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HOW TO DEFINE THE DIFFERENTIABLE GRAPH OF A SINGULAR FOLIATION

By Jean PRADINES

RÉSUMÉ. Pour une large classe de feuilletages singuliers au sens de Štefan, nous construisons un groupoïde différentiable qui généralise le "graphe" d'un feuilletage régulier ; ceci attache à chaque feuille singulière un espace fibré principal différentiable, qui est une extension du revêtement d'holonomie défini par Ehresmann pour les feuilletages topologiques localement simples.

La construction utilise une description par diagramme des équivalences régulières et des isomorphismes transverses entre elles, ainsi que de la composition de leurs graphes réguliers. Ensuite cette description est affaiblie pour tenir compte des éventuelles singularités. Ceci conduit aux concepts de germes de "convections" et de "convecteurs", et leur composition. Finalement les feuilletages de Štefan assez bons admettent une convection intrinsèque.

CONTENTS.

NOTICE. We present here the full version (with some minor corrections) of a text written in october 1984 for the Proceedings of the Fifth International Colloquium on Differential Geometry (Santiago de Compostela, September 1984).

Only a shortened version, taken from the Introduction, will appear in these Proceedings (Research Notes in Mathe., Pitman Ed., 1985). Another summary was published in :

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0. INTRODUCTION.

As recalled in the lecture delivered by Haefliger at the present Conference [12], to any regular foliation is associated its *holonomy pseudogroup,* well defined up to a suitable equivalence (considered independently by W.T. van Est and by A. Haefliger), which bears all the topological and differentiable information on the transverse structure of this foliation, in other words on its leaf space, which in general fails to exist as a manifold.

As a matter of fact, this holonomy pseudogroup may be viewed as a pseudogroup representative (under a suitable equivalence relation between differentiable groupoids) of the *holonomy groupoid,* introduced by C. Ehresmann in 1961 [9] as a topological groupoid (and considered by the author, in a wider context, with its manifold structure in 1966 [15]), later rediscovered (using a different construction) by Winkelnkemper [23] and popularized by A. Connes [5], under the name of *graph of the foliation.*

On the other hand, as explained in the address by A. Lichnerowicz [13], the study of *symplectic geometry* has focused attention on those foliations with possible *singularities,* which are generated by families of vector fields.

Such foliations were encountered previously by H. Sussmann in the context of control theory [21], and a nice geometric characterization was discovered independently by P. Štefan [20].

We give an equivalent geometric formulation in the Appendix B, where we prove too a basic theorem connecting Štefan foliations and general differentiable groupoids (in a way that is not implied by nor implies Štefan Theorem), which completes a result stated by the author in 1966 [15].

If we drop the differentiable structure, Štefan foliations may be viewed as special examples of *topological foliations* in the sense of Ehresmann [9] and as a consequence their holonomy groupoid has been defined as a *topological groupoid* by this author, at least under the assumption of (topological) "*local simplicity*", which ensures the existence of a "germ of leaf space".

It should be noticed that the much more restrictive assumptions of local stability and "almost regularity", recently used by P. Dazord [6] and M. Bauer [1] respectively, imply Ehresmann's local simplicity. Under these assumptions these authors show that the Ehresmann holonomy characterizes the structure of the foliation around a singular leaf.

However we point out that in general the above process involves a considerable *loss of information*, as it follows from the two subsequent remarks concerning the singular case :

1° the loss of differentiability is irreversible, the holonomy groupoid being no longer a manifold, which is related to the fact that *even the local quotient spaces fail to exist as manifolds ;*

2° the holonomy group of any singular leaf which is reduced to a single point *will always vanish* and therefore brings no information on the vicinity of such a leaf, which contradicts the intuition that a kind of whirl should be associated with such leaves, involving a continuous local group action in some loose sense.

The *purpose of the present paper* is to extend (by means of completely new methods) the construction of the differentiable graph to a rather wide class of Štefan foliations (expected to be generic in some suitable sense) and to derive from this construction the holonomy groups, which, for singular leaves, will be *continuous*, and will not arise from the fundamental groups of the leaves.

This implies that the "graph" is no longer equivalent in any sense (even the algebraic one) to a pseudogroup, and the full definition of differentiable groupoids cannot be avoided.

More precisely we attach to any singular leaf a *principal "holonomy bundle"* which generalizes the holonomy covering of a regular leaf. By *"squeezing"* the connected components of the fibres, we recover Ehresmann's groups and the associated covering.

This opens the way for defining and studying the *transverse structure* of a singular foliation as the equivalence class (in a suitable sense) of its "differentiable graph", which we postpone to future papers.

We proceed through successive steps :

First, as a heuristic introduction, we recall a new construction of the graph in the *regular* case, which we published recently [19], and which is adapted to the desired extension. It consists in dealing first with "all" the germs of regular foliations rather than with a specified one, and playing with the *two equivalent descriptions* of the local structure by means of the *local regular quotient space* and of the *local regular graph*. As the former description is going to vanish in the singular case, we notice some intrinsic properties of the latter. In this way the holonomy groupoids of individual foliations appear as connected components (in a suitable sense) of a universal differentiable groupoid called the *universal graphoid*.

Then tackling the *singular case,* the properties of regular graphs have to be weakened one by one.

Dealing first with a *unicity property,* we are led in Section 2 to the notion of *monographs,* and, by localisation, of *faithful graphs,* which are not purely set-theoretic nor topological, but involve the differentiability by means of a categorical trick.

Sections 3 to 8 develop a somewhat systematic account of the machinery of morphisms of *differentiable graphs* (to be used in future papers too) and bring out the important notions of *"differentiable equivalences"* and *"differentiable actors",* and the *"decomposition diagram"* for a morphism.

Then, to supply the missing local regular equivalence and quotient space, we introduce the notions of *local rule of three* and of *germs of convectors and convections,* which are defined by the existence of (germs of) differentiable commutative squares (or *"ratios"*), describing the equivalent "fractions".

It is a highly remarkable and non obvious fact that these germs of convectors may be *composed* in a way which is the exact generalization of the set-theoretic composition of graphs, but cannot be expressed but in terms of *diagrams* in the category of *germs* of differentiable maps.

The proofs are by *diagram chasing* through a beautiful *hypercubic diagram* whose prototype is presented and discussed in Section 10, using some formal rules that are stated in Appendix A, and some consequences listed in the previous sections. These rules rely themselves on a few numbers of elementary properties of submersions and embeddings that we listed in [18] under the name of "Godement dyptich", because of the axiomatic use of Godement's characterization of regular equivalences by regular graphs.

This leads to a differentiable groupoid, called the *" universal convector",* whose units are the germs of convections, which contains the universal graphoid as an open subgroupoid. The essential property of this groupoid is that its structure is *uniquely determined by its underlying graph structure,* though it has non trivial isotropy groups.

However, as the terminology suggests, a convection is something more than the underlying Štefan foliation, and may be viewed intuitively as a certain class of (multidimensional) flows along the leaves, which are the stream lines.

Unfortunately it may happen that there is no way, or no *canonical* way of associating a convection to a singular foliation. So in order to recover fully the nice situation of the regular case, we have to limit ourselves to the class of those Štefan foliations (called *" holonomous")* which admit an intrinsic or *extremal convection,* in a sense to be made precise.

It seems that this class is broad enough for a wide number of applications, though we must confess that at present we cannot make precise to what extent it is "generic" (if it is), and it may happen that the definition of "holonomous" we use here, will require some further adjustment in view of new examples.

With this reservation, we have an exact generalization of the differentiable graph of the regular case.

The present work was achieved in partial collaboration with my student B. Bigonnet in whose thesis examples of the holonomy bundle will be found.

0a. SOME CONVENTIONS AND NOTATIONS.

In the sequel, a class of differentiability C^k is fixed throughout, with $k \geqq 1$. The manifolds may be *non Hausdorff* and non connected, but are locally finite-dimensional. "*Map*" means "differentiable map of the fixed class C^k".

As in [3], "*submanifold*" means "regular submanifold" and *embedding*" means "regular embedding".

We make a distinction between "*transversal* " and "*transverse*" (which requires supplementary tangent spaces).

Often but not always we use the following types of arrows in our diagrams :

- $\rightarrow$ for a (differentiable) map ;
- $\twoheadrightarrow$ for a *surmersion* (= surjective submersion) ; or sometimes for submersive germs ;
- $\hookrightarrow$ for an open inclusion ;
- $\rightarrowtail$ for an embedding ;
- $\dashrightarrow$ for a locally defined map ;
- $\circ\circ\!\!\rightarrow$ for a set-theoretic map between (abstract) sets ;
- $\bullet\bullet\!\!\rightarrow$ for an arrow to be constructed in the course of the proof.

A universe U is fixed throughout, and M denotes the " *universal manifold* ", i.e., the disjoint sum of manifolds belonging to U.

We recall that, when an equivalence relation is defined on a (non small) subset of U, an *equivalence class* is defined by its canonical representative (chosen by the Hilbert symbol), so that it is still an *element* of U [2].

This convention applies in particular when defining *germs* of manifolds, maps, and (finite) diagrams.

The symbol $| \; |$ is sometimes used, when necessary, to *forget* extra structures on a manifold.

Letters A, B are used to refer to the *Appendices.* Appendix A is used throughout.

1. THE REGULAR CASE : A UNIVERSAL GRAPH.

As announced in the Introduction, we just sketch here roughly the ideas of the construction detailed in [19].

A *transverse isomorphism* between two simple foliations defined on manifolds A, B is just a diffeomorphism between the leaf spaces (which are here manifolds by assumption). It may be identified with a pair of surmersions (i.e., surjective submersions) from A and B onto the same target Q, considered up to a diffeomorphism on Q : this is called a *regular isonomy* in [19] and might have been called too a *regular cograph.*

Taking the pullback of this pair, we get a commutative square :

$$\begin{tikzcd} & R \arrow[dl, "b"'] \arrow[dr, "a"] & \\ A \arrow[dr] & & B \arrow[dl] \\ & Q & \end{tikzcd}$$

which has very nice categorical properties : it is not only a pullback but a pushout too (cf. Proposition A 6).

As a consequence, *each of the two lower and upper halves of the square uniquely determines the other one, up to isomorphism.* Moreover the map $p = (b, a) : R \to B{\times}A$ is a (regular) *embedding.*

Now let A and B run through the open subsets of a fixed (big enough) manifold M (in fact it is more convenient to work with universes, using some logical cautions), and consider the set of all germs of isonomies. We claim that it bears a *canonical structure of differentiable groupoid H* (which we call the *universal graphoid) ;* the *algebraic structure* comes immediately from the cograph or *lower half* of the square while the *manifold structure* comes naturally from the graph or *upper half* : it is an open subset of the manifold $\mathcal{J}$ of germs of submanifolds of $M{\times}M$, and more precisely of the open submanifolds $\mathcal{J}_r$ of germs of regular graphs (denoted by J_S in [19]).

The *units* of this groupoid are the germs of regular equivalences, alias the *germs of foliations* of M, which lie in a manifold F, étale on M.

Now any foliation F may be identified with a *section* of F over an open subset M of M. The differentiable groupoid I *induced* on F by the universal graphoid H is called in [19] the *isonomy groupoid* of F, and the *holonomy groupoid* may now be identified with its "α-*connected component* ", which means the union of the connected components of the units in the fibres of the source projection (cf. Appendix B).

Moreover if we consider those differentiable groupoids G with base B for which the map

$$\pi = (\beta, \alpha) : G \to B{\times}B$$

(where α, β denote the source and target maps) is an immersion (resp. a subimmersion), the canonical factorization of this (sub)immersion, in the sense of [3], defines a functor and a local diffeomorphism (resp. submersion) from G onto an open subgroupoid of the universal graphoid H. We call these groupoids local graphoids (resp. epigraphoids), and *graphoids* when this factorization is injective (in which case π is called a "faithful immersion").

This means essentially that *the groupoid structure is uniquely determined by its underlying graph structure* (i.e., the pair β, α).

Now, as announced in the Introduction, it is interesting, with the extension to the singular case in mind, to get some intrinsic (i.e., independant from the lower half) characterization of the upper half of the square : it is given by the following purely algebraic (or set-theoretic) condition (added to the submersion and embedding conditions already mentioned) : $RR^{-1}R = R$, in which the composition rule is the usual one for graphs. The graphs satisfying this condition are called *isonomous* in [19].

The geometric interpretation of this condition is, picturing the elements of R as arrows, that there is a unique way of filling a square with 3 given edges in R. A more algebraic interpretation is that there is defined on R a *3-terms composition law*, which is a groupoid analogue to the elementary *rule of three* $t = zy^{-1}x$ in a multiplicative group.

Finally we have a nice, intrinsic characterization of the local properties of R, which says that the two (simple) foliations defined by the projections a and b (which are induced by the canonical projections in BxA) *generate a regular foliation*. (Compare with Theorem B 8 below.)

Note that the groupoid law of H, which we derived from the (trivial) composition of cographs comes too from the (less trivial) composition of graphs, which is the usual one, except it is convenient for us to reverse it. But observe that the composite of any two regular graphs *is not in general regular,* though *this is true for isonomous graphs.* To get coherent conventions, we reverse likewise the usual definition of *the graph of a map* f *:* $A \to B$; it will be the set of pairs $(f(x), x) \in B \times A$.

