a, b \in X} a \text{ minimal } \{z \mid r < z \wedge z \leq x\} \wedge b \text{ minimal } \{z \mid r < z \wedge z \leq y\} \\ \wedge \{z \mid z \leq x\} \text{ is l.or.} \wedge \{z \mid z \leq y\} \text{ is l.or.} \vee x < y \vee y < x \vee x = y$$

$$\begin{aligned}
&\Rightarrow (\exists r, a, b \in X \text{ } r \text{ maximal } \{z \mid z < x \wedge z < y\} \wedge a = b \\
&\quad \wedge r < a \wedge a \in x \wedge r < b \wedge b \in y) \vee x < y \vee y < x \vee x = y \\
&\Rightarrow (\exists r, a \in X \text{ } r \text{ maximal } \{z \mid z < x \wedge z < y\} \\
&\quad \wedge r < a \wedge a \in x \wedge a \in y) \vee x < y \vee y < x \vee x = y \\
&\Rightarrow (\exists r, a \in X \text{ } r \text{ maximal } \{z \mid z < x \wedge z < y\} \wedge (a = x \vee a < x) \\
&\quad \wedge (a = y \vee a < y) \wedge r < a) \vee x < y \vee y < x \vee x = y \\
&\Rightarrow (\exists r, a \in X \text{ } r \text{ maximal } \{z \mid z < x \wedge z < y\} \wedge a < x \wedge a < y \\
&\quad \wedge r \in a \wedge r \neq a) \vee x = y \vee x < y \vee y < x \vee x = y \\
&\Rightarrow (\exists r, a \in X \text{ } r = a \wedge r \neq a) \vee x < y \vee y < x \vee x = y \\
&\Rightarrow \text{false} \vee x = y \vee x < y \vee y < x \\
&\Rightarrow x = y \vee x < y \vee y < x.
\end{aligned}$$

Resuming all the preceding arguments we have:

$$\begin{aligned}
&\models x \in X \wedge y \in X \wedge \{z \mid z \in x\} \text{ is l.or.} \wedge \{z \mid z \in y\} \text{ is l.or.} \\
&\Rightarrow x \in X \wedge y \in X \wedge \{z \mid z \in x\} \text{ is l.or.} \wedge \{z \mid z \in y\} \text{ is l.or.} \\
&\quad \wedge \{z \mid z \in y\} \in K'(X) \\
&\Rightarrow x \in X \wedge y \in X \wedge \{z \mid z \in x\} \text{ is l.or.} \wedge \{z \mid z \in y\} \text{ is l.or.} \\
&\quad \wedge \exists_{p \in X} p \text{ minimal } \{z \mid z \in y\} \\
&\Rightarrow x \in X \wedge y \in X \wedge \{z \mid z \in x\} \text{ is l.or.} \wedge \{z \mid z \in y\} \text{ is l.or.} \\
&\quad \wedge \exists_{p \in X} p \text{ minimal 'X'} \\
&\Rightarrow \exists_{p \in X} x = y \vee x < y \vee y < x \Rightarrow x = y \vee x < y \vee y < x.
\end{aligned}$$

LEMMA 8. Let X be decidable, $\in \subset X \times X$ a partial order. If $r: X \rightarrow \Omega^X$ factors through $K(X)$ and $(X, \in)$ is strongly transitive then:

$$\models \forall_{x, y \in X} (x < z \wedge y < z \Rightarrow x = y \vee x < y \vee y < x).$$

PROOF. If

$$S = \{y \in X \mid \{x \in X \mid x < y\} \text{ is l.or.}\}$$

we want to prove that $S = X$:

$$\begin{aligned}
&\models \{z \mid z < w\} \subset 'S' \wedge x < w \wedge y < w \\
&\Rightarrow x \in X \wedge y \in X \wedge \{z \mid z < x\} \text{ is l.or.} \wedge \{z \mid z < y\} \text{ is l.or.} \\
&\Rightarrow x = y \vee x < y \vee y < x.
\end{aligned}$$

Therefore:

$$\begin{aligned}
&\models \{z \mid z < w\} \subset 'S' \Rightarrow \forall_{x, y \in X} (x, y \in \{z \mid z < w\} \Rightarrow x = y \vee x < y \vee y < x) \\
&\Rightarrow \{z \mid z < w\} \text{ is l.or.} \Rightarrow w \in 'S'.
\end{aligned}$$

Since $(X, \in)$ is strongly inductive we have that $S = X$.

THEOREM 9. *Let X be decidable, $\leq \subset X \times X$ a partial order. If $r: X \rightarrow \Omega^*$ factors through $K(X)$ and $(X, \leq)$ is strongly transitive, then $(X, \leq)$ is linearly ordered.*

PROOF.

$$\begin{aligned} & \models x \in X \wedge y \in X \\ \Rightarrow \{z \mid z < x\} \in {}^1S^1 \wedge \{z \mid z < y\} \in {}^1S^1 \wedge x \in X \wedge y \in X \\ \Rightarrow x = y \vee x < y \vee y < x. \end{aligned}$$

The converse is also true. Since complemented non-empty subobjects of X have a minimum, $(X, \leq)$ is extensional. Let Z be the image of

$$\text{id} \times r: X \times X \rightarrow X \times \Omega^* ;$$

since $1 = \text{pr}_1 \mid Z$ is a finite cardinal in E/X by Corollary 9 of [3], 1 is strongly inductive. It is easy to prove that if 1 is strongly inductive so is X .

Let $E_{K, d}$ be the full subcategory of K -finite decidable objects of E . This category is a boolean topos, see [4].

THEOREM 10. *Let X be K -finite decidable and $\leq \subset X \times X$ a partial order. If $r: X \rightarrow \Omega^*$ factors through $K(X)$ then the following propositions are equivalent:*

- (a) $(X, \leq)$ is linearly ordered in E .
- (b) $(X, \leq)$ is extensional in E .
- (c) $(X, \leq)$ is strongly transitive in E .

PROOF. (c) $\Rightarrow$ (a): Theorem 9.

(a) $\Rightarrow$ (c): X is a finite cardinal and by Corollary 9 of [3] we have that $(X, \leq)$ is strongly inductive. Since $(X, \leq)$ has a successor function (Proposition 2 of [3]), then it is extensional.

(b) $\Rightarrow$ (a): by Proposition 1 of [2] we know that $(X, \leq)$ is linearly ordered in E iff it is linearly ordered in $E_{K, d}$. On the other hand applying Proposition 5 to $(X, \leq)$ in $E_{K, d}$ we have that $(X, \leq)$ is strongly inductive in $E_{K, d}$ and by Theorem 9, $(X, \leq)$ is linearly ordered since $K(X) = 2^X$ and $(X, \leq)$ is extensional in $E_{K, d}$.

(a) $\Rightarrow$ (b): already proved.

In [2] we defined X to be a finite cardinal when K is K -finite, decidable and linearly ordered.

COROLLARY 11. *Let X be a K -finite decidable object of E . The following propositions are equivalent:*

- (a) *X is a finite cardinal.*
- (b) *There exists $\leq \subset X \times X$ a partial order such that $(X, \leq)$ is extensional in E and $r: X \rightarrow \Omega^*$ factors through $K(X)$.*
- (c) *There exists $\leq \subset X \times X$ a partial order such that $(X, \leq)$ is strongly transitive and $r: X \rightarrow \Omega^*$ factors through $K(X)$.*

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