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ROSS STREET

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CORRECTION TO "FIBRATIONS IN BICATEGORIES"

by Ross STREET

Given homomorphisms of bicategories J: $\mathcal{A} \to Cat$ and S: $\mathcal{A} \to \mathcal{K}$, the bilimit {J, S} of S indexed by J can be constructed using biproducts, cotensor biproducts and biequalizers. However, the construction described in Section 1 (1.24), page 120, of the paper "Fibrations in bicategories" (these *Cahiers*, XXI-2, 1980, 111-160) is wrong. I am grateful to Max Kelly for noticing this error. He also recognized that {J,S} could be constructed if $\mathcal{K}$ admitted some further bilimits of a simple kind. In fact, no further bilimits are needed: I shall show that they can be constructed from those at hand.

Certain small categories **Iso, End, Aut,** T will be required. A functor from one of these into a category amounts to an isomorphism, endomorphism, automorphism, composable pair of isomorphisms, respectively, in the category. Notice that **Iso,** T are equivalent to 1 while **Aut** is not.

The *biequifier* of 2-cells

$$\theta, \phi: f \Rightarrow g : A \to B$$

in $\mathcal{K}$ is an arrow h: X → A which, for all objects X, induces an equivalence of categories between $\mathcal{K}$(K, X) and the full subcategory of $\mathcal{K}$(K, A) consisting of those a: K → A for which $\theta a = \phi a: fa \Rightarrow ga$. (If θ, φ are invertible this is the same as the *biequinverter* of θ, φ as used later (4.2) in the paper.) The *biidentifier* of an endo-2-cell

$$\psi: f \Rightarrow f: A \to B$$

is the biequifier of ψ and the identity of f. If φ is invertible, the biequifier of θ, φ is the biidentifier of $\phi^{-1}\theta$. So to construct biequifiers of invertible pairs it suffices to construct biidentifiers of auto-2-cells.

Consider an auto-2-cell $\psi: f \Rightarrow f: A \to B$. Let ψ^{-}, $f^{-}: A \to \{\mathbf{Aut}, B\}$ be the arrows corresponding to the automorphisms ψ, 1_f in $K(A, B)$. Let $h: H \to A$, $\sigma: \psi^{-}h \simeq f^{-}h$, be the biequalizer of ψ^{-}, f^{-}. Let $k: K \to A$, $\tau: fk \simeq fk$, be the biequalizer of f, f. Let $e: \{\mathbf{Aut}, B\} \to B$ be induced by the unique functor $1 \to \mathbf{Aut}$. There exist $l: H \to K$ and $v: kl \simeq h$ rendering $e\sigma$ isomorphic to τl by definition of K. Similarly, we obtain $d: A \to K$ and $kd \simeq 1_K$ rendering 1_f isomorphic to τd. Now form the bi-pullback

$$\begin{array}{ccc} P & \xrightarrow{\ v\ } & H \\ {\scriptstyle u}\downarrow & = & \downarrow{\scriptstyle l} \\ A & \xrightarrow[\ d\ ]{} & K \end{array}$$

of d, l ; this is just the biequalizer of the two arrows from the biproduct of A, H into K which use d, l and the projections. I claim $u: P \to A$ is the biidentifier of $\psi: f \Rightarrow f$. Since bilimits are defined representably, we only need to check the construction in *Cat*. Then K can be taken to be the category of pairs (a, τ) where a is an object of A and $\tau: fa \simeq fa$ in B, while H can be taken to be the full subcategory of K consisting of the pairs (a, σ) with $\sigma.\psi a = \sigma$. Since σ is invertible, the last equation implies $\psi a = 1$. Also l is the inclusion and d takes a to $(a, 1_{fa})$. With this we see that the objects of P are pairs (σ, ρ) where

$$\rho: a \simeq a' \text{ in A}, \ (a, \sigma) \in \text{H and} \quad f\rho.\sigma = f\rho.$$

This last condition implies $\sigma = 1_{fa}$, and, since ψ is natural, $\psi a' = 1$. So P is equivalent to the full subcategory of A consisting of those a with $\psi a = 1$.

(The above construction with **Aut** replaced by **End** yields the biidentifier of any endo-2-cell ψ.)

The next bilimit required is the *descent object* Desc(X) of a truncated bicosimplicial diagram

$$X: \quad X_0 \;\underset{\delta_1}{\overset{\delta_0}{\underset{\iota}{\leftleftarrows\!\!\!\rightarrow}}}\; X_1 \;\overset{\delta_0}{\underset{\delta_2}{\overset{\delta_1}{\rightrightarrows\!\!\!\rightarrow}}}\; X_2$$

$$\sigma_{ij} : \delta_i\delta_{j-1} \simeq \delta_j\delta_i \qquad \text{for} \quad i < j, \qquad n_i : 1 \simeq \iota\delta_i$$

in a bicategory *K*. When *K* is *Cat*, the category Desc(X) has objects pairs (x, θ) where x is an object of X_0 and $\theta: \delta_0 x \simeq \delta_1 x$ in X_1 such that