The previous considerations on the regular case have to be kept in mind as a heuristic guide for the singular case and allow us to be now more dogmatic for the following unpublished construction.

2. DIFFERENTIABLE GRAPHS, MONOGRAPHS AND FAITHFUL GRAPHS

The first step consists in weakening the monomorphism condition implied by the embedding condition on $R \to B \times A$ for a regular graph.

Let us fix two manifolds $A, B \in U$. *A differentiable isogeny* (from A to B) is just a differentiable map

$$p = (b, a) : R \to B \times A,$$

where a, b are *surmersions.*

It is convenient to consider p as defining an extra structure, briefly denoted by $\overleftarrow{R}$ or sometimes ${}_BR_A$, on the underlying manifold R. The notation $|\ |$ will be used to emphasize (when necessary) that this structure has to be forgotten, so that $|\overleftarrow{R}| = R$.

When several isogenies are involved, we use the notations $p_{\overleftarrow{R}}$, $a_{\overleftarrow{R}}$, $b_{\overleftarrow{R}}$, or sometimes p', a', b', etc... Later on, we shall also use arrows with various directions.

The isogenies are the objects of a *category* ${}_B I_A$. An arrow $r : \overleftarrow{R}' \to \overleftarrow{R}$ is defined by a map from R' to R commuting with p and p'. There is a *terminal object* $\overleftarrow{B \times A}$. We speak of a *weak morphism*, when k is only assumed to be C^{k-1}.

When A = B, the diagonal map defines an isogeny denoted by $\overleftarrow{O}_A$, called the *null isogeny*.

The equivalence class (cf. 0a) of an object of ${}_B I_A$ *up to isomorphism* is called a *differentiable graph from* A *to* B.

When much accuracy is required, $[\overleftarrow{R}]$ will denote the graph defined by $\overleftarrow{R}$. Note that there is no quotient category of ${}_B I_A$ with the graphs as objects. However in the following *we shall often mix up the notions of isogeny and graph.*

A regular graph is a differentiable graph for which p is an embedding.

Proposition 2.1. *1° If* $\overleftarrow{R}$ *is a regular graph from* A *to* B, *then any morphism in* ${}_B I_A$ *with source* $\overleftarrow{R}$ *is an embedding.*

2° Let $p : R \to B \times A$ *be an isogeny,* $z \in R$, $x = a(z)$, $y = b(z)$. *Then there exists a submanifold* Z *of* R *containing* z *on which* p *induces a regular graph (from* U *to* V, *open subsets of* A, B *).*

If moreover $\dim_x A = \dim_y B$, *then we can choose for* Z *a graph of diffeomorphism.*

Proof. 1° It is well known that if $p = p'u$ is an embedding, then so is u.

2° Take Z transverse to Ker $T_z p$ in the first case, and tangent to a common supplementary of Ker $T_z a$ and Ker $T_z B$ in the second, and shrink Z if necessary. ◊

To handle the singular case, we need the following fundamental generalization.

A differentiable graph $\overleftarrow{R}$ is called a *monograph* (of class C^k) if it is a subobject (sous-truc in the sense of Grothendieck [10]) of the terminal object $\overleftarrow{B \times A}$: this just means that given two *surmersions* $(g, f) : Z \to B \times A$, there exists at most one differentiable map $u : Z \to R$ such that $au = f$, $bu = g$. (Note that it is meaningful to speak of morphisms of monographs.)

The same unicity property then remains valid whenever f, g are (possible non surjective) *submersions*, as it is readily seen by extending f, g, u in an obvious way to the disjoint sum of R and Z.

In order $\overleftarrow{R}$ to define a monograph, it is *necessary* that there exist a dense open subset of R in which p be an immersion, and *sufficient* that this immersion be injective.

We turn now to a necessary and sufficient criterion.

For any manifold X let us denote by J(X) the (non Hausdorff!)

manifold of germs of submanifolds of X of class C^k.

Given an isogeny $p : R \to B\times A$, let U be the open subset of J(R) consisting of those germs on which p induces a regular graph, and, when dim A = dim B (= constant), let V be the open subset of those germs, which are transverse to the fibres of both a and b.

Let $\tilde{p}$ (resp. $\hat{p}$) denote the maps induced by p from U (resp. V) to J(B×A). Then using Proposition 2.1 one proves :

Proposition 2.2. *With the above notations,* $\overleftarrow{R}$ *defines a monograph iff* $\tilde{p}$ *(resp.* $\hat{p}$*) is injective.* ◊

Letting now A, B run through the set of manifolds belonging to $\overleftarrow{U}$, it follows from the above remarks that if $\overleftarrow{R}$ defines a monograph, *any open subset of* R *bears an induced monograph structure* (from the image of a to the image of b).

We have a notion of *germ* (cf. 0a) of differentiable isogeny, graph, and monograph. From the obvious (non Hausdorff!) manifold structure on the set of germs of isogenies, we derive a *canonical structure of manifold* $\mathcal{J}_m$ *for the set of germs of monographs,* which bears a *canonical differentiable graph structure* $\mathcal{J}_m$ *from* M *to* M.

Warning! There is no good manifold structure on the set of all germs of differentiable graphs!

The manifold of germs of regular graphs is canonically identified with an open submanifold of $\mathcal{J}_m$.

When much accuracy is needed, we use the notations $(\overleftarrow{R})_x$, $[\overleftarrow{R}]_x$ for germs of isogenies and graphs at x, and sometimes $(\overleftarrow{R})_.$, $[\overleftarrow{R}]_.$ when we do not want to name x, but *we shall often denote a germ improperly by one of its representatives.*

The following example and counterexample shed light on the subtlety of these notions.

Example. A monograph structure of class C^k ($k = 1$) from **R** to **R** is defined on $\mathbf{R}_+^*\times\mathbf{R}$ by setting $p(y, x) = (yx, x)$. Note that p is not injective and that the two simple foliations defined by the two projections have a leaf in common. Though apparently trivial, this example will be useful to understand how a group structure on the singular fibre may arise from the differentiable graph structure, which is the key idea for building our holonomy groups.

Counter-example. Consider now the differentiable graph from **R** to **R** defined on **R**x**R** by setting

$$p(y, x) = (x - \exp(-1/y^2)\sin(1/y), x).$$

Observing that the graphs induced on the submanifolds Z_n defined by $y = (2\pi n)^{-1}$ are equal, one deduces immediately that the germ of this differentiable graph at the origin is not a germ of monograph. However one can prove that the monograph property would be true if, in the previous definitions of the category $_B\mathfrak{I}_A$ and of the monographs, the maps had been replaced everywhere by germs of maps.

The counter-example shows that this latter definition would not be the good one (for no good manifold structure could be defined on such germs) though it would be much simpler to handle. This is a source of technical difficulty in proving that a germ of graph is a germ of monograph, for this property cannot be proved by working directly in the category of germs of maps, and requires the construction of a monograph representative of the germ.

The following proposition is very useful ; it generalizes the property for a graph that p be a subimmersion.

Proposition 2.3. *Given an isogeny $\overleftarrow{R}$, there exists at most one factorization u of p through $\mathcal{J}_m$ which is a submersion.*

Definition 2.4. If such is the case, the graph defined by $\overleftarrow{R}$ is said to be *sub-faithful* (sub-fidèle). It is called *faithful* (resp. a *local monograph*) if the factorization u is injective (resp. étale).

Proof. Let u_i (i = 1, 2) : $R \to \mathcal{J}_m$ be two factorizations. Take any $r_0 \in R$ and set $m_i = u_i(r_0) \in \mathcal{J}_m$. The germs m_i may be represented by open sets M_i of $\mathcal{J}_m$ which define monographs. Consider local sections s_i of u_i such that $s_i(m_i) = r_0$ and set

$$h = u_2 s_1 , \qquad k = u_1 s_2 .$$

The composite kh (which is defined on a neighborhood V_1 of m_1) defines a local automorphism of the monograph M_1 and therefore has to be the identity of V_1 ; likewise hk is the identity of an open set V_2. This means that m_1 and m_2 are *isomorphic* germs of isogenies and hence are *equal* as germs of monographs. ◊

3. VERTICIAL MONOGRAPHS.

A monograph $[\overleftarrow{R}]$ from A to B is called *verticial* if A = B and if there exists a morphism $o : \overleftarrow{O}_A \to \overleftarrow{R}$, otherwise a bi-section of both a and b. Note that o is unique ; we call it the *vertex map*. It induces a diffeomorphism of A onto a submanifold R_0 of R.

Let us denote by $\mathcal{J}_{m_0} \subset \mathcal{J}_m$ the subset of germs of verticial monographs ; clearly these germs are characterized by the existence of a germ of bi-sections.

Proposition 3.1. *$\mathcal{J}_{m_0}$ is a submanifold of $\mathcal{J}_m$ étale on M.*

Proof. A germ $x_0 \in \mathcal{J}_{m_0}$ has an open neighborhood $\overleftarrow{R} \subset \overleftarrow{\mathcal{J}}_m$ which is a verticial monograph from A to A (an open subset of M). The canonical section o induces a diffeomorphism of A onto a submanifold R_0 of R. We have clearly $x_0 \in R_0 \subset \mathcal{J}_{m_o} \cap R$.

Conversely if x lies in $\mathcal{J}_{m_o} \cap R$, there is by the previous argument a submanifold Z of R such that one has $x \in Z \subset \mathcal{J}_{m_o}$. If we set

$$y = oa(x) = ob(x) \in R_0,$$

the germs of induced isogenies $(\overleftarrow{Z})_x$ and $(\overleftarrow{R}_0)_y$ are clearly isomorphic ; hence the germs of graphs $[\overleftarrow{Z}]_x$ and $[\overleftarrow{R}_0]_y$ are *equal*, which implies $x = y$.

So we have $\mathcal{J}_{m_o} \cap R = R_0$, which is a submanifold, and $\mathcal{J}_{m_o}$ is itself a submanifold of $\mathcal{J}_m$, on which a and b agree, and induce a local diffeomorphism onto M. Note that by composition with the local inverses we get *retractions of an open neighborhood of* $\mathcal{J}_{m_o}$ *onto* $\mathcal{J}_{m_o}$. ◊

When we have $A = R_0$ for a verticial monograph $\overleftarrow{R}$, we say that it is *in canonical form*.

4. REDUNDANT COMPOSITION.

Let $\overleftarrow{R} = (b_{\overleftarrow{R}}, a_{\overleftarrow{R}})$ be a differentiable graph from A to B, and $\overleftarrow{S} = (b_{\overleftarrow{S}}, a_{\overleftarrow{S}})$ a differentiable graph from B to C. Then their *redundant composite* $\overleftarrow{S} * \overleftarrow{R}$ from A to C is defined by considering the fibre product $(S \times_B R, v, u)$ of $a_{\overleftarrow{S}}$ and $b_{\overleftarrow{R}}$ and then taking

$$(b_{\overleftarrow{S}}v, a_{\overleftarrow{R}}u) : S \times_B R \to C \times A.$$

The redundant composition defines a *category* (with the open sets of M as objects), but *not a groupoid : the symmetric graph* of $\overleftarrow{R}$, which will be suggestively denoted by $\overrightarrow{R}$, and which is defined by

$$a_{\overrightarrow{R}} = b_{\overleftarrow{R}}, \quad b_{\overrightarrow{R}} = a_{\overleftarrow{R}},$$

is not in general an inverse.

Note that the redundant composition of isogenies would not be associative! However we go on using often improperly $\overleftarrow{R}$ instead of $[\overleftarrow{R}]$.

The following *notations* are very suggestive, once one agrees to picture an element of R by an arrow pointing *downwards* and a sequence of arrows by a broken line :

$$V(\overleftarrow{R}) = \overrightarrow{R} * \overleftarrow{R}, \quad \Lambda(\overleftarrow{R}) = \overleftarrow{R} * \overrightarrow{R}, \quad N(\overleftarrow{R}) = \overleftarrow{R} * \overrightarrow{R} * \overleftarrow{R}.$$

We sometimes abbreviate

$$V \text{ for } |V(\overleftarrow{R})|, \quad \Lambda \text{ for } |\Lambda(\overleftarrow{R})|, \quad N \text{ for } |N(\overleftarrow{R})|.$$

Note that the canonical projections (which are surmersions) of Λ and V onto R define on these manifolds a second structure of (regular isonomous in the sense of [19]) graph *from* R *to* R, and the redundant composite of these graphs has the same underlying manifold as $N(\overleftarrow{R})$.

5. CHANGE OF BASES. DIFFERENTIABLE EQUIVALENCES.

Letting now A, B run through the manifolds belonging to our universe U, we have an obvious notion of morphisms of isogenies (but not of graphs!). A morphism $r : \overleftarrow{R'} \to \overleftarrow{R}$ is defined by a commutative diagram :

$$
Q(r) \qquad
\begin{tikzcd}
R' \arrow[r, dashed, "r"] \arrow[d, "p'"'] & R \arrow[d, "p"] \\
B' \times A' \arrow[r, "g \times f"] & B \times A
\end{tikzcd}
$$

Note that f, g are uniquely determined by the map r ; they are called the *changes of bases* and we say r is a *morphism over* $g\times f$.

