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THE FREE ADJUNCTION

by Stephen SCHANUEL and Ross STREET

RÉSUMÉ. Les adjonctions dans une 2-catégorie K correspondent aux 2-foncteurs de la 2-catégorie Adj "adjonction libre" vers K. On donne ici une description explicite de cette 2-catégorie Adj.

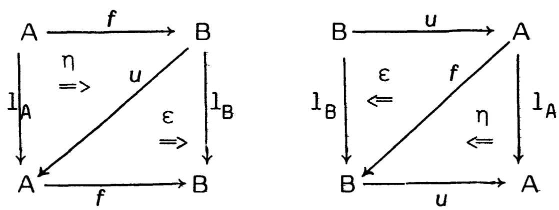
In obtaining the unit-counit expression of adjunction, Kan [2] made it clear that adjunctions can be defined in an arbitrary 2-category K. An adjunction in K consists of arrows

$$f : A \to B, \quad u : B \to A$$

and 2-cells

$$\eta : 1_A \Rightarrow uf, \quad \varepsilon : fu \Rightarrow 1_B$$

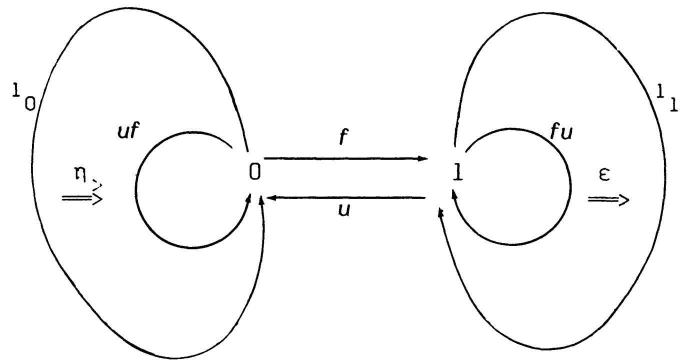
such that the 2-cells, obtained by pasting [3] the following diagrams, are the identity 2-cells of f, u respectively.



Using presentations by computads [4], one can see that there is a 2-category Adj such that a 2-functor Adj $\to K$ precisely amounts to an adjunction in K. The purpose of this Note is to provide a concrete description of the 2-category Adj : *the free adjunction.*

In generating monads (= standard constructions) from adjoint functors, Huber [1] made it clear that describing Adj is related to describing the *free monad* Mnd . A concrete description of Mnd is easy : it has one object 0, the hom-category Mnd(0, 0) is the category Δ of finite ordinals and order-preserving functions, and composition $\Delta \times \Delta \to \Delta$ is ordinal sum.

The 2-category Adj has two objects 0, 1, say, since an adjunction in K involves two objects A, B. The full sub-2-category with object 0 should be the free monad and the full sub-2-category with object 1 should be the free comonad; hence,



$$\mathrm{Adj}(0, 0) \simeq \Delta, \qquad \mathrm{Adj}(1, 1) \simeq \Delta^{op}.$$

(Auderset [0] has clarified the above aspects of Adj.)

It has been observed by many people (although [5], p. 131, is the only reference which comes to mind) that Δ^{op} is isomorphic to the category of non-empty ordinals and first-and-last-element-preserving order-preserving functions. It turns out that Adj(0, 1), Adj(1, 0) can also be taken as categories whose objects are the non-empty ordinals and whose arrows are respectively last-element-preserving, first-element-preserving arrows in Δ.

This leads to the following 2-category Badj. The objects are finite-ordinals $p = \{0, 1, ..., p-1\}$. Objects of the category Badj(p, q) amount to $m : p \to q$ where m is a finite ordinal with both $p \leq m$ and $q \leq m$. An arrow $\vartheta : m \Rightarrow m'$ in Badj(p, q) is an order-preserving function $\vartheta : m \to m'$ satisfying

$$\vartheta(i) = \begin{cases} i \text{ for } 0 \leq i < p, \\ m' - m + i \text{ for } m-q \leq i < m; \end{cases}$$

composition of such arrows is composition of functions. The composition functor

$$\mathrm{Badj}(q, r) \times \mathrm{Badj}(p, q) \to \mathrm{Badj}(p, r)$$

takes

$$(n : q \to r,\ m : p \to q) \quad \text{to} \quad m-q+n : p \to r$$

and

$$(\varphi : n \Rightarrow n',\ \vartheta : m \Rightarrow m') \quad \text{to} \quad \psi : m-q+n \Rightarrow m'-q+n'$$

given by the formula

$$\psi(i) = \begin{cases} \vartheta(i) \text{ for } i < m, \\ \varphi(i-m+q) + m' - q \quad \text{for } i \geq m-q. \end{cases}$$

This composition is functorial, associative, and the identity of p is $p : p \to p$.

The full sub-2-category of Badj consisting of the objects 0, 1 is Adj. There is a distinguished adjunction $f : A \to B$, $u : B \to A$, η, ε in Adj given as follows :

$$A = 0, \quad B = 1, \quad f = 1 : 0 \to 1, \quad u = 1 : 1 \to 0,$$
$$\eta : 0 \Rightarrow 1 = uf, \qquad \varepsilon : fu = 2 \Rightarrow 1$$

are the unique such functions.

It is now possible to prove the universal property.

Proposition. *For each adjunction in a 2-category K, there exists a unique 2-functor* Adj $\to$ K *whose value at the distinguished adjunction is the given adjunction.* ◊

Kan's adjoint functors gave us the setting for "free" constuctions in Mathematics; Adj gives the "free expression of freeness".

The inclusion Mnd $\to$ Adj induces a 2-functor

$$[\text{Adj}, K] \to [\text{Mnd}, K]$$

from the 2-category of adjunctions in K to the 2-category of monads in K. The technique for obtaining adjoints to such a 2-functor is also due to Kan [2] (*Kan extensions*). In this case (as pointed out in [0]) the adjoints yield the Kleisli and Eilenberg-Moore constructions on monads in K [3].

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