$$\iota\theta = \mu_1\mu_0^{-1}, \quad \sigma_{12}.\delta_1\theta.\sigma_{01} = \delta_2\theta.\sigma_{02}.\delta_0\theta,$$

and has arrows $\chi: (x, \theta) \to (x', \theta')$ where $\chi: x \to x'$ is an arrow of X_0 such that $Q'.\delta_0\chi = \delta_1\chi.\theta$. For a general *K*, the descent object of X consists of an object D, an arrow $h: D \to X_0$ and an invertible 2-cell $\omega: \delta_0 h \simeq \delta_1 h$ inducing an equivalence between $K(K, D)$ and $\mathrm{Desc}(K(K,X))$. Notice that X can be regarded as a homomorphism from an appropriate category *A* into *K* and, if we take $J: A \to Cat$ to be the functor amounting to the diagram

$$1 \;\substack{\longrightarrow \\ \longleftarrow \\ \longrightarrow}\; \mathrm{Iso} \;\substack{\longrightarrow \\ \longrightarrow \\ \longrightarrow}\; T$$

the bilimit $\{J, X\}$ is equivalent to Desc(X).

The descent object can be constructed using biequalizers and biidentifiers of auto-2-cells. First, take the biequalizer

$$h: H \to X_0, \qquad \theta: \delta_0 h \simeq \delta_1 h,$$

of δ_0, δ_1, then the biequifier $k: K \to H$ of the two invertible 2-cells

$$\begin{array}{ccc} \iota\delta_0 h & \xrightarrow{\quad \iota\theta \quad} & \iota\delta_1 h \\ & {\scriptstyle \mu_0^{-1}h}\searrow \quad \nearrow{\scriptstyle \mu_1 h} & \\ & h & \end{array}$$

and then, the biequifier $m: M \to L$ of the two invertible 2-cells

$$\begin{array}{ccccc} & \delta_1\delta_0 hk & \xrightarrow{\delta_1\theta k} & \delta_1\delta_1 hk & \\ {\scriptstyle \sigma_{01}hk}\nearrow & & & & \searrow{\scriptstyle \sigma_{12}hk} \\ \delta_0\delta_0 hk & & & & \delta_2\delta_1 hk \\ {\scriptstyle \delta_0\theta k}\searrow & & & & \nearrow{\scriptstyle \delta_2\theta k} \\ & \delta_0\delta_1 hk & \xrightarrow[\sigma_{02}hk]{} & \delta_2\delta_0 h & \end{array}$$

Then L, hkm, θkm form Desc(X).

The bilimit $\{J,S\}$ can be obtained as the descent object Desc(X) where

$$X_0 = \Pi\Pi_A \{JA, SA\}, \qquad X_1 = \Pi\Pi_{A,B} \{A(A, B) \times JA, SB\},$$

$$X_2 = \Pi\Pi_{A,B,C}\ \{A(B,\ C)\times A(A,\ B)\times JA,\ SC\}.$$

In (1.25) it was stated that indexed pseudo-limits in a 2-category could be constructed from cotensor products, products and equalizers. This is certainly true since pseudo-limits are particular indexed limits and all indexed limits can be so constructed (see [14], using the *Bibliography* of the paper). The proof outlined in (1.25) was a modification of (1.24). Using the corrected (1.24), we can squeeze out more from the method. Many naturally occuring 2-categories have *iso-inserters*. The iso-inserter of the diagram $f,\ g\colon A \rightrightarrows B$ is its *limit* (not pseudo-limit) indexed by the diagram

$$1 \rightrightarrows \mathrm{Iso} \quad \text{in } \mathit{Cat}.$$

An iso-inserter is a biequalizer but not conversely. The *strict descent object* of a truncated simplicial object X (this time μ_i, σ_{ij} are identities) is defined as for the descent object except that we insist on an isomorphism between $K(K,\ D)$ and $\mathrm{Desc}(K(K,\ X))$, not merely an equivalence. It can be constructed using an iso-inserter and identifiers of auto-2-cells (the latter are defined as were biidentifiers except that we ask for an isomorphism in the representation property). Then psdlim(J, S) is the strict descent object for X as before with biproducts and cotensor biproducts replaced by their "non-bi" versions. However, it does not seem possible to construct identifiers of auto-2-cells using a "non-bi" version of the construction of biidentifiers. The object P we are led to does support an idempotent whose splitting gives the identifier; but this is already true of the iso-inverter of ψ^-, $f^\cdot$. So *products, cotensor products, iso-inserters, and, either identifiers of auto-2-cells or splittings of idempotents, imply all indexed pseudo-limits.*

I would like to stress that I am currently using the word "weighted" in preference to "indexed" in this context.

Finally, there is a typographical error in (4.2) on page 140. The functors between 1 and Iso should have their directions reversed.

Department of Mathematics
Macquarie University
N.S.W. 2109
AUSTRALIA