It is clear that the previous symbols Λ, V, N now define *functors*.

Warning! The canonical morphism $\overleftarrow{R} \to \overleftarrow{B\times A}$ is no longer a monomorphism in the whole category of morphisms when it is a monomorphism of ${}_A I_B$!

However we note the obvious but useful :

Remark 5.1. *Given changes of bases* $f : A' \to A$, $g' : B' \to B$, *and an isogeny* $p' : R' \to B'\times A'$, *then* $(g\times f)p'$ *defines an isogeny from* A *to* B *iff* f *and* g *are surmersions.* ◊

As a consequence we have :

Proposition 5.2. *If* f, g *are surmersions and* $\overleftarrow{R}$ *is a monograph from* A *to* B , *there exists at most one morphism from* $\overleftarrow{R'}$ *to* $\overleftarrow{R}$ *over* $g\times f$.

Now given an isogeny $R \to B\times A$ and changes of bases f, g we can consider the induced manifolds $R^* = f^*(R)$, ${}^*R = {}^*g(R)$ defined by the pullbacks :

$$
\begin{tikzcd}
 & R^* \arrow[r, "a^*"] \arrow[d, "b^*"] \arrow[dd, bend right=60] & A' \arrow[d, "f"] \\
{}^*R \arrow[r] \arrow[rr, bend left=15, "{}^*a"] \arrow[d, "{}^*b"'] & R \arrow[r, "a"'] \arrow[d, "b"] & A \\
B' \arrow[r, "g"] & B &
\end{tikzcd}
$$

Definition 5.3. We say f is *right* (resp. g is *left*) *transversal* to $\overleftarrow{R}$ if b^* (resp. ${}^*a$) is again a surmersion. A sufficient condition is that f (resp. g) be a surmersion. Then there is defined the *right* (resp. *left*) *induced* isogeny $\overleftarrow{R}^* = f^*(\overleftarrow{R})$ (resp. ${}^*\overleftarrow{R} = {}^*g(\overleftarrow{R})$).

When both are defined, we say $g\times f$ is *transversal* to $\overleftarrow{R}$. By composition of pullbacks one has readily :

Proposition 5.4. *If* $g\times f$ *is transversal to* $\overleftarrow{R}$, *then* ${}^*g(f^*(\overleftarrow{R}))$ *and* $f^*({}^*g(\overleftarrow{R}))$ *are canonically isomorphic, which defines the induced graph* $[{}^*\overleftarrow{R}^*]$.

Keeping the preceding notations, let now be given three surmersions

$$f : A' \to A, \qquad g : B' \to B, \qquad h : C' \to C,$$

and set $\overleftarrow{R}' = {}^*g(\overleftarrow{R})$, $\overleftarrow{S}' = g^*(\overleftarrow{S})$. Denote by $u : R' \to R$, $v : S' \to S$ the canonical surmersions. In the following proposition $S\times_B R$, $S'\times_{B'}R'$ are viewed as (isonomous regular) graphs from R to S, R' to S'.

Proposition 5.5. *One has*

1° ${}^*h(\overleftarrow{S}) * f^*(\overleftarrow{R}) = {}^*hf^*(\overleftarrow{S} * \overleftarrow{R})$;

2° $S'\times_{B'}R' = {}^*vu^*(S\times_B R)$ *and* $S'\times_{B'}R' \to S\times_B R$ *is a surmersion.*

This results from the following commutative diagrams :

$$\begin{tikzcd}
? \arrow[r] \arrow[d] \arrow[dr] & {} \arrow[r] \arrow[d] & f^*R \arrow[r] \arrow[d] & A' \arrow[d, "f"] \\
{} \arrow[r] \arrow[d] & S * R \arrow[r] \arrow[d] & R \arrow[r, "a"] \arrow[d, "b"] & A \\
h^*S \arrow[r] \arrow[d] & S \arrow[r, "a"] \arrow[d, "b"] & B & \\
C' \arrow[r, "h"] & C & &
\end{tikzcd}
\qquad
\begin{tikzcd}
S'\times_{B'}R' \arrow[rr] \arrow[dr, dotted] \arrow[dd] & & R' \arrow[dr, "u"] \arrow[drr, "a'"] \arrow[dd] & & \\
& S\times_B R \arrow[rr] \arrow[dd] & & R \arrow[r, "a"] \arrow[dd, "b"] & A \\
S' \arrow[rr] \arrow[dr, "v"] \arrow[ddr, "b'"'] & & B' \arrow[dr, "g"] & & \\
& S \arrow[rr, "a"] \arrow[d, "b"] & & B & \\
& C & & &
\end{tikzcd}$$

in which all the squares are pullbacks, using the properties of the composition of pullback squares. In the second diagram, the dotted arrow is constructed by means of the universal property of the front square, and the two new squares which arise are again pullbacks, which proves this arrow to be a surmersion. ◊

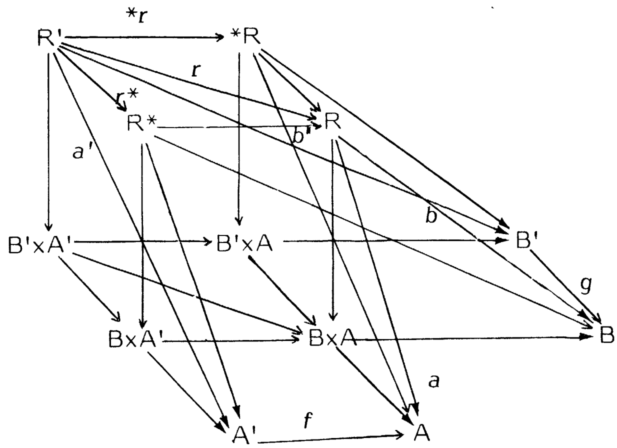
By repeated use of the proposition, we get the useful following corollary, which is also a consequence of Proposition 6.4 below :

Corollary 5.6. *If* $r : {}^*\overleftarrow{R}{}^* \to \overleftarrow{R}$ *is the canonical morphism over surmersive changes of bases then* $\mathrm{V}r$, Λr, $\mathrm{N}r$ *are again surmersions.*

We consider now a morphism $r : \overleftarrow{R}' \to \overleftarrow{R}$ over (not necessarily surmersive) changes of bases f, g. The *differentiable version of the algebraic notion of essential surjectivity* is given by the :

Definition 5.7. The morphism r is said to be *essentially surmersive* if $g\times f$ is transversal to $\overleftarrow{R}$.

Without any assumption on f, g we can draw the important *decomposition diagram :*



When $g x f$ is transversal to $\overleftarrow{R}$, then $\overleftarrow{R}*$ and $*\overleftarrow{R}$ are isogenies, and $r*$ and $*r$ are morphisms.

Playing with Corollary A 3 through out this diagram, we get :

Proposition 5.8. *The four squares*

$$Q(r), \qquad R'R**RR, \qquad R'*RA'A, \qquad R'R*B'B,$$

have the same properties of versality and monicity. (Cf. App. A.)

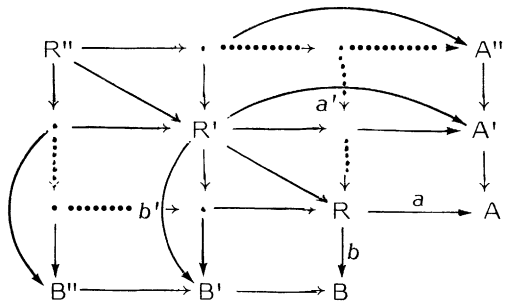
This leads to the following definition, in which *the word "differentiably" will be often omitted* (but should be restored when there is a risk of confusion with the underlying purely set-theoretic conditions).

Definition 5.9. 1° We say the morphism r is [*locally*] *differentiably full* (resp. [*regularly*] *faithful*) when the square $Q(r)$ is [locally] versal (resp. [regularly] monic). (App. A)

2° It is called a [*local*] *differentiable equivalence* if it is [locally] differentiably full, faithful and essentially surmersive (this last condition being fulfilled in particular when f and g are surmersions).

Note that this definition is meaningful even when $*R*$ is not defined ; when it is, the canonical morphism is a special case of differentiable equivalence. (This notion of differentiable equivalence is especially useful in the case of groupoids, and may be used to define the transverse structure of a foliation ; in another paper, we'll study the link with the van Est-Haefliger equivalence for pseudogroups [12, 22] and the Skandalis-Haefliger equivalence for groupoids [11]).

From Proposition A 3 results the composition of [regularly] faithful morphisms as well as of [locally] full *submersive* ones. The composition of [local] equivalences results from the diagram :



These definitions can be stated for *germs* too.

6. ACTORS.

We turn now to the properties of the two squares $Q^*(r) = (\mathrm{I})$, $^*Q(r) = (\mathrm{II})$:

$$\begin{tikzcd}
B' \arrow[d] & R' \arrow[l, "b'"'] \arrow[r, "a'"] \arrow[d, "r"] \arrow[l, phantom, shift right=4, "(\mathrm{II})"] \arrow[r, phantom, shift right=4, "(\mathrm{I})"] & A' \arrow[d] \\
B & R \arrow[l, "b"] \arrow[r, "a"'] & A
\end{tikzcd}$$

which amounts to the properties of the (always defined) canonical maps r^*, $^*r : R' \to \underset{\leftarrow}{R}{}^*$, $^*R$; they are linked to the property that r arises from "actions" of R on A', B'.

Definition 6.1. 1° We say the morphism r is *right* (and/or *left*) [*locally*] *differentiably full* (resp. [*locally*] *active*, resp. [*regularly*] *faithful*) when the square (I) (and/or (II)) is [locally] versal (resp. [locally] universal, resp. [regularly] monic).

2° It is called a [*local*] *actor* if it is *surmersive* and [locally] right *and* left active.

By Proposition A 2, all these properties are stable by composition.

Warning. Again these definitions are transferrred to germs. But when defining a *germ of actor,* it is convenient to require that it admits a right active representative as well as a left active one, but in general *it will not admit a representative which is both,* i.e., which is an actor. This is a source of important technical difficulties which cannot be removed. However we can find a representative which is a *local* actor.

From the decomposition diagram and Proposition A 2, we deduce the following implications *(in opposite directions!).*

Proposition 6.2. *1° If f is differentiably full (resp. locally full) and g/f is a surmersion (resp. a submersion), then r is right/left differentiably* *full (resp. locally full).*

2° If r is right or left regularly faithful, then it is regularly faithful.

(Note : in the more restrictive context of functors, Ehresmann uses "*well faithful* " for our "right faithful" .)

From Remark 5.1 and Proposition 5.2 one gets :

Proposition 6.3. *Assume $r : \overleftarrow{R'} \to \overleftarrow{R}$ is a [germ of] faithful morphism over two surmersions. Then if $\overleftarrow{R}$ is a [germ of] monograph, so is $\overleftarrow{R'}$.* ◊

Proposition 6.4. *Assume $r : \overleftarrow{R}' \to \overleftarrow{R}$ is right and left locally full. Then if it is submersive so are $\wedge r$, $\vee r$, $\mathrm{N}r$. (More precisely right is enough for $\wedge r$ and left for $\vee r$.)*

Proof. Consider the "prismatic" diagrams whose base consists in the three pullbacks :

$$\begin{tikzcd}
\mathrm{N}(\overleftarrow{R}) \arrow[r] \arrow[d] & \Lambda(\overleftarrow{R}) \arrow[r] \arrow[d] & R \arrow[d] \\
\mathrm{V}(\overleftarrow{R}) \arrow[r] \arrow[d] & R \arrow[r] \arrow[d] & A \\
R \arrow[r] & B &
\end{tikzcd}$$

and top is the analogous diagram for R', the vertical arrows being f, g, r, $\vee r$, $\mathrm{N}r$. By Proposition A 2, the versality of the vertical sides R'RA'A, R'RB'B, is transferred step by step to all the vertical sides and the submersivity of r to all the vertical edges. ◊

Using Proposition A 2 6° d, one has :

Proposition 6.5. *Let be given a morphism $r : \overleftarrow{R}' \to \overleftarrow{R}$.*

1° Assume r is regularly faithful (in particular a local actor or a local equivalence). Then if $\overleftarrow{R}$ is regular, so is $\overleftarrow{R}'$.

2° Assume r is a surmersive equivalence. Then if $\overleftarrow{R}'$ is regular, so is $\overleftarrow{R}$.

This applies to germs. ◊

Remark 6.6. When gxf is *transversal* to $\overleftarrow{R}$, Proposition 5.8 may be restated by saying that the squares $Q(r)$, $*Q(r^*)$, $Q^*(*r)$ have the same properties as R'R**RR. In turn this means that

the properties of [local] fullness and [regular] faithfulness of r are equivalent to the same left properties for r^* or right properties for $*r$.

7. FIBRE PRODUCTS.

By a *universal square of morphisms,* we mean a commutative square of morphisms such that the underlying square of maps is universal in the sense of App. A, as well as the two base squares. This clearly implies the pullback property in the category of morphisms.

For instance, if r is differentiably full and faithful, $Q(r)$ is universal.

We fix the following notations for a commutative square of morphisms :

$$\begin{tikzcd}
& S' \arrow[rr, "s"] \arrow[dl, "u'"'] \arrow[dd] & & S \arrow[dl, "u"'] \arrow[dd] \\
R' \arrow[rr, "r"] \arrow[dd] & & R \arrow[dd] & \\
& D'\times C' \arrow[rr, "k\times h"] \arrow[dl, "n'\times m'"] & & D\times C \arrow[dl, "n\times m"] \\
B'\times A' \arrow[rr, "g\times f"'] & & B\times A &
\end{tikzcd}$$

By a repeated use of Proposition A 2 and Corollary A 3, one proves :

Proposition 7.1. *Let be given a universal square of morphisms as above.*

1° Assume f, g, m, n are surmersions. Then if $\overleftarrow{R}$, $\overleftarrow{R'}$, $\overleftarrow{S}$ are [verticial] monographs, so is $\overleftarrow{S'}$.

2° Assume m, n are surmersions. Then if u is respectively [locally] full, [regularly] faithful, a [local] equivalence, right/left [locally] full, right/left [regularly] faithful, a [local] actor, so is u'.

3° Assume r, hence f, g, are surmersions. Then :
a) if u' is respectively [regularly] faithful, right/left [regularly] faithful, right/left active, right/left full, a [local] actor, so is u ;
b) Assume moreover m, n (or equivalently m', n') are surmersions. Then if u' is [locally] full or a [local] equivalence, so is u.

This proposition remains valid for germs.

We give now a basic *existence criterion* (keeping the same notations) :

Proposition 7.2. *A sufficient condition for the existence of the fibre product of (u, r) is that u be a right and left full surmersion (for instance a local actor or a surmersive equivalence). The pullback square is then universal.*

This remains valid for germs.

Proof. We construct s, u', h, k, m', n' by fibre product of (differentiable) maps, and the universal property gives the map $S' \to D'\times C'$. *The non obvious point is that this map defines an isogeny,* i.e., that one gets *surmersions* when composing with the canonical projections. This comes from a repeated use of Proposition A 2 in the two cubic diagrams with top side RR'SS' and bottom sides AA'CC' and BB'DD'. The versality of the vertical side is transferred to the parallel one, which in turn implies the surmersivity of both projections of S'. ◊

Remark 7.3. In the case of germs we point out again that we do not require the existence of a representative satisfying *both* right and left fullness. We get a pullback in the category of germs of morphisms, but in general we cannot use the *same* representative for the two cubic diagrams of the above proof.

However it is important to notice that there is a *basis* of representatives of the germ of S' consisting of fibre products of representatives of S, R'. This relies on the fact that the topology of S' is induced by the product topology of S and R'. But when *several* universal squares are involved, we cannot in general make *simultaneous* choices of such representatives!

The proposition applies in particular when $u = n \times m$, where m, n are surmersions. As a first application, we have :

Proposition 7.4. *1° Assume $\underleftarrow{r} : \overleftarrow{R}' \to \overleftarrow{R}$ is a surmersive equivalence. Then $\underleftarrow{R}'$ is a monograph iff so is $\underleftarrow{R}$. This remains valid for germs.*

2° Assume moreover $f = g$. Then the germ of $\overleftarrow{R}'$ is verticial iff so is the germ of $\overleftarrow{R}$.

Proof. Use the following diagrams :

$$\begin{tikzcd}
Z' \arrow[r, dotted, "z"] \arrow[d, dotted, shift right, "v'"'] \arrow[d, dotted, shift left, "u'"] \arrow[dd, dotted, bend right=60, "q'"'] & Z \arrow[d, shift right, "v"'] \arrow[d, shift left, "u"] \arrow[dd, bend left=60, "q"] \\
R' \arrow[r, "r"] \arrow[d, "p'"] & R \arrow[d, "p"'] \\
B' \times A' \arrow[r, "g \times f"] & B \times A
\end{tikzcd}
\qquad
\begin{tikzcd}
A' \arrow[r, "f"] \arrow[d, "o'"] \arrow[dd, bend right=60] & A \arrow[l, dotted, bend right, "s"'] \arrow[d, dotted, "o"'] \arrow[dd, bend left=60] \\
R' \arrow[r, "r"] \arrow[d] & R \arrow[d] \\
A' \times A' \arrow[r, "f \times f"] & A \times A
\end{tikzcd}$$

In the left diagram we construct z and the isogeny q' by pullback, and we lift the given u, v by the universal property of $Q(r)$. We conclude $u' = v'$, and then $u = v$ because z is epimorphic.

In the right diagram we use a right inverse s of the submersive germ f, an define o as $ro's$ (composition of germs).

The validity of 1° for germs relies on Remark 7.3.

This proves the "only if" part.

The "if" part of 1° comes from Proposition 6.3, and of 2° from the versality of $Q(r)$. ◊

Remark 7.5. The above argument has proved the following :

Let $\overleftarrow{R}$, $\overleftarrow{R}'$ be germs of monographs and $r : \overleftarrow{R}' \to \overleftarrow{R}$ a germ of morphism over $f = g$. Then

1° If f is a submersive germ and $\overleftarrow{R}'$ is verticial, so is $\overleftarrow{R}$;
2° If r is full and $\underleftarrow{R}$ is verticial, so is $\overleftarrow{R}'$.

8. REGULAR DOUBLE GRAPHS.

The constructions involved in the next sections are significantly cleared by replacing them in the slightly more general context described here, intended to emphasize the symmetries which are at work. The double graphs appear as a powerful tool for lifting singular graphs up to regular ones and for carrying over properties from one graph to another one.

A *double isogeny* H^{+} on the underlying manifold H is a commutative diagram of *surmersions* :

$$\begin{tikzcd}
C & R \arrow[l] \arrow[r] & A \\
Q \arrow[u] \arrow[d] & H \arrow[l, "b^{\leftarrow}"'] \arrow[r, "a^{\leftarrow}"] \arrow[u, "a^{\downarrow}"'] \arrow[d, "b^{\downarrow}"] \arrow[ul, "a^{\nwarrow}"'] \arrow[ur, "a^{\swarrow}"] \arrow[dl, "b^{\swarrow}"] \arrow[dr, "b^{\nwarrow}"'] & P \arrow[u] \arrow[d] \\
D & S \arrow[l] \arrow[r] & B
\end{tikzcd}$$

This may be viewed equivalently, and *in two different ways* as an "isogeny in the category of morphism of isogenies".

The less pedantic expression "*double graph*" should be kept for an isomorphism class of such structures with A, B, C, D fixed, but, as for graphs, *we shall often use loosely one for the other,* except when more accuracy is needed.

We have a corresponding notion of *germs.*

There are two underlying graph structures on H : the *horizontal* one $H^{\leftarrow}$ from P to Q, and the *vertical* one $H^{\downarrow}$ from R to S (and of course their symmetrics), as well as graph structures $P^{\downarrow}$ from A to B, $Q^{\downarrow}$ from C to D, $R^{\leftarrow}$ from A to C, $S^{\leftarrow}$ from B to D, and morphisms

$$a^{\leftarrow},\ b^{\leftarrow} : H^{\downarrow} \to P^{\downarrow},\ Q^{\downarrow}, \qquad a^{\downarrow},\ b^{\downarrow} : H^{\leftarrow} \to R^{\leftarrow},\ S^{\leftarrow} .$$

There are also *oblique* graph structures $H^{\nwarrow}$ from B to C, $H^{\swarrow}$ form A to D (and their symmetrics).

H^{+} is called a *bi-actor* if $a^{\leftarrow}$, $b^{\leftarrow}$ (or equivalently $a^{\downarrow}$, $b^{\downarrow}$) are actors, and a *bi-valence* if $a^{\leftarrow}$, $b^{\leftarrow}$, $a^{\downarrow}$, $b^{\downarrow}$ are equivalences.

When defining the *germs* of these notions, we cannot require the representative squares involved to be *simultaneously* universal : they will only be *locally* universal.

Proposition-Definition 8.1. *Assume* $P^{\downarrow}$, $Q^{\downarrow}$, $R^{\leftarrow}$, $S^{\leftarrow}$, *are regular. Then the following are equivalent :*

$a^{\leftarrow}$) $a^{\leftarrow}$ is regularly faithful ; $a^{\downarrow}$) ; $b^{\leftarrow}$) ; $b^{\downarrow}$) ;
$c^{\leftarrow}$) $H^{\leftarrow}$ is regular ; $c^{\downarrow}$) ;

d) H *is a submanifold of* DxCxBxA ;
e) H *is a submanifold of* SxRxQxP.

When all these conditions are fulfilled, H^+ is called *regular.*

Proof. We consider the commutative diagram

$$\begin{tikzcd}
Q\times P \arrow[rrrr, bend left, start anchor=north, end anchor=north] & & H \arrow[ll] \arrow[rr] \arrow[lld] \arrow[d] \arrow[rrd] & & D\times C\times B\times A \arrow[d, tail] \\
R\times Q\times P \arrow[u] & & S\times R\times Q\times P \arrow[ll] \arrow[rr] & & (D\times C\times B\times A)^2
\end{tikzcd}$$

Even without the first assumption, we always have :

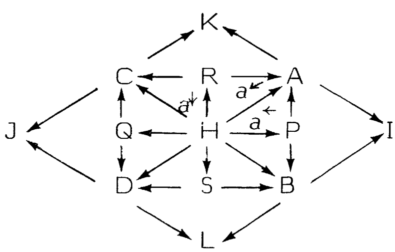
$$d \Rightarrow \overset{\leftarrow}{c} \Rightarrow \overset{\downarrow}{a} \Rightarrow e$$

as well as the analogous ones. Finally the regularity assumptions on $P^{\downarrow}$, $Q^{\downarrow}$, $R^{\leftarrow}$, $S^{\leftarrow}$ give e => d, which loops the loop. ◊

We make now the assumption :

(RI) $P^{\downarrow}$, $Q^{\downarrow}$, $R^{\leftarrow}$, $S^{\leftarrow}$ *are regular and isonomous.*

Then we can complete the commutative diagram by adding the four "cographs" (unavoidably distorted on the figure) :



This diagram defines on H a *second double graph structure,* the *"oblique"* one, denoted suggestively by $\overset{\times}{H}$. We have also oblique graph structures on A, B, C, D.

Note that the oblique maps $a^{\swarrow}$, $b^{\swarrow}$ of $\overset{\times}{H}$ are *right and left full. Conversely any double graph structure* $\overset{\times}{H}$ *having this property comes from a well defined* H^+.

Proposition 8.2 (Turn-table Lemma). *(Lemme de la plaque tournante) Under assumption (RI), the following conditions are equivalent :*

(i) $\overset{\leftarrow}{a}$ *and* $\overset{\leftarrow}{b}$ *(or equivalently* $\overset{\downarrow}{a}$ *and* $\overset{\downarrow}{b}$*) are right and left full (resp. right and left [regularly] faithful, resp. are actors) ;*

(ii) $a^{\nwarrow}$, $b^{\nwarrow}$, $a^{\swarrow}$, $b^{\swarrow}$ *are full (resp. [regularly] faithful, resp. are equivalences).*

Proof. The squares PABI, RACK being universal by construction, we may view HPRA as the decomposition square (cf. § 5) of the morphism $a^{\swarrow}: (a^{\swarrow})^*$ and $a^{\leftarrow}$, $^*(a^{\swarrow})$ and $a^{\downarrow}$ have the same underlying maps. So by Proposition 5.8, each property (ii) for $a^{\swarrow}$ is equivalent to the same right property (i) for $a^{\leftarrow}$ or for $a^{\downarrow}$, because this property is expressed by the same property of the square HPRA. Likewise for the four analogous squares. ◊

These equivalences remain true for *local* properties of the squares and morphisms, so that *Proposition 8.2 remains valid for germs* (with the usual warnings!). In particular we state :

Corollary 8.3. *Let* H^+ *be a germ of double graph satisfying (RI). Then the following conditions are equivalent :*

(i) H^+ *is a germ of bi-actor ;*
(ii) $H^{\times}$ *is a germ of bi-valence.*

They imply H^+ *is regular.*

(The last assertion comes from Propositions 8.1 and 6.5.) ◊

The study of this very rich and beautiful diagram will be completed in Section 10 and will be the key of our constructions. We shall need the following lemma, which concerns a diagram *extracted from the above one :*

Lemma 8.4. *Let* H^+ *be a germ of double graph such that* HPRA *is versal, and suppose the diagram is completed by two squares* QCDJ, SBDL *which are monic, with submersive edges. Assume* $\llcorner D \lrcorner$ *is a germ of monograph.*

Now let be given four germs of maps p, q, r, s, from a germ of manifold Z *to* P, Q, R, S, *such that the whole diagram be commutative. Denote the composed germs by a, b, c, d, j, l, from* Z *to* A, B, C, D, J, L.

Then if j, l are surmersive, there exists a germ h from Z *to* H *making the whole diagram commutative.*

Proof. We construct h using the versality of HRPA. By composition we get only three (possibly) new germs of maps q', s', d' from Z to Q, S, D. Now the monograph property of $\llcorner D \lrcorner$ gives $d = d'$ and then the monicity of SBDL, QCDJ gives $s = s'$, $q = q'$. ◊

As an immediate consequence of Proposition 8.1, we have :

Proposition 8.5. *With the notations of Proposition 7.1, let be given a universal square of morphisms with surmersive edges. Then if* $\overset{\leftarrow}{R}'$ *and* $\overset{\leftarrow}{S}$ *are regular, so is* $\overset{\leftarrow}{S}'$.

In fact, taking into account the change of notations, the assumption of Proposition 8.1 is satisfied and the conclusion follows from $\overset{\leftarrow}{c} \Leftrightarrow \overset{\downarrow}{c}$. ◊

9. THE LOCAL RULE OF THREE : GERMS OF CONVECTORS AND CONVECTIONS.

The next step consists in generalizing foliations and graphoids by extending the isonomy condition $RR^{-1}R = R$ for a regular graph.

First let $(\overleftarrow{R}, b, a)$ be a *regular* graph from A to B. With the above notations the characterization of isonomous graphs recalled in §1 may be restated (rather pedantically) as follows :

The following statements are equivalent :
a) there exist surmersions $u : A \to Q$, $v : B \to Q$ such that the square (a, b, u, v) be a pullback ;
b) there exists a (unique) morphism of isogenies $\vartheta : N(\overleftarrow{R}) \to \overleftarrow{R}$;
b') $RR^{-1}R = R$.

Note that the graphs of *regular equivalences* are just the *verticial isonomous regular graphs.*

In the singular case, we have to give up the first formulation, and the second condition cannot be required but *locally* which leads to the following fundamental generalization :

Definition 9.1. A germ of monograph (of class C^k) $[\overleftarrow{R}]_x$ is called *isonomous* if there exists a (necessarily unique) germ of morphism from $(N(\overleftarrow{R}))_{xxx}$ to $(\overleftarrow{R})_x$.

A germ of isonomous monograph is called a *germ of convector,* and a *germ of convection* if it is moreover verticial.

The germs of convectors (resp. convections) define an *open submanifold* C of $\mathcal{J}_m$ (resp. C_o of $\mathcal{J}_{m_o}$). The next objective is to construct on C a canonical structure of differentiable groupoid with base C_o, containing the universal graphoid H as an open subgroupoid.

Definition 9.2. An open subset of C is called a *convector.* A *local convector* is a graph whose all germs lie in C. A *convection* is a local section of the canonical projection $C_o \to M$.

A regular foliation may be identified with its canonical convection.

Definition 9.1 means precisely that we can find a monograph representative $\overleftarrow{R}$ from A to B, an open subset U of the manifold $N = |N(\overleftarrow{R})|$ containing (x, x, x), and a (differentiable) map $\vartheta : U \to R$ commuting with the projections into A and B. The following *notation* may be suggestive : $\vartheta(z, y, x) = (z : y : x)$.

Note that there is a *maximum* U denoted by $\Theta(\overleftarrow{R})$ on which such a ϑ is defined (for two local ϑ's have to agree in the intersection of their domains).

This unique ϑ (assumed to exist) is uniquely characterized by the properties

$$a(z : y : x) = a(x), \qquad b(z : y : x) = b(z).$$

The graph (see the above convention at the end of Section 1) of this map $\vartheta_{\leftarrow}$ is a submanifold of R^4, diffeomorphic to $\Theta(\overleftarrow{R})$, denoted by $H = H(\overleftarrow{R})$, and called the *ratio manifold* (variété des proportions) of $\overleftarrow{R}$.

In general there is no representative $\overleftarrow{R}$ of a germ of convector such that the rule of three is defined on the whole of $N(\overleftarrow{R})$, so that, in the regular case, the terminology of Definition 9.1 requires a justification, which is given by the important following *semi-global* proposition :

Proposition 9.3. *Let $[\overleftarrow{R}]_{x_o}$ be a germ of regular graph. Then the following conditions are equivalent :*

a) it is a germ of convector ;

b) it admits a representative which is a regular isonomous graph.

Only a $\Rightarrow$ b has to be proved. Let $\overleftarrow{R}$ be a regular representative of the germ, and set

$$a_o = a(x_o), \qquad b_o = b(x_o).$$

From the fact that $N = |N(\overleftarrow{R})|$ is a submanifold of R^3 and R a submanifold of $B\times A$, we can assume that ϑ is defined on $Z = T^3 \cap N$, where T is an open neighborhood of x_o in R, and is itself the trace on R of $W\times V$, where V, W are open neighborhoods of a_o, b_o in A, B. If $\overleftarrow{T}$ is the regular graph induced by $\overleftarrow{R}$ on T, we note that Z is the whole of $|N(\overleftarrow{T})|$. On the other hand ϑ, which is well defined by its projections $a\vartheta$, $b\vartheta$, takes its values in $R \cap (W\times V) = T$. So $\overleftarrow{T}$ is a (globally) regular isonomous representative of the germ. ◊

It is worth noting too that for a regular germ being isonomous, *the set-theoretic existence of ϑ is enough.*

Warning 9.4. In the following we shall encounter a germ of manifold bearing two (and more) germs of regular isonomous graphs. In spite of the above semi-global proposition, it will not be possible in general to find a representative which is a (global) regular isonomous graph for *both* structures! (That would simplify the proofs significantly, but they would be false!)

The consideration of the ratio manifold enlightens the symmetry properties of the rule of three :

Proposition 9.5. *The germ of the ratio manifold* H *is invariant by the Klein group of permutations of* R^4.

Proof. Let us denote by u, v the maps $N \to R$ induced by the canonical projections pr_3, pr_2 : these are submersions. The maps

$$\sigma: (z, y, x) \mapsto (y, z, t), \qquad \tau : (z, y, x) \mapsto \vartheta(x, t, z), \quad \text{where } t = (z, y, x),$$

from Θ to R, induce maps with range N, hence *local* maps from Θ to

Θ. So we have well defined local maps $u' = \vartheta\sigma$, $v' = \vartheta\tau$. Now it is immediate that one has

$$au' = au, \quad bu' = bu, \quad av' = av, \quad bv' = bv,$$

and these maps are submersions ; so by the monograph property of $\overleftarrow{R}$, we have

$$u' = u, \quad v' = v, \quad \text{i.e.,} \ \vartheta(y, z, t) = x, \ \vartheta(x, t, z) = y$$

whenever these are defined, which expresses the invariance of the germ of H by the generators of the Klein group. ◊

Corollary 9.6. *There are on H two associated canonical germs of double graphs $H^{\times}$ and H^{+} described in the diagram below. Moreover H^{+} is a germ of bi-actor and $H^{\times}$ a germ of bi-valence, and H^{+} is regular.*

$$\begin{tikzcd}
& & A & & \\
& R \arrow[ur, "a"] \arrow[dl, "b"'] & \Lambda R \arrow[l] \arrow[r] & R \arrow[ul, "a"'] \arrow[dr, "b"] & \\
B & VR \arrow[u] \arrow[d] & H \arrow[l, dotted] \arrow[u, dotted] \arrow[r] \arrow[d] \arrow[ul, "\vartheta'"] \arrow[ur] \arrow[dl] \arrow[dr, "(\mathrm{I})"] & VR \arrow[u] \arrow[d] & B \\
& R \arrow[ul, "b"] \arrow[dr, "a"'] & \Lambda R \arrow[l] \arrow[r] & R \arrow[ur, "b"'] \arrow[dl, "a"] & \\
& & A & &
\end{tikzcd}$$

Identifying H with Θ, we have first the *locally* universal square (I) with submersive edges and the map ϑ' (identified with the rule of three). We get the two other diagonal arrows by composition, and then the dotted maps by the pullback properties. Now Proposition 9.5 forces ϑ' to be a (possibly non surjective!) submersion too as well as the dotted maps, and the three other squares to be *locally* universal (but not universal in general!). The last conclusion comes from Corollary 8.3. ◊

10. THE PULLBACK-PUSHOUT HYPERCUBE.

For a better understanding of the properties of the diagram just above, and an analogous one to be encountered later, we state first some general stability properties of convectors and we come back to the slightly more general context of Section 8.

Proposition 10.1. *Let $r : \overleftarrow{R}' \to \overleftarrow{R}$ be a germ of equivalence over two submersive germs f, g (with $f =_{\leftarrow} g$). Then $\overleftarrow{R}'$ is a germ of [regular] convector (convection) iff so is $\overleftarrow{R}$.*

Proof. First the properties of regularity, monograph and verticiality come from Propositions 6.5 and 7.4. Now to the rule of three :

Using the functoriality of N, the "if" part comes from the versality of $Q(r)$, while the "only if" part is a consequence of the right invertibility of the germ of map Nr (which results from Proposition 6.4 or Corollary 5.6). ◊

We can now complete Proposition 7.1 for universal squares of morphisms (keeping the same notations) :

Proposition 10.2. *Given a germ of universal square of morphisms over submersive germs f, g, m, n [with $f = g$, $\overleftarrow{m} = n$], then if $\overleftarrow{R}$, $\overleftarrow{R}'$, $\overleftarrow{S}$ are germs of convectors [convections], so is $\overleftarrow{S}'$.*

Proof. Applying the functor N, we get a new commutative square. Because of the monograph property of $\overleftarrow{R}$, the diagram is still commutative when we add the three given germs of rules of three. The fourth rule of three now results from the pullback property. The proof of the verticiality assertion follows the same lines. ◊

We now complete Corollary 8.3 :

Proposition 10.3. *Given a germ of diagram $(H^{+}, H^{\times})$ satisfying the (equivalent) conditions of Corollary 8.3, then $H^{\leftarrow}$, $H^{\downarrow}$ are both germs of regular convectors.*

Proof. We already know $H^{\leftarrow}$ and $H^{\downarrow}$ are regular. To construct the rule of three, we apply the functor N to the germs of morphisms $H^{\downarrow} \to P^{\downarrow}$, $Q^{\downarrow}$, which by Proposition 6.4 gives submersive germs, and we compose with the germs of rules of three (which are again submersive). So we get germs from $N(H^{\downarrow})$ to $P^{\downarrow}$, $Q^{\downarrow}$, which are submersive, as well as the canonical germ from $N(H^{\downarrow})$ to R, S, and the whole diagram is clearly commutative. The Lemma 8.4 applies and gives the conclusion for $H^{\downarrow}$. Likewise for $H^{\leftarrow}$. ◊

As a consequence we have by Proposition 9.3 well defined germs of "cographs" $(Q, P \to E)$, $(S, R \to F)$, which complete two new germs of pushout universal squares.

Moreover, using the pushout property of these squares (cf. Remark A 7), we define on E, F graph structures

$$_{L}E_{K} = E^{\downarrow}\,, \qquad _{J}F_{I} = F^{\leftarrow}\,.$$

So we have built two germs of universal squares of graph morphisms

$$H^{\downarrow}Q^{\downarrow}P^{\downarrow}E^{\downarrow}\,, \qquad H^{\leftarrow}S^{\leftarrow}R^{\leftarrow}F^{\leftarrow}$$

(which are pushouts too) and by Proposition 7.1 3° a we know that the germs $P^{\downarrow}$ (or $Q^{\downarrow}$) $\to E^{\downarrow}$, $R^{\leftarrow}$ (or $S^{\leftarrow}$) $\to F^{\leftarrow}$ are again germs of actors, which means that the four new squares that arise PAEK, RAFI, PBEL, RCFJ, are again germs of universal squares.

Now the full symmetry of the superb commutative diagram that has just arisen cannot be restored but by a *four-dimensional figure,* which the extra-terrestrial reader will draw easily, while the sublunary one

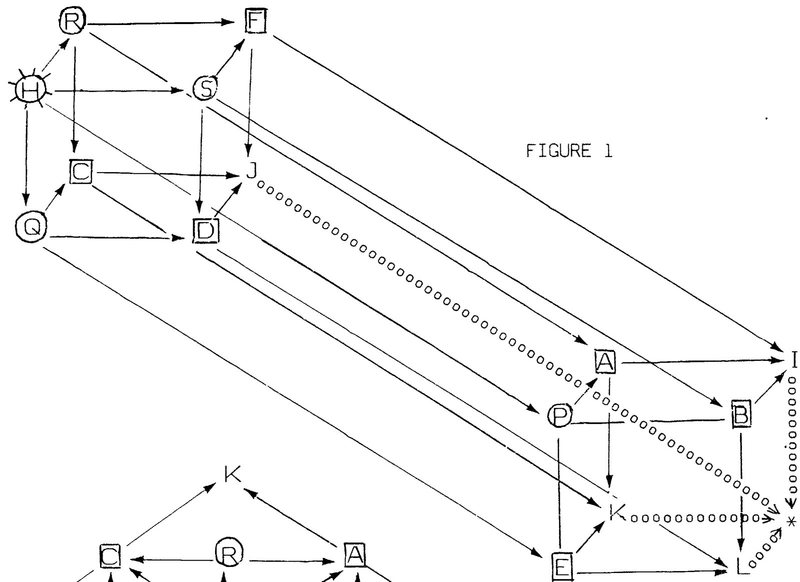


FIGURE 1

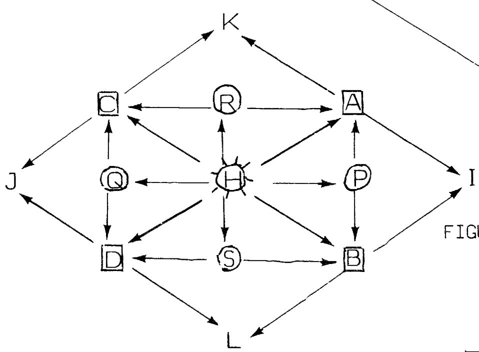


FIGURE 2

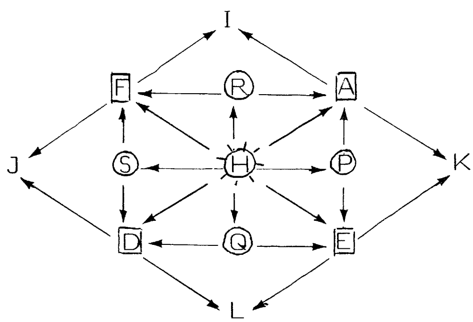


FIGURE 3

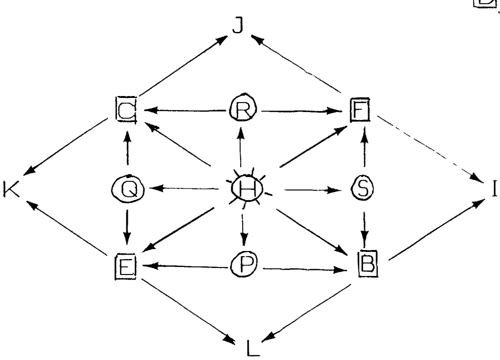


FIGURE 4

will be content with the poor Figure 1 on page 26 and its three extracted diagrams.

This diagram reveals four levels of hierarchy among the structures involved, *the structures at the same level playing equivalent roles :*

1° The "sunned" letter H is the origin of *four* edges generating by pairs *six graph structures* (up to symmetries) and *three double graph structures* with their associated "double cross" structures pictured on the extracted diagrams ;

2° the four circled letters P, Q, R, S are the origin of three edges defining *three graph structures* ;

3° the six squared letters A, B, C, D, E, F bear *graph structures*, which now play *symmetric roles ;*

4° the four letters I, J, K, L are non-structured *manifolds ;*

5° finally the starred vertex and the dotted arrows ending at it are "*virtual*": they don't exist as manifold and differentiable maps. However, using the pushout property of the pre-existing square sides, it is possible to construct them in the category of *sets* in such a way that the six new squares arising to complete the "*hypercube*" be simultaneous pushouts. According to the general philosophy of the description of "virtual quotients", the "structure" of this set is described precisely by the differentiable diagrams situated above it.

We are led to the following definition.

Definition 10.4. A *germ of regular superconvector* is a germ of commutative diagram as in Figure 1 (without the dotted part) such that all the edges are submersive and all the square sides are germs of universal squares.

Scholium. We can then summarize the above discussion by saying that such a germ is determined :

1° in three different ways by a germ of bi-valence whose four squares are germs of versal squares, such as $\overset{\times}{H}$;

2° in three different ways by a germ of bi-actor satisfying condition (RI) such as $\overset{+}{H}$;

3° in six different ways by pullback of a pair of germs of actors with common target whose sources are germs of regular convectors, such as $P^{\downarrow}$, $Q^{\downarrow}$.

Applying Proposition 10.1, we have as a consequence the basic :

Proposition 10.5. *If one of the germs* $\overleftarrow{A}$, $\overleftarrow{B}$, $\overleftarrow{C}$, $\overleftarrow{D}$, $\overleftarrow{E}$, $\overleftarrow{F}$, *is a germ of convector, all of them are.*

In fact, by the oblique equivalences, the convector property is transferred from $\overleftarrow{A}$ (for instance) to ${}_{B}H_{C}$, then to $\overleftarrow{D}$, then to ${}_{F}H_{E}$, then to $\overleftarrow{B}$ and $\overleftarrow{C}$, then to ${}_{D}H_{A}$, then to $\overleftarrow{E}$ and $\overleftarrow{F}$, using successively the diagonals of Figures 2, 3, 4, 2, 3. ◊

In the next section, we are going to build two fundamental germs of superconvectors, playing with the equivalence of the various ways of generating them.

11. CONSTRUCTION OF THE UNIVERSAL CONVECTOR.

Given a germ of convector $\overleftarrow{R}$, we first construct the *units* from the diagram of Corollary 9.6 which is a special case of regular superconvector.

Proposition 11.1. $H^{\downarrow}$ *and* $H^{\leftarrow}$ *are regular, isonomous, and verticial.*

Proof. By Section 10, only the verticiality is left to be proved. The map

$$(y, x) \mapsto (y, y, x) \qquad \text{from } R^2 \text{ to } R^3$$

induces a map from VR to NR. Arguing as in Proposition 9.5, one proves that *locally* one has $(y : y : x) = x$. So the germ of the diagonal map from R^2 to R^4 induces a *germ* from VR to H which is the desired bisection for $H^{\leftarrow}$.The proof is similar for $H^{\downarrow}$. ◊

So we have well defined *germs* of regular equivalences in VR and ΛR having H as their common graph. But *this does not mean* (in spite of Proposition 9.3) that we can find a representative of our germ $\overleftarrow{R}$ such that the two (global) equivalences on VR and ΛR have a common global graph!! We can only say that the intersection of these graphs is an open subset of each one.

By the general theory of superconvectors the quotient germs bear well defined graph structures (denoted by $\overleftarrow{E}$, $\overleftarrow{F}$ in Section 10, while $\overleftarrow{A}$, $\overleftarrow{B}$, $\overleftarrow{C}$, $\overleftarrow{D}$ coincide all four with the present $\overleftarrow{R}$), which we denote by

$$\overleftarrow{X} = \alpha(\overleftarrow{R}), \qquad \overleftarrow{Y} = \beta(\overleftarrow{R}),$$

respectively from A to A and from B to B. Applying Proposition 10.5 and Remark 7.5, we have :

Proposition 11.2. $\alpha(\overleftarrow{R})$ *and* $\beta(\overleftarrow{R})$ *are germs of convections.*

They are called the source and target of the germ of convector $\overleftarrow{R}$ (*not to be confused* with the germs A, B).

(Note that what we have done is just mimicking the set-theoretic definition of the composites RR^{-1}, $R^{-1}R$, in terms of diagrams.)

From the previous proposition we get well defined global applications $\alpha, \beta : C \to C_0$, which provide factorizations of the canonical surmersions $a, b : C \to M$ through the étale map $C_0 \to M$. This proves that α, β are again *surmersions*, so that we get on C a *new graph structure : (β, α) from C_0 to C_0*.

From now on $\overleftarrow{C}$ will denote this new graph structure. It would be pedantic to make a distinction between the germs of graphs induced by $\overleftarrow{\mathcal{J}}_m$ and by $\overleftarrow{C}$: the latter are deduced from the former by a canonical change of bases and we say that they are put *in canonical form ;* we can assume this has always been done from now on.

Proposition 11.3. *The surmersions* α , β *define retractions of* C *onto* C_0.

Proof. This means that if $x = [\overleftarrow{R}]_x$ is a germ of convection, then $\alpha(\overleftarrow{R})$ and $\overleftarrow{R}$ define the same germ.

This follows from the commutative diagram (to be read in the category of germs) :

$$\begin{tikzcd}
 & \overleftarrow{R} \arrow[dl, "1_R"', "\simeq"] \arrow[d] \arrow[dd, dotted, bend left] \arrow[r, "a"] \arrow[ddr, "a"] & A \arrow[d, "o"] \arrow[dd, bend left=60, "1_A", "\simeq"'] \\
R \arrow[d, "b"'] & VR \arrow[l, "pr_1"] \arrow[r, "pr_2"] \arrow[d] \arrow[dl, "b"] & R \arrow[d, "a"] \\
A & \alpha R \arrow[l, "b"] \arrow[r, "a"] & A
\end{tikzcd}$$

in which the squares are germs of universal squares, hence the vertical composite too. This implies the dotted vertical map to be a germ of diffeomorphism, and more precisely a germ of isogeny isomorphism, therefore the identity for the germs of graphs. ◊

More precisely the diagram proves that α and β extend to the whole of C the local retractions of $\mathcal{J}_m$ considered in Section 3.

Now to the *composition* of germs of convectors *(not to be confused* with the redundant one of Section 4).

For more symmetry it will be convenient to define first the *"difference* (or quotient) *law"*

$$\delta(y, x) = yx^{-1} \qquad \text{from } \Lambda(\overleftarrow{C}) \text{ to } C.$$

So we consider two germs of convectors $\overleftarrow{R}$, $\overleftarrow{S}$ with $\alpha(\overleftarrow{R}) = \alpha(\overleftarrow{S}) = \overleftarrow{X}$, $\overleftarrow{R}$ from A to B, $\overleftarrow{S}$ from A to C.

Let us denote by $V\overleftarrow{R}$, $V\overleftarrow{S}$, $\overleftarrow{\Lambda}(S, R)$ the (germs of) graphs we get by considering the (germs of) manifolds $|V(\overleftarrow{R})|$, $|V(\overleftarrow{S})|$, $|\overleftarrow{S} * \overrightarrow{R}|$, as (germs of) graphs *from* R *to* R, S *to* S , R *to* S *respectively.*

We know that the canonical germs of morphisms from $V\overleftarrow{R}$ and $V\overleftarrow{S}$ to $\overleftarrow{X}$ are germs of actors. So taking their pullback, we construct, by the scholium of Section 10, a germ of regular superconvector associated to the diagram :

$$\begin{tikzcd}
 & & A & & \\
 & S \arrow[ur] \arrow[dl] & \Lambda(S, R) \arrow[l] \arrow[r] & R \arrow[ul] \arrow[dr] & \\
C & VS \arrow[u] \arrow[d] & H \arrow[l] \arrow[r] \arrow[u] \arrow[d] & VR \arrow[u] \arrow[d] & B \\
 & S \arrow[ul] \arrow[dr] & \Lambda(S, R) \arrow[l] \arrow[r] & R \arrow[ur] \arrow[dl] & \\
 & & A & &
\end{tikzcd}$$

By the general theory of Section 10, the vertical structure $H^{\downarrow}$ is regular, isonomous, and verticial (because $\overleftarrow{V}R$, $\overleftarrow{V}S$, $\overleftarrow{X}$ are verticial). It defines a *germ of regular equivalence* on $\Lambda(S, R)$ whose quotient germ gives the sixth lacking squared vertex (denoted F in § 10) with its germ of graph structure from B to C, which will be a germ of convector by Proposition 10.5. We denote it by $\delta(\overleftarrow{S}, \overleftarrow{R}) = \overleftarrow{S}\overrightarrow{R}$.

This gives the announced difference law δ in *C*.

(Once more we have translated the set theoretic composition of graphs in terms of diagrams.)

To prove the *differentiability* of δ, we use the fact that the manifold of germs of monographs is, by construction, *locally* diffeomorphic to the manifold of germs of isogenies (here it is very important to make the distinction !), which is itself locally diffeomorphic to the representative manifolds R, S and so on. So we get a chart for δ which is just the canonical map from $S \times_A R$ to its quotient $|\overleftarrow{S}\,\overrightarrow{R}|$, which is differentiable, and even a *submersion.*

Now we have to check the *axioms of groupoids.*

Proposition 11.4. *One has :*

$$\alpha(\overleftarrow{S}\,\overrightarrow{R}) = \beta(\overleftarrow{R}), \quad \beta(\overleftarrow{S}\,\overrightarrow{R}) = \beta(\overleftarrow{S}).$$

Proof. Keeping the same notations, let us consider the two morphisms :

$$\overleftarrow{S} \leftarrow \overleftarrow{\Lambda}(S, R) \rightarrow \overleftarrow{S}\,\overrightarrow{R}.$$

By Proposition 6.4, we get two submersive germs of maps when applying the functor Λ. Writing Y for the mid term, we can write the commutative diagram :

$$\begin{tikzcd}
\Lambda(\overleftarrow{S}) \arrow[d] & Y \arrow[l] \arrow[r] \arrow[d] \arrow[dl] \arrow[dr] & \Lambda(\overleftarrow{S}\,\overrightarrow{R}) \arrow[d] \\
\beta(\overleftarrow{S}) \arrow[r] & C \times C & \beta(\overleftarrow{S}\,\overrightarrow{R}) \arrow[l]
\end{tikzcd}$$

which gives two factorizations of the same germ of isogeny through a submersion on a monograph. So the conclusion follows from Proposition 2.3.

The proof is similar for α. ◊

Note that, by the very construction, we have

$$\overleftarrow{R}\,\overrightarrow{R} = \beta(R), \quad \overrightarrow{R}\,\overleftarrow{R} = \alpha(R).$$

So if we set

$$\sigma(\overleftarrow{R}) = \overrightarrow{R} \quad \text{and} \quad \gamma(\overleftarrow{S}, \overleftarrow{R}) = \delta(\overleftarrow{S}, \sigma(\overleftarrow{R})) = \overleftarrow{S}\,\overleftarrow{R},$$

we have an *inverse law,* and, by Proposition 11.4, we have

$$\alpha(\overleftarrow{S}\,\overleftarrow{R}) = \alpha(\overleftarrow{R}), \quad \beta(\overleftarrow{S}\,\overleftarrow{R}) = \beta(\overleftarrow{S}).$$

The *associativity law* comes from the commutative diagram :

$$\begin{tikzcd}
\overleftarrow{T}\times(\overleftarrow{S}*\overleftarrow{R}) \arrow[d, dashed] & & \overleftarrow{T}*\overleftarrow{S}*\overleftarrow{R} \arrow[ll, tail] \arrow[rr, tail] \arrow[dl] \arrow[ddl] \arrow[dr] \arrow[ddr] & & (\overleftarrow{T}*\overleftarrow{S})\times\overleftarrow{R} \arrow[d] \\
\overleftarrow{T}\times\overleftarrow{S}\,\overleftarrow{R} & \overleftarrow{T}*\overleftarrow{S}\,\overleftarrow{R} \arrow[l, tail] \arrow[d] & & \overleftarrow{T}\,\overleftarrow{S}*\overleftarrow{R} \arrow[r, tail] \arrow[d] & \overleftarrow{T}\,\overleftarrow{S}\times\overleftarrow{R} \\
 & \overleftarrow{T}(\overleftarrow{S}\,\overleftarrow{R}) \arrow[dr] & & (\overleftarrow{T}\,\overleftarrow{S})\overleftarrow{R} \arrow[dl] & \\
 & & D\times A & &
\end{tikzcd}$$

The two lateral squares are written only in order to justify the submersions by their universality. The conclusion follows again from Proposition 2.3.

In the same way the *unit law* comes from the diagram :

$$\begin{tikzcd}
(\beta\overleftarrow{R})\overleftarrow{R} \arrow[dr] & (\overleftarrow{R}*\overleftarrow{R})*\overleftarrow{R} = N(\overleftarrow{R}) \arrow[l] \arrow[r, "\vartheta"] & \overleftarrow{R} \arrow[dl] \\
 & B\times A &
\end{tikzcd}$$

and Proposition 2.3. Likewise for α.

Thus we have just defined on C a *canonical structure of differentiable groupoid* (in the sense of Ehresmann) with base C_0, whose underlying graph structure is the canonical one.

On the other hand, given any differentiable groupoid G (not necessarily a monograph), we can define a *global rule of three* by

$$\vartheta(z, y, x) = zy^{-1}x.$$

It is differentiable and moreover a *submersion* : in fact, we can write $\vartheta = ps$, where p is the second projection of NG, which is submersive, and $s : NG \to NG$ is defined by

$$s(z, y, x) = (x, \vartheta(z, y, x), z),$$

which is clearly differentiable and involutive, hence a diffeomorphism.

So, here, again by Proposition 2.3, the global rule of three of C *must extend the local canonical one, as well as any given one.* Since the composition law is in turn uniquely determined by the global rule of three, we have that the groupoid structure just defined on C is the only one with the canonical graph structure as the underlying one.

Summarizing, we have proved the first two parts of the basic :

Theorem 11.5. *We consider the manifold C of germs of convectors and the submanifold C_0 of germs of convections. Then :*

1° there is defined on C a canonical structure of verticial differen-

tiable graph with C_0 as base ;

2° C admits one and only one differentiable groupoid structure with C_0 as its base whose underlying graph is the canonical one ;

3° given any differentiable groupoid G *with base* B *whose underlying graph is sub-faithful (Definition 2.4) (more precisely a local monograph or faithful), there is a canonical convection* c : B → C_0 *and a canonical functor f over* c *from* G *to C, which is a submersion (more precisely étale or injective) ; in particular the universal graphoid H is identified with an open subgroupoid of C.*

We call C the universal convector.

Proof of 3°. By assumption we have a canonical factorization f of the map $\pi_G = (\beta_G, \alpha_G)$ from G to B×B through $\mathcal{J}_m$, which takes its values in C (because of the rule of three of G) and sends the units into C_0. This gives the desired section c : B → C_0. Since f is a submersion (over a diffeomorphic change of base), so is Nf : NG → NC. By Proposition 2.3, f and Nf commute with the rules of three, which implies f is a functor. ◊

As a consequence any *convector* C in the sense of Definition 9.2 *generates an open subgroupoid* G *of* C. The *open subgroupoid of* C induced on the base of G will be called the *envelop* of C.

Given now a *convection* c : B → C_0 (where B is an open subset of M), we can identify B with its image in C_0.

Definition 11.6. The open subgroupoid H of C induced on c(B) is called the *isonomy groupoid* of the convection ; its α-connected component is called its *holonomy groupoid* H^C. We define the *total (or Godbillon) holonomy groupoid* H^T as the saturation [19] of H^C (which may be strictly larger).

Scholium 11.7. Now Theorem B 8 applies. So there are *canonical Štefan foliations* C^t, C_0^t *of C,* C_0 such that C^t is a local Lie groupoid over C_0^t. C_0^t is induced by C^t and induces it by means of α or β .

Denoting the manifold of germs of Štefan foliations by $F^\vee$, we have a canonical factorization

$$C_0 \longrightarrow F^\vee \longrightarrow M$$

by étale maps.

According to Remark B 7a, to each leaf F defined by a convection is associated a well defined (up to isomorphism) principal bundle H_F^C over F which we call the *holonomy bundle,* whose structural group is the (full) *holonomy group,* and we have a factorization

$$H_F^C \longrightarrow H_F^- \longrightarrow F$$

where the first arrow is a principal bundle with connected structural group, and the second one a normal covering whose group is the *squeezed* *holonomy group.*

12. EXTREMAL CONVECTIONS. HOLONOMOUS ŠTEFAN FOLIATIONS.

Up to now we have ignored the existence properties involved in the pullback-pushout square associated to a *regular* isonomy, save for the existence of the rule of three. Of course these have to be considerably weakened, as can be seen even on the trivial example of Section 2.

Let us fix a Štefan foliation F and consider the (possibly empty) set of convections which define F, partially ordered by the morphisms. A terminal object, when it exists, is called an *extremal convection*. The *germs* of extremal convections define an open subset F^{h} of F^{V}, endowed with a canonical section into C_0, and a groupoid induced by C.

We do not attempt here to study to what extent this open set may be "generic" in some sense, and shall be content with the following criterion, which seems to cover a rather wide range of situations. We take here $k = \infty$.

Let G be a convection over A. The kernel of $T\alpha$ induces on the vertex section a vector bundle $\boldsymbol{g}$, and we denote by $\boldsymbol{p}$ the restriction of $T\pi$. The sheaf $\tilde{\boldsymbol{g}}$ of germs of sections of $\boldsymbol{g}$ has a canonical structure of Lie algebra sheaf (which we considered in [16]) and $\boldsymbol{p}$ defines a morphism $\tilde{\boldsymbol{p}}$ of Lie algebra sheaves into the Lie algebra sheaf of germs of vector fields on A.

Now a *sufficient condition* for the germ of G to be extremal is that $\tilde{\boldsymbol{p}}$ *induce a bijection of* $\tilde{\boldsymbol{g}}$ onto the sheaf of germs of vector fields which are tangent to F. The proof uses the equivalence of the category of vector bundles with the category of locally free sheaves of modules, and the local integration of morphisms of Lie algebroids [16].

For those Štefan foliations which admit an extremal convection, the concepts of holonomy of Section 11 become intrinsic : we call them *holonomous Štefan foliations.* The Ehresmann holonomy groups coincide with our *squeezed* groups when both concepts are defined.

By lack of time and of space, we must postpone the discussion of the invariance of these notions by differentiable equivalence to future papers and to the thesis of my student B. Bigonnet. We just mention here that *a wide variety of holonomous Štefan foliations and holonomy bundles may be built up by suspension from singular foliations generated by vector fields.*

APPENDIX A. VERSAL AND MONIC SQUARES

In order to avoid repetitions, we draw up here a list of some formal properties of commutative squares of manifold morphisms, which are of constant use.

In the following, we consider two composable commutative squares P, Q :

$$\begin{tikzcd} A' \arrow[r, "f'"] \arrow[d, "u"'] \arrow[dr, phantom, "P"] & B' \arrow[r, "g'"] \arrow[d, "v"'] \arrow[dr, phantom, "Q"] & C' \arrow[d, "w"] \\ A \arrow[r, "f"] & B \arrow[r, "g"] & C \end{tikzcd}$$

Definition A 1. 1° The commutative square P is called *locally versal* if A' is non empty and :
a) the set-theoretic fibre product $R = A \times_B B'$ is a *submanifold* of $A \times B'$;
b) the canonical map $i : A' \to R$ is a *submersion*.

If moreover i is surjective (injective, bijective), then P is called *versal (locally universal, universal)*.

2° P is called [*regularly*] *monic* if $(u, f') : A' \to A \times B'$ is injective [and is an embedding].

Note that condition a implies that R is a *pullback* and is satisfied whenever f and v are transversal and in particular when f or v is a submersion, a condition often fulfilled in the applications in view, and that condition 2° is satisfied whenever u or f' is injective [and is an embedding].

Proposition A 2. *1° All the previous notions are stable by product of squares.*

2° Assume P *is locally versal (resp. versal, resp. locally universal, resp. regularly monic). Then if* v *is a submersion (resp. and is surjective, resp. and is injective, resp. is an embedding), so is u.*

3° Assume gf and w are transversal. Then if P *and* Q *are [locally] [uni-]versal, so is* QP. *The assumption may be dropped when* Q *is universal.*

4° Assume Q *is locally universal. Then if* QP *is [locally] [uni-] versal, so is* P.

5° Assume f is a surmersion, the pair (g, w) is transversal, and P *is versal. Then if* QP *is [locally] [uni-]versal, so is* Q.

6° a) If P *and* Q *are [regularly] monic, so is* QP.
b) If QP *is [regularly] monic, so is* P.
c) Assume f is surjective [and a submersion] and P *is a set-theoretic pullback. Then if* QP *is [regularly] monic, so is* Q.
d) Assume P *is regularly monic (more particularly locally universal). Then if* v *is an embedding, so is u.*

Corollary A 3. *If* Q *is universal, then* P *and* QP *have the same proper-*

ties of versality (and monicity).

This applies in particular to the squares :

$$\begin{tikzcd}
A' \arrow[r] \arrow[d] & A\times B' \arrow[r] \arrow[d] & B' \arrow[d] \\
A \arrow[r] & A\times B \arrow[r] & B
\end{tikzcd}$$

Proposition A2 is especially powerful *in cubic commutative diagrams.* The detailed proof is left to the reader and consists in a repeated use of purely formal elementary properties of submersions, embeddings and pullbacks, through the following commutative diagram :

$$\begin{tikzcd}
A' \arrow[d] \arrow[dr, "f'"] \arrow[ddd, bend right, "u"'] & & \\
R \arrow[r] \arrow[d] & B' \arrow[d] \arrow[dr, "g'"] \arrow[dd, bend right, "v"'] & \\
T \arrow[r] \arrow[d] & S \arrow[r] \arrow[d] & C' \arrow[d, "w"] \\
A \arrow[r, "f"] & B \arrow[r, "g"] & C
\end{tikzcd}$$

where the three squares are pullbacks whose existence derives from the various assumptions involved. ◊

This proposition will be used throughout several hundreds of times.

Remark A 4. Let us consider two maps $Z \to A, B'$ making the whole diagram commutative. Then :

1° if P is monic, there exists at most one map $Z \to A'$ making the diagram commutative ;

2° if P is regularly monic and if there exists a set-theoretic map $Z \to A'$ making the diagram commutative, this map is differentiable ;

3° if P is versal, such a map exists *locally*, around any $z \in Z$;

4° the statement of Proposition A 2 is simplified when all the edge maps are surmersions, a frequent situation in the applications.

Warning A 5. The above definitions and proposition have their counterpart in term of *germs of squares.* Using the fact that the topology of the fibre product R is *induced by the product topology* of $A\times B'$, it is readily seen that a germ of locally [uni-]versal square admits a representative which is [uni-]versal. If moreover some of the edges are submersive germs, the representative may be chosen in such a way that they are represented by surmersions. However when *several* squares occur in a commutative diagram, it will not be in general possible to find a representative of the whole diagram with these properties *simultaneously* verified for all the squares : one has to be content with *local* [uni-] versality and *sub*mersions. This is a source of technical difficulties.

Proposition *A 6. If* P *is a versal square with f,* v *(hence f', u too) surmersions, then it is a pushout.*

Proof. First the pullback square RAB'B with surjective edges is a set-theoretic pushout, hence, using the fact that *f* is a surmersion, a differentiable one, and we conclude by using the epimorphism property of A' → R. ◊

Remark A 7. *The square* P *is also a pushout in the category of surmersions.*

Remark A 8. In the category Dif there exist pullback squares which don't arise from the transversality condition and pullbacks which are not universal in the present sense. It seems to us that the universal ones are the best adapted for a wide range of applications.

APPENDIX B. ŠTEFAN FOLIATIONS AND DIFFERENTIABLE GROUPOIDS

It is convenient to introduce first an auxiliary notion :

Definition B 1. A *prefoliation* of a manifold X is a second manifold structure X' on the same underlying set (called the *fine structure*) such that the identity mapping *i'* : X' → X be an immersion.

Trivial examples are the coarse foliation X and the *discrete* one, denoted by $X^{\bullet}$.

We say the prefoliation X" of X is *finer* than X' if it defines a prefoliation of X', or equivalently if its underlying topology is finer than that of X'.

The connected components of the fine structure are called the *leaves* of the prefoliation F = (X, X'). *If* X *is paracompact, any leaf is second countable.*

There is an obvious notion of *morphisms* of prefoliations.

The following "*elementary operations*" for prefoliations are defined in an obvious way : product, inverse image by a map *f* : Y → X which is transversal to the identity *i'*, gluing, suspension.

A prefoliation is called an *elementary foliation* if it is isomorphic to the product of a coarse one and a discrete one.

Within this framework, a *(regular) foliation* may be defined as a prefoliation which is locally elementary.

Likewise, we can restate the important definition of Štefan [20] :

Definition B 2. A prefoliation F = (X, X') is called a *Štefan foliation* if for any *a* ∈ X, there exists a neighborhood U of *a* and a regular foliation U" of U which is finer than the induced prefoliation U' and such that *a* has the same leaf (with the same manifold structure) in U' and U".

This notion is clearly *stable by the elementary operations* above (but not by composition of identity maps!). More precisely, we note :

Remark B 2 a. Let Z be a submanifold of X. If the transversality condition holds at $x \in X$, then it still holds on the trace of a *neighborhood of the topology of* X (and the induced prefoliation is a Štefan foliation).

Any Štefan foliation is locally the *product* of a coarse one by the Štefan foliation induced on a transverse submanifold, which admits a singular leaf consisting of a single point.

In the subsequent proposition, an immersion $j : F \to X$ is called a *weak embedding* if given any manifold Z and any set-theoretic mapping $f: Z \to F$, the condition "jf is of class C^k" implies "f is continuous" (hence C^k). Any subset $F \subset X$ bears at most one manifold structure such that the canonical injection be a weak embedding ; if such is the case, we say F is a *weak submanifold.*

By the same argument as in the regular case [4], one proves :

Proposition B 3. *Any leaf of a Štefan foliation which is second countable is a weak submanifold. This applies in particular to any leaf of a paracompact manifold.*

(Actually the universal property above is still valid for continuous maps, replacing Z by any locally connected topological space.)

Definition B 3 a. A *prefoliation* whose all leaves are weak submanifolds is called *strict.*

Corollary B 4. *Given an equivalence relation on a paracompact manifold* X, *there is at most one Štefan foliation on* X *whose leaves are the given equivalence classes.*

The importance of Štefan foliations lies in the following facts :

- *To any pseudogroup* G *of local diffeomorphisms of* X, *Štefan has associated a canonical Štefan foliation for which the orbits of* G *are unions of leaves. In particular, given any family of vector fields on* X, *its classes of accessibility are exactly the leaves of a Štefan foliation.* [20]

The subsequent Theorem B 8 for differentiable groupoids in NOT a generalization of Štefan Theorem. It completes the statement of Theorem 4 in [15], and we sketch its unpublished proof.

First we recall briefly some basic (but not well known enough) facts about differentiable groupoids in the sense of Ehresmann [7, 8, 15, 16, 17].

Given a groupoid G with base B, we denote by α, β the source and target projections from G to B, $\omega : B \to G$ the canonical *identification* of objects with units, γ the composition law, σ the inverse law, δ the "difference law" $(y, x) \mapsto yx^{-1}$ from $\wedge G$ to G. For any $e \in B$, we set

$$G_e = \alpha^{-1}(e) \ , \qquad {}_eG = \beta^{-1}(e),$$

${}_eG_e$ the isotropy group, $G^t(e)$, $B^t(e)$ the *intransitivity classes* of e in G, B. We set

$$\pi = (\beta , \alpha) : G \to B{\times}B.$$

A *differentiable groupoid* structure on G consists in manifold structures on G and B such that :

(i) $\alpha : G \to B$ is a *surmersion* (which implies ΛG is a submanifold of $G{\times}G$) ;

(ii) $\omega : B \to G$ is differentiable, hence an embedding (which *identifies* B and $\omega(B)$) ;

(iii) $\delta : \Lambda G \to G$ is differentiable.

Considering the maps

$$x \mapsto \delta(\alpha x, x), \qquad (y, x) \mapsto (yx^{-1}, x^{-1}),$$

which are involutive, hence diffeomorphisms, one sees that σ is a diffeomorphism and β, γ, δ are *surmersions,* hence *open* maps. In particular π defines a *differentiable* isogeny (and *graph*) (§ 2).

We say G is *étale* if α is étale ; this notion is equivalent to that of *van Est manifold scheme* [22].

Ehresmann has defined a *canonical étale prolongation,* which we denote by SG, together with an immersive projection functor $s : SG \to G$. It consists of those germs of submanifolds of G which are *bi-transverse,* i.e., transverse to both α and β, with a suitable composition law. It admits two canonical representations into the pseudogroup of germs of local diffeomorphisms of G, by means of germs of *"local right or left translations"*. The image of s is an open subgroupoid G^h of G, which we call the *homogeneous component* of G.

We denote by G^α, G^β the (regular) foliations defined by α, β, and by G^c (called the *α-connected component* of G) the union of the leaves of G^α meeting B. It is not hard to prove :

Proposition B 5. *G^c is an open subgroupoid of G, contained in G^h, and generated by any neighborhood of B in G^c.*

We observe that the local left translations are local automorphisms of both G^α and G^β. Moreover (G^h being open in G) they act locally transitively in G^α. Denoting by i^α the identity map $G^\alpha \to G$, this implies that the rank of βi^α is locally constant. Together with the symmetric statement, this gives the ("well unknown"!) :

Proposition B 6. *The rank of π is constant on the leaves of G^α and of G^β.*

This defines a canonical prefoliation G^π of G which defines a regular foliation of both G^α and G^β (but not of G in general!).

An immediate consequence is that, for any $e \in B$, *the isotropy group* *of e is a Lie group* which acts differentiably on G_e. Moreover this action defines a *principal bundle* structure on G_e, whose base space is the intransitivity class $B^t(e)$, which inherits in that way a *manifold structure, immersed in* B. Using right translations (which are diffeomorphisms of the α-fibres commuting with β), one sees that this structure does not depend on the choice of e in the intransitivity class. Thus we have defined a *prefoliation* B^t of B.

Now we can view the intransitivity class $G^t(e)$ as the *associate bundle* with fibre type G_e, and this defines on it a manifold structure which, by the same argument, does not depend on the choice of e. The construction of the associate bundle is described by means of a universal square, which is the front side of the following commutative cubic diagram :

$$\begin{tikzcd}
& \Lambda G \arrow[rr, "pr_2"] \arrow[dd, "\delta"] & & G \arrow[dd, "\beta"] \\
G_e \times G_e \arrow[ur] \arrow[rr, "pr_2"] \arrow[dr] \arrow[dd] & & G_e \arrow[ur] \arrow[dr] \arrow[dd] & \\
& G \arrow[rr, "\alpha"] & & B \\
G^t(e) \arrow[ur, dotted] \arrow[rr, "\alpha^t"] & & B^t(e) \arrow[ur, dotted] &
\end{tikzcd}$$

The differentiability of the dotted arrows comes from the differentiability of the diagonal arrows and the universal property of the vertical surmersions. Now, by Proposition A 2, the universality of the front, rear, and top sides implies the *universality of the bottom side* too.

This means that G^t is also a *prefoliation of* G, *induced by* α from the prefoliation B^t of B, and the maps induced by α and ω are still a surmersion and an embedding.

We consider now the new cubic commutative diagram :

$$\begin{tikzcd}
& \Lambda G \arrow[rr, "pr_2"] \arrow[dd, "\delta"] & & G \arrow[dd, "\beta"] \\
\Lambda G^t(e) \arrow[ur] \arrow[rr, "pr_2"] \arrow[dd, dotted, "\delta^t"'] & & G^t(e) \arrow[ur] \arrow[dd, "\beta^t"] & \\
& G \arrow[rr, "\alpha"] & & B \\
G^t(e) \arrow[ur] \arrow[rr, "\alpha^t"] & & B^t(e) \arrow[ur] &
\end{tikzcd}$$

Here the differentiability of the dotted arrow comes from the universality of the bottom side.

This completes the proof that (G^t, B^t) *is a differentiable groupoid,* and the identity $i^t : G^t \to G$ is an immersive functor. Note that the symmetry induces a diffeomorphism of G^t, so that a posteriori G^t *is induced by* β *too.*

Moreover $\pi^t : G^t(e) \to B^t(e) \times B^t(e)$ is a *surmersion* ; this comes from the fact that, when composing with the surmersion $G_e \times G_e \to G^t(e)$, we get the surmersion $\beta^t \times \beta^t$. This may be expressed using the following :

Definition B 7. The differentiable groupoid G is called a *local Lie groupoid* (or *locally transitive* groupoid) (resp. a *local Galois groupoid* (or *locally coarse* groupoid), resp. a *local graphoid*) if $\pi_G : G \to B \times B$ is a submersion (resp. an étale map, resp. an immersion). We define Lie and Galois groupoids by adding the surjectivity condition for π_G.

We open here a short parenthensis.

Remark B 7 a. For a Lie (resp. Galois) groupoid, the α-fibres are principal bundles (resp. Galois coverings) over B.

Given any Lie groupoid G, the kernel of π is a differentiable subgroupoid N on which α and β agree (it seems convenient to call this a *multigroup*). Now the α-connected component N^c is invariant in G, and this defines an *exact sequence* of differentiable groupoids

$$N^c \longrightarrow G \longrightarrow G^{-},$$

where G^{-} is a Galois groupoid which we call the *condensed* (or *squeezed*) groupoid. For the α-fibres, we have a sequence

$$G_e \longrightarrow G_e^{-} \longrightarrow B,$$

where the first arrow is a principal fibration with connected structural group and the second a Galois (or normal) covering.

We come back now to the situation described before Definition B 7. We have the following *universal property :*

Given any local Lie groupoid L over the base C, and any differentiable functor $u : L \to G$, we have a unique factorization $u = i^t u^t$, where $u^t : L \to G^t$ is a differentiable functor.

Proof. The intransitivity classes of L are open, hence closed, so that we can assume L is transitive. Let us take any $a \in C$, and denote its image by $e \in B$. The universal square :

$$\begin{array}{ccc} L_a & \rightarrowtail & L \\ \downarrow & & \downarrow{\scriptstyle \pi} \\ C \times \{a\} & \rightarrowtail & C \times C \end{array}$$

shows that the restriction of β to L_a is a surmersion, which implies

$$C = C^t(a) = C^t\,, \qquad \text{hence} \qquad L = L^t\,.$$

Then we know that the restriction of δ to $L_a \times L_a$ is a surmersion, and now the differentiability of the set-theoretic mapping u^t follows from the commutative diagram :

$$\begin{tikzcd} L_a \times L_a \arrow[r] \arrow[d, "\delta"'] \arrow[dr] & G_e \times G_e \arrow[d] & \\ L \arrow[r, dotted, "u^t"] & G^t(e) \arrow[r, "i^t"] & G \end{tikzcd}$$

◊

The above discussions can now be summarized by the following synthetic statement :

Theorem B 8. *The full subcategory of local Lie groupoids is coreflective* [14] *in the category of differentiable groupoids and functors. Moreover the unit of the adjunction $i^t : G^t \to G$ defines a Štefan foliation.*

Proof. Only the last statement is still to be proved. The proof we give does not use Štefan Theorem for vector fields and *is still valid in class* C^1. It is enough to prove that the prefoliation B^t of B is Štefan, for G^t is the prefoliation induced by α (or by β).

Let e lie in $B \subset G$. We consider a small manifold Z through e which is transverse to the kernel of $T_e\pi$. Then Z is *transversal* to the regular foliations G^α, G^β, so that there are induced *regular foliations* Z^α, Z^β which are moreover transverse to one another. We denote by α_Z, β_Z the submersions induced by α, β.

We take now a small submanifold T of Z which is transverse to $\operatorname{Ker} T\alpha_Z \oplus \operatorname{Ker} T\beta_Z$ and we set $W = \beta_Z^{-1}(\beta_Z(T))$, which is a submanifold of Z. Then (locally) β_Z defines a regular foliation W^β of W while α_Z induces a diffeomorphism onto an open submanifold U of B, with a regular foliation U'' inherited from W^β.

It is now an easy matter to check that U'' satisfies the conditions of Definition B 2 : to compare U'' and U^t, it is enough to compare the prefoliations induced by α on G ; the leaves of the former are unions of leaves of G^α meeting a connected submanifold of a leaf of G^β, therefore contained in a leaf of G^t. ◊

Remark B 9. 1° The identity functor i^t is not a *differentiable* equivalence in the sense of Definition 6.9, because the condition of essential surmersivity of Definition 5.7 is not fulfilled. However the square $Q(i^t)$ is clearly universal.

2° The theorem is *uninteresting for pseudogroups : G^t is discrete.*

3° The theorem still applies for non-Hausdorff and non countable manifolds ; *we did not use Proposition B 3.* However *if* B *is paracompact,* B^t *hence* G^t *are strict* (Definition B 3a).